15th Triennial World Congress, Barcelona, Spain

ROBUST PREDICTIVE CONTROL OF THE FURUTA PENDULUM

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Abstract: In this paper, a Linear Parameter Varying (LPV) model of the Furuta pendulum is derived. Based on this model, a balancing controller is designed using robust predictive control techniques. Invariant set theory is used to accurately characterise the region of the state space in which the balancing controller is effective. An energy-based control law is used to swing up the pendulum. Invariant sets calculated for the balancing controller can be exploited to systematically determine the switching condition between swing up and balancing controllers. In practice, the pendulum may be swung towards the upright position with varying speed and the speed of the rotating arm may also vary. Based on this physical insight, the speed of the rotating arm is chosen as gain scheduling variable. It is shown that this strategy is effective in achieving a more consistent swing up and balancing behaviour which is not sensitive to the performance of the swing up controller.

Keywords: Furuta pendulum, invariant sets, predictive control, linear parameter varying systems.

1. INTRODUCTION

The inverted pendulum has been a classic benchmark for illustrating various control ideas and techniques. In this work, swing up and balancing of the Furuta pendulum is considered. The Furuta pendulum has the pendulum attached to a rotating arm instead of a cart moving on a straight line. This gives a nice property that there are no end points which makes it convenient for experimentation and especially when velocity control of the arm speed is performed. The pendulum is open loop unstable in the upright configuration and the motor driving the rotating arm has limited authority. Thus the pendulum serves as a suitable process for the study of an unstable system with actuator saturation.

To design a control strategy for swinging up and balancing the Furuta pendulum, it is advantageous to split the control problem into two sub-problems: swing up and balancing, since systematic design methodologies of these low-level controllers are available. In this paper, the energy-based approach proposed in (Astrom and Furuta, 1999) is used to swing up the Furuta pendulum while predictive control techniques (Chisci *et al.*, 2001*a*; Chisci *et al.*, 2001*b*) are employed to design the balancing controller. To perform the task of swing up and balancing the pendulum at its upward position, it is necessary to determine the switching condition so as to guarantee that the balancing controller can catch and balance the pendulum. In (Fierro *et al.*, 1999), Lyapunov theory is used to determine the switching boundaries. In this work, the switching boundaries are calculated using the approach of invariant sets (Blanchini, 1999). Invariant sets can be used to calculate the region of the state space in which the balancing law is guaranteed stabilisable. This

[1] The first author would like to thank the Association of Commonwealth Universities for the fellowship at Cambridge University Engineering Department where this work was carried out.

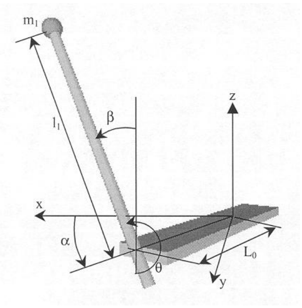

Fig. 1. Schematic diagram of the Furuta pendulum

requires an accurate description of the Furuta pendulum which exhibits nonlinear dynamics. A Linear Parameter Varying (LPV) model of the Furuta pendulum is derived for this purpose and is used in the design of the balancing controller. Polytopic techniques, which assume that the state-space coefficients evolve in a prescribed (bounded) polytope, are used to account for the time-varying and nonlinear nature of the plant. Robust model predictive control techniques are employed to guarantee robust stability as the parameters of the model change with time and with operating points, while statisfying the constraints imposed by the motor driving the rotating arm. A larger region of the state space in which the balancing controller is applicable means that the swing up and balancing performance will be less sensitive to the values of the parameters chosen for the swing up law. To enlarge this region, the speed of the rotating arm is selected as the scheduling variable since the dynamics of the plant system is influenced by the speed of the rotating arm. The LPV synthesis technique provides a framework to incorporate gain scheduling strategies in the controller (Kajiwara *et al.*, 1999). Simulation results show that with this gain scheduling strategy, the swing up and balancing behaviour of the pendulum is less sensitive to the changes in the parameters of the swing up law.

