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CLOSED FORM DIRECT KINEMATICS OF A CLASS OF STEWART PLATFORM

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Abstract: This paper presents the closed form of direct kinematics for a class of Stewart platform. Algebraic technique is explored to establish the model of a class of Stewart platform and the sets of nonlinear polynomial equations are derived. A Modified Dialytic elimination method is applied to solve the sets of nonlinear polynomial equations. Finally a 24th degree polynomial equation was obtained. This new result has been numerically verified by the inverse kinematics. *Copyright©2002 IFAC*

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1. INTRODUCTION[1]

Stewart platform was presented for a few decade years. It has high accuracy, sturdiness and capacity of loading comparing with serial manipulators. Generally, each of these mechanisms consists of two platforms, the mobile platform and the base platform. In the parallel manipulator of this paper, the mobile platform is connected to the base via six identical links consisting of a revolute joint attached to the ground followed by an actuated prismatic joint that is connected to the platform by a revolute joint. Only the prismatic joints are actuated (Merlet, 1998).

Griffs and Duffy (1989) investigated a forward displacement analysis for 3-3 Stewart platforms by geometry method. Merlet (1992) used vector technique to obtain a Triangular Symmetric Simplified Manipulator. Innocenti and Parenti-castelli, (1993) solved 5-5 parallel mechanisms with respect to triangle technique. Sreenivasan and Waldron, (1994) proposed the problem that the mobile platform is similar to the base based on vector and matrix methods. Analytical techniques usually tried to change the sets of original multivariate polynomial equations into a high degree polynomial equation in an unknown by elimination, but elimination process is very difficult. At present, there are several methods to solve sets of multivariate polynomial equations, such as Groebner Bases (Buchberger, 1990) method, Wu (Wu, 1984) elimination method, and Dialytic (McNamee, 1993; Roth, 1993) elimination method. They exist computational complexity problem for complex practical problems in some extent. How to efficiently calculate and how to decrease computational time are crucial problems. Although the classical Dialytic elimination method has been known for a long time, its use has been limited because it is not practical for problems with more than two or three unknowns or equations of high degree (Roth, 1993). Modified Dialytic elimination method attempts to make itself a practical computational tool. In this paper, the establishment of the coordinates and the construction of functions satisfy the decrease of the numbers of variables and the most exponents of variables so that the sets of multivariate polynomial equations can be converted into the type which the modified Dialytic elimination could solve. Then the sets of multivariate polynomial equations in terms of modified Dialytic elimination method were reduced and a 24th degree polynomial equation in one unknown was derived. Moreover, Extraneous solutions were not introduced by taking advantage of the modified Dialytic elimination method.

A brief current state of research and the development of direct kinematics were introduced in section1. Section 2 established the kinematic equations based on an algebraic method. Then the sets of nonlinear polynomial equations were reduced with respect to the modified Dialytic elimination and a 24^{th} degree polynomial equations in one unknown was obtained. In addition, the result was verified by virtue of the inverse kinematics. Finally, the conclusions were drawn.

2. ESTABLISHMENT OF THE DIRECT KINEMATICS

For the direct kinematics of parallel manipulators, articular variables can be expressed in general as a nonlinear algebraic function of links l_i. Then have

$$V = F_i(l_i) \quad i \in [1,6] \tag{1}$$

Parallel manipulators that consist of an equilateral triangular mobile platform and an equilateral hexagonal base platform linked by six extensible length links are considered (see Figure 1). A reference frame O, x, y, z to the base and a mobile frame O_1, x_1, y_1, z_1 are attached