2. MODELLING OF THE FURUTA PENDULUM

In this section, we will show how an LPV model can be obtained from the nonlinear model of the Furuta pendulum. The LPV model will then form the basis for the robust predictive control strategy described in Section 4. Fig. 1 shows a schematic and the co-ordinate system used for the derivation of the dynamic model for the Furuta pendulum. The pendulum system consists of two sections, namely the rotating arm and the pendulum whose angular positions are denoted respectively by α and $\psi \triangleq \theta - \pi$ ($\psi = 0$ in the upward position). Using the method of Lagrange, the nonlinear model of the Furuta pendulum is

$$J(\psi)\begin{bmatrix}\ddot{\alpha}\\ \ddot{\psi}\end{bmatrix} + C(\psi,\dot{\psi},\dot{\alpha})\begin{bmatrix}\dot{\alpha}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}t_3 v\\ t_6\sin\psi\end{bmatrix} \tag{1}$$

where

$$J(\psi) \triangleq \begin{bmatrix}1+t_7\sin^2\psi & -t_1\cos\psi\\ -t_4\cos\psi & 1\end{bmatrix} \tag{2}$$

$$C(\psi,\dot{\psi},\dot{\alpha}) \triangleq \begin{bmatrix}t_2+\frac{1}{2}t_7\dot{\psi}\sin 2\psi & t_1\dot{\psi}\sin\psi + \frac{1}{2}t_7\dot{\alpha}\sin 2\psi\\ -\frac{1}{2}t_8\dot{\alpha}\sin 2\psi & t_5\end{bmatrix} \tag{3}$$

t_i, $1 \le i \le 8$, are suitable coefficients depending on the physical parameters of the system and v is the motor voltage input.

2.1 *Feedback pre-compensation*

To obtain an LPV model for the Furuta pendulum, we first use feedback pre-compensation to obtain a simpler dynamics.

$$\begin{cases}\ddot{\alpha} = u\\ \ddot{\psi} = f(\psi,\dot{\psi},\dot{\alpha},u)\end{cases} \tag{4}$$

This can be achieved by a suitable feedback pre-compensation $u = g(\psi,\dot{\psi},\dot{\alpha},v)$ where

$$\begin{aligned}
&g(\psi,\dot{\psi},\dot{\alpha},v) = \frac{t_3 v - f_2}{f_1} \triangleq u\\
&f_1 = (1 - t_1t_4 + (t_7 + t_1t_4)sin^2(\psi))\\
&f_2 = -t_1t_6cos(\psi)sin(\psi) - \frac{1}{2}t_1t_8\dot{\alpha}^2 sin(2\psi)cos(\psi) +\\
&\qquad + t_1t_5\dot{\psi}cos(\psi) + t_2\dot{\alpha} + t_7\dot{\alpha}\dot{\psi}sin(2\psi) + t_1\dot{\psi}^2 sin(\psi)
\end{aligned} \tag{5}$$

and it turns out that

$$\ddot{\psi} = t_6\sin(\psi) + t_4\cos(\psi)u + t_8\dot{\alpha}^2\sin(\psi)\cos(\psi) - t_5\dot{\psi} \tag{6}$$

At equilibrium, $u = 0$. Thus there is a continuum of equilibria $\psi_e = 0$ or π, $\dot{\psi}_e = 0$, $\dot{\alpha}_e = \frac{t_3}{t_2}v_e$ where v_e is arbitrary. Since, from physical considerations, $t_1t_4 \le 1$, it turns out that $f_1 \ne 0$, $\forall\psi$. This implies that the relationship between the actuator input v and u is invertible. It is proved in (Teel, 1996) that stabilising the system (4) implies stabilising the system (1).

2.2 *Quasi-LPV Model and LPV Relaxation*

From (4), we obtain the state space model

$$\begin{bmatrix}\ddot{\alpha}\\ \ddot{\psi}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & -t_5 & \phi_1(t)\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\dot{\alpha}-\dot{\alpha}_e\\ \dot{\psi}\\ \psi\end{bmatrix} + \begin{bmatrix}1\\ t_4\phi_2(t)\\ 0\end{bmatrix}u \tag{7}$$

with state vector $x = [\dot{\alpha} - \dot{\alpha}_e,\ \dot{\psi},\ \psi]'$. Thus, the dynamics of the Furuta pendulum is characterised by two time-varying parameters

$$\phi_1(t) = (t_6 + t_8\dot{\alpha}^2\cos(\psi))\frac{\sin(\psi)}{\psi},\quad \phi_2(t) = \cos(\psi) \tag{8}$$

Note that the system described by (7) is known as a "quasi-LPV" model (Tu and Shamma, 1998) since its parameters are functions of the states. However, in LPV modelling, the state dependence of the parameters is relaxed, i.e. ϕ_1 and ϕ_2 are treated as independent, in the hope that the relaxed model behaves closely to the quasi-LPV model (7).

2.3 *Discretisation and Constraints*

Since the aim is to implement the control on a digital computer and construct the control action on-line, we discretise (7) with a sampling time h, yielding a discrete time LPV model

$$x(t+1) = A(\phi_1(t))x(t) + B(\phi_2(t))u(t) \qquad (9)$$

where[2] $A(\phi_1(t)) = (I + hA_c(\phi_1(t)))$ and $B(\phi_2(t)) = hB_c(\phi_2(t))$

We shall assume magnitude bounds on the continuous-time state variables $\dot{\alpha}, \psi, \dot{\psi}$ which in turn induce the discrete-time constraints

$$|\dot{\alpha}(t)| \le \dot{\alpha}_u, \quad |\psi(t)| \le \psi_u, \quad |\psi(t+1) - \psi(t)| \le h\dot{\psi}_u \qquad (10)$$

These finally imply magnitude and rate constraints constraints on the parameters

$$\phi_{il} \le \phi_i(t) \le \phi_{iu},\ \delta\phi_{il} \le \phi_i(t+1) - \phi_i(t) \le \delta\phi_{iu},\ i = 1,2 \qquad (11)$$

The behaviour of the relaxed model is close to that of the quasi-LPV model (7) if the constraints (11) are suitably chosen.

2.4 *Polytopic set description of uncertain LTV/LPV systems*

Since the parameters ϕ_1 and ϕ_2 are actually functions of the states, there will be variations in the values of the parameters as the states of the system evolve. In order to account for these possible variations in the parameters, an uncertain LTV/LPV system is considered. From (9), for a given interval of continuous time variables it is possible to compute bounds on the parameters (11) via (8) to obtain a difference inclusion. There will be four vertices characterised by $\{\phi_{1l},\ \phi_{1u},\ \phi_{2l},\ \phi_{2u}\}$. Thus the nonlinear model of the Furuta pendulum can be modelled as a discrete-time uncertain LTV system

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ [A(t), B(t)] &= \sum_{j=1}^{q} \lambda_j(t)[A_j, B_j], \quad \forall t \ge 0 \\ \sum_{j=1}^{q} \lambda_j(t) &= 1,\ \lambda_j(t) \ge 0,\ j = 1,2,\ldots,q. \end{aligned} \qquad (12)$$

Here $x(t) \in \mathbb{R}^3$ is the state, $u(t) \in \mathbb{R}$ is the control input and $q = 4$. The system (12), referred to as *polytopic system*, provides a classical description of model uncertainty. Since the parameters ϕ_1 and ϕ_2 of the Furuta model are measurable, it is convenient to consider a discrete-time LPV system (Shamma and Xiong, 1999)

$$\begin{aligned} x(t+1) &\in \mathcal{F}(p(t)) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},\ x(0) = x_0 \\ p(t+1) &\in Q(p(t)) \qquad\qquad p(0) = p_0 \end{aligned} \qquad (13)$$

where, $p(t) = [\phi_1(t), \phi_2(t)]'$ is a time-varying parameter which belongs to the discrete set $P = \{p_1, p_2, \cdots, p_l\}$ and evolves according to the set-valued map $Q : P \rightsquigarrow P$. Finally the map $\mathcal{F} : P \rightsquigarrow \mathbb{R}^{n\times(n+m)}$ is also set-valued in order to represent additional uncertainty in the system dynamics.

Remark 1. The LPV description reduces to the LTV description when the values and evolution of the parameters are not considered.

[2] The reason for choosing this particular discretisation scheme is to keep A and B linear in p_1 and p_2.

3. PREDICTIVE CONTROL OF LPV SYSTEMS

In this section we briefly recall a predictive control technique (Chisci *et al.*, 2001*b*) for constrained LPV systems that will be subsequently exploited, with appropriate modifications, in the balancing controller of the Furuta pendulum. With reference to the LPV system (13), let us assume that

A1. The model is subject to pointwise-in-time control and state constraints

$$u(t) \in \mathcal{U},\ x(t) \in \mathcal{X} \qquad \forall t \ge 0 \qquad (14)$$

for appropriate polytopes $\mathcal{U}$ and $\mathcal{X}$ containing the origin in the interior.

A2. For each value $p_j \in P$ of the parameter there exists a linear feedback $u(t) = F_j x(t)$ such that the closed-loop polytopic system

$$\begin{aligned} x(t+1) &\in \mathcal{F}(p_j) \begin{bmatrix} x(t) \\ F(p(t))\, x(t) \end{bmatrix} \\ F(p) &\stackrel{\triangle}{=} F_j \quad \text{if } p = p_j \end{aligned} \qquad (15)$$

is absolutely asymptotically stable (Gurvits, 1995).

Let us consider the LPV system

$$\begin{cases} x(t+1) \in \mathcal{F}(p(t)) \begin{bmatrix} x(t) \\ F(p(t))\, x(t) \ +\ c(t) \end{bmatrix} \\ p(t+1) \in Q(p(t)) \end{cases} \qquad (16)$$

for which the actual input $u(t)$ is the sum of the gain-scheduled linear feedback $F(p(t))\ x(t)$ plus the correction term $c(t)$. Let $s(t) \stackrel{\triangle}{=} [x'(t), p(t)]'$ denote the extended state-parameter vector. Let $\mathcal{S}_0$ be the set of initial conditions $s(0) = [x'(0), p(0)]'$ for which the plant state is asymptotically steered to the origin, without violating constraints (14), under the linear gain-scheduled feedback $u(t) = F(p(t))x(t)$. Assuming that $\mathcal{S}_0$ is non empty, it is possible to compute recursively the invariant sets $\mathcal{S}_N$, $N \ge 0$, of vectors $[s'(0), \underline{c}'(0)]' = [s'(0), c(0), c(1), \ldots, c(N-1)]'$ such that $\{c(0), c(1), \ldots, c(N-1)\}$, depending on $s(0)$ only, steers $s(0)$ to $\mathcal{S}_0$, in N steps, while satisfying the constraints (14). Notice that $\mathcal{S}_N$ is actually a collection of sets $\mathcal{S}_N^1, \mathcal{S}_N^2, \ldots, \mathcal{S}_N^l$ corresponding to the parameter values $p_1, p_2, \ldots, p_l$. For details on the computation of $\mathcal{S}_N$ the reader is referred to (Chisci *et al.*, 2001*b*). In order to enlarge the domain of attraction of the gain-scheduled controller, the following predictive control algorithm can be exploited.

- Parameter Varying - Predictive Control (PV-PC)
At each sample time t, given $s(t) = [x'(t), p(t)]'$, find

$$\underline{\hat{c}}(t) \;=\; \arg\min_{\underline{c}(t)} \; \underline{c}'(t)\,\underline{c}(t) \text{ subj. to } \begin{bmatrix} s(t) \\ \underline{c}(t) \end{bmatrix} \in S_N. \tag{17}$$

Then apply to the system the control signal

$$u(t) \;=\; F(p(t))\,x(t) \;+\; \hat{c}(t|t) \tag{18}$$

where $\underline{\hat{c}}(t)' = [\hat{c}(t|t)', \ldots, \hat{c}(t+N-1|t)']$. $\diamondsuit$

Theorem 1. Provided that $s(0)$ is feasible, the receding-horizon control (17)-(18) guarantees that
(1) the constraints (14) are satisfied and
(2) $\lim_{t\to\infty} x(t) = \mathbf{0}$.

Algorithm PV-PC therefore ensures asymptotic stability with domain of attraction $\underline{\Sigma}_N$, where $\underline{\Sigma}_N$ is the projection of $\mathcal{S}_N$ onto the plant state space.

4. CONTROL OF THE FURUTA PENDULUM

4.1 *Swing-up control*

The control law used to swing up the Furuta pendulum is the following (Astrom and Furuta, 1999)

$$u = \mathrm{sat}_{\mathrm{ng}}(kE\,\mathrm{sign}(\dot{\psi}\cos\psi)) \tag{19}$$

where $\mathrm{sat}_{\mathrm{ng}}$ denotes a linear function which saturates at $\pm ng$. The energy of the uncontrolled pendulum, defined to be zero when the pendulum is in the upward position ($\psi = 0$), is calculated as

$$E = \frac{1}{2}J\dot{\psi}^2 + mgl(\cos\psi - 1),$$

where J is the moment of inertia of the pendulum with respect to the pivot point and l is the distance from the pivot to the centre of mass. In other words, the swing up control law has two parameters: n and k. The parameter n influences the behaviour of the swing up; it gives the maximum control signal and thus the maximum rate of energy change. The parameter k determines the region where the swing up law behaves linearly. For different values of n and k, the pendulum approaches the upright position with different speed and the speed of the rotating arm will also be different.

4.2 *Balancing control*

To facilitate the switching strategy between the swing up and balancing controllers, the latter should have the following properties:
(1) a priori known region of the state space in which the pendulum is guaranteed stabilisable;
(2) this region should be as large as possible.

To meet both requirements, we exploit invariant sets. It is important to stress that the determination of invariant sets for the Furuta pendulum is computationally tractable thanks to the embedding of the original nonlinear model into an LPV model carried out in Section 2. Hereafter, we will present two different techniques for balancing control based on a different discretisation of the parameter p. The first technique will use the pendulum angle ψ as scheduling variable, while the second technique will use the arm speed $\dot{\alpha}$ for the same purpose.

4.2.1. *Scheduling on* ψ The idea is to use the LPV model (13) with a discrete parameter set P obtained by discretisation of the angle $\psi \in [-\psi_u, \psi_u]$. More precisely, the interval $\Omega = [-\psi_u, \psi_u]$ is partitioned into subintervals Ω_i, $i = -\ell, \ldots, -1, 0, 1, \ldots, \ell$, and a polytope $\mathcal{P}_i$ is associated to each Ω_i. Since the objective is to stabilise the equilibrium $\psi = 0$, in this case it is not necessary to find stabilising gains F_i for all polytopes $\mathcal{P}_i$; a single gain F, for the polytope $\mathcal{P}_0$ associated to the subinterval Ω_0 containing the origin, suffices. Then free control moves can be used in order to enlarge the domain of attraction of the linear controller F. To this end, let us introduce the special robust controlled invariant sets $\overline{\Sigma}_i$, $i \geq 0$, of all initial states $x(0)$ which can be robustly steered into Σ_0 by an i-steps feedback control sequence $\{u(0), u(1), \ldots, u(i-1)\}$, where each $u(k) \in \mathcal{U}_k$ is allowed to depend on the current state $x(k)$. The sets $\overline{\Sigma}_i$, $i > 0$, can be computed recursively as follows: $for\; i = 1, 2, \ldots$

$$\overline{\Sigma}_i = \mathcal{X}_i \cap \{x | \exists u \in \mathcal{U}_i : A_j x + B_j u \in \overline{\Sigma}_{i-1},\, j = 1, 2, \ldots, q\}.$$

where: (1) the initial condition is $\overline{\Sigma}_0 = \Sigma_0$; (2) the pairs $[A_j, B_j]$ are the vertices of the polytope P_i associated to Ω_i; (3) $\mathcal{U}_i$ and $\mathcal{X}_i$ are the input and, respectively, state constraint sets relative to the region Ω_i. Notice that, by symmetry, $\overline{\Sigma}_{-i} = -\overline{\Sigma}_i\}$ for $i = 1, 2, \ldots, \ell$.

Remark 2. Notice that for each subinterval Ω_i there are obviously different state constraints $\mathcal{X}_i$, but also different input constraints $\mathcal{U}_i$ since the control input u is obtained from the plant input v via the state feedback pre-compensation (5). The bounds $\mathcal{U}_i$ can be obtained numerically considering the imposed state bounds (10) and (5).

For on-line control, the following strategy is used.

Algorithm 1. Given $x(t)$, locate the region $\overline{\Sigma}_i$ to which $x(t)$ belongs and implement the following control:

$$\begin{array}{ll} \text{If} \quad x \in \Sigma_0, & u(t) = Fx(t) \\ \text{else} & u(t) = \hat{u}(t) \end{array}$$

where $\hat{u}(t) = \arg\min_u \, u(t)^2$ subject to

$$u(t) \in \mathcal{U}_i, \quad A_j x(t) + B_j u(t) \in \overline{\Sigma}_{i-1} \quad j = 1, 2, \ldots, q. \;\diamondsuit$$

4.2.2. *Scheduling on* $\dot{\alpha}$ Recall that the Furuta pendulum admits a continuum of equilibria

$$[\dot{\alpha}_e,\ \dot{\psi}_e,\ \psi_e]' = \left[\frac{t_3}{t_2}v_e,\ 0,\ 0\right]', \quad \forall\, v_e$$

Our aim is to enlarge the basin of attraction of the equilibrium $[\dot{\alpha}_e,\ \dot{\psi}_e,\ \psi_e]' = \mathbf{0}$ in order to improve the catching ability of its locally stabilizing controller. It turns out that an effective way of addressing this issue is by using a finite bank of predictive controllers relative to different equilibria and by switching among them so as to ultimately drive the system to an invariant neighborhood of the origin. So it is possible to choose a finite number, m, of equilibrium points $x_{e,i} = [\dot\alpha_{e,i},\ 0,\ 0]'$, $1 \le i \le m$, and define the set and relative subintervals Ω_i containing $\dot\alpha_{e,i}$ so as to cover a sufficiently large range of arm speeds. For each equilibrium, bounds on the input and the state allow to compute the corresponding difference inclusion and invariant set S_N^i (Chisci *et al.*, 2001*b*). In particular in order to implement the on-line scheme that will be described hereafter, it is important to have overlapping invariant sets. For this reason, overlapping subintervals Ω_i are chosen. In this manner each $\dot\alpha(t)$, that satisfies constraints, belongs to one or two intervals Ω_i and the algorithm selects among feasible equilibria the one that is nearest to the origin. In order to avoid set-point jumps, which would result from this simple strategy, an additional continuous variable is considered in the optimization procedure. This results in a filtered and smoothed input fed into the predictive controller so as to achieve a bumpless transfer. The control algorithm is the following.

Algorithm 2. At each sample time t, given the current state $x(t)$ and an admissible equilibrium $\dot\alpha_{e,i}$ with the respective invariant set S_N^i, find

$$[\hat{\dot\alpha}_e(t),\ \hat{\underline{c}}(t)] = \arg \min_{\underline{c}(t),\dot\alpha_e(t)} \left\{\underline{c}'(t)\underline{c}(t) + \rho\dot\alpha_e^2(t)\right\},\ \ \rho > 0,$$
$$\text{subject to}$$
$$\begin{bmatrix} x(t) - \begin{bmatrix} \dot\alpha_e(t) \\ 0 \\ 0 \end{bmatrix} \\ \underline{c}(t) \end{bmatrix} \in S_N^i, \quad \begin{bmatrix} \begin{bmatrix} \dot\alpha_e(t) - \dot\alpha_{e,i}(t) \\ 0 \\ 0 \end{bmatrix} \\ \underline{c}(t) \end{bmatrix} \in S_N^i, \tag{20}$$
$$|\dot\alpha_e(t) - \dot\alpha_e(t-1)| < h\ddot\alpha_u.$$

where $\dot\alpha_{e,i}$ is chosen according to the measured scheduling variable $\dot\alpha$. ◇

Remark 3. The constraints reflect the idea of choosing a continuously varying equilibrium so that the state belongs to its feasibility region and the equilibrium itself is close enough to one of the preselected operating points. Since bounds on $\ddot\alpha$ have been imposed to formulate the design procedure, it is necessary to take into account the maximum variation of the state variable $\dot\alpha$, derived from bounds on u, which in turn provides bounds on the variation of the equilibrium point $\dot\alpha_e$.

5. SIMULATION RESULTS

Simulations were performed using the Matlab/Simulink environment. The control strategies described in Section 4 have been applied to the nonlinear model (1). The numerical values used were obtained from a system identification experiment on a real pendulum (Png, 1999):

$$t_1 = 0.0265,\ t_2 = 1.0524,\ \ t_3 = 0.4549,\ t_4 = 0.6777,$$
$$t_5 = 0.6058,\ t_6 = 48.5267\ \ t_7 = 0.0304,\ t_8 = 0.7776$$

The chosen sampling time is $h = 0.01$ seconds. In addition, the following actuator and state constraints were imposed for all subsequent designs:

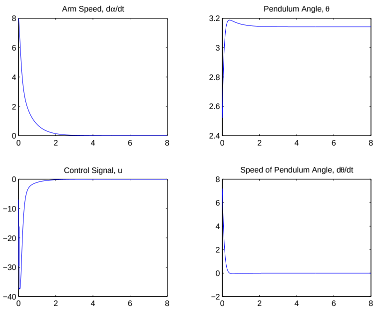

Fig. 2. Simulation for algorithm 1 with initial condition $x(0) = [8.00\ \ 7.20\ \ -0.62]'$

$$-200 \le v \le 200 \quad -10 \le \dot\psi \le 10 \quad \text{rad/s}$$

Fig. 2 shows the balancing behaviour using the strategy of algorithm 1. The relevant data are reported in Table 1; futher it has been assumed that $-11 \le \dot\alpha \le 11$ rad/s.

Table 1. Algorithm 1 (Scheduling on ψ)

Regions	Feedback Gains F_i	# of Constraints
$\Omega_{-2} : -\frac{36}{180}\pi < \psi < -\frac{33}{180}\pi$	-	177
$\Omega_{-1} : -\frac{33}{180}\pi < \psi < -\frac{30}{180}\pi$	-	99
$\Omega_0 : -\frac{30}{180}\pi < \psi < \frac{30}{180}\pi$	$[0.04 - 0.23 - 2.18] \times 10^3$	34
$\Omega_1 : \frac{30}{180}\pi < \psi < \frac{33}{180}\pi$	-	99
$\Omega_2 : \frac{33}{180}\pi < \psi < \frac{36}{180}\pi$	-	177

5.1 *Scheduling on* $\dot\alpha$

The simulation results obtained using Algorithm 2 are discussed here. We chose

$$X_q = \{-17.5,\ -14,\ -10.5,\ -7,\ 0,\ 7,\ 10.5,\ 14,\ 17.5\}.$$

Fig. 3 shows the balancing behaviour using the strategy of algorithm 2. The relevant data are reported in Table 2; futher it has been assumed that $-40\frac{\pi}{180} \le \psi \le 40\frac{\pi}{180}$ rad/s.

Table 2. Algorithm 2 (scheduling on $\dot\alpha$)

Regions	Feedback Gains F_i	# of Constraints
$\Omega_{-4} : -21 \le \dot\alpha \le -14$	$[0.03 - 0.25 - 3.90] \times 10^3$	127
$\Omega_{-3} : -17.5 \le \dot\alpha \le -10.5$	$[0.03 - 0.23 - 3.11] \times 10^3$	140
$\Omega_{-2} : -14 \le \dot\alpha \le -7$	$[0.03 - 0.22 - 2.51] \times 10^3$	80
$\Omega_{-1} : -10.5 \le \dot\alpha \le -3.5$	$[0.03 - 0.21 - 1.96] \times 10^3$	72
$\Omega_0 : \lvert\dot\alpha\rvert < 7$	$[0.02 - 0.18 - 1.47] \times 10^3$	114
$\Omega_1 : 3.5 \le \dot\alpha \le 10.5$	$F_1 = F_{-1}$	72
$\Omega_2 : 7 \le \dot\alpha \le 14$	$F_2 = F_{-2}$	86
$\Omega_3 : 10.5 \le \dot\alpha \le 17.5$	$F_3 = F_{-3}$	140
$\Omega_4 : 14 \le \dot\alpha \le 21$	$F_4 = F_{-4}$	127

5.2 *Swing up and balancing*

In this section, simulation results of the swing up and balancing behaviour are presented. The simulations start with the pendulum at the downward position. First the energy-based swing up law (Astrom and Furuta, 1999) is applied. Control will be switched to the balancing law when the state of the system is inside the invariant set. Recall that the parameter n in the swing up law (19) determines the maximum rate of energy change

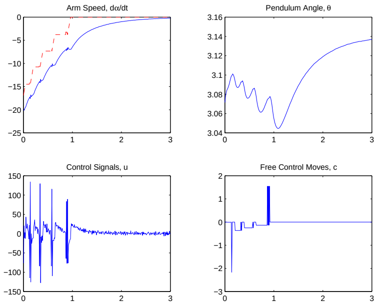

Fig. 3. Simulation with initial condition $x_0 = [-20 \quad 0.68 \quad -0.07]'$, $\dot{\alpha}$ (blue, solid), $\dot{\alpha}_e$ (red, dashed)

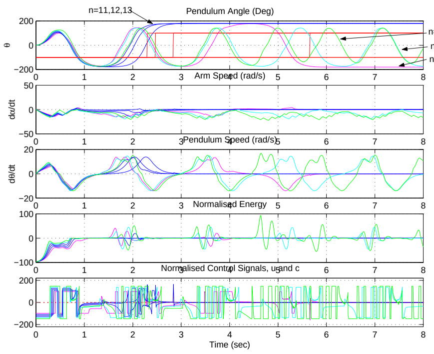

Fig. 4. Swing up and balancing performance using algorithm 1

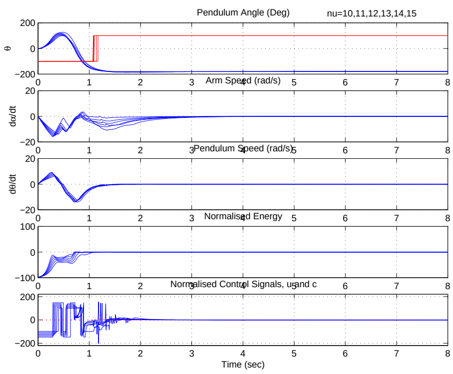

Fig. 5. Swing up and balancing performance using algorithm 2

allowable during swing up. To test the sensitivity of the design, different values $n = 10,\ 11,\ 12,\ 13,\ 14,\ 15$ were used. Figs. 4 and 5 show the performance with the balancing controllers of algorithm 1 and, respectively, 2. It is clear that with the enlarged region due to the gain scheduling strategy, the overall swing up and balancing performance is less sensitive to the design of the swing up law. Noisy simulations have also been carried out but are not reported here due to lack of space; the presence of disturbances non only did not affect stability but also slightly degraded performance.

6. CONCLUSIONS

In this paper, a LPV model of the Furuta pendulum is derived. Based on this model, a balancing controller is designed using robust predictive control techniques. The LPV model captures well the nonlinear dynamics of the system and allow the use of invariant set theory to accurately characterise the region of the state space in which the balancing controller is effective. As a result, the switching condition between swing up and balancing controllers can be determined systematically. In practice, the pendulum may approach the upright position with varying speed and the speed of the rotating arm may also vary. Based on this physical insight, the speed of the rotating arm is chosen as gain scheduling variable. It is shown that this strategy is effective in achieving a more consistent swing up and balancing behaviour which is not sensitive to the performance of the swing up controller. The experimental evaluation of the proposed control strategies is currently in progress.

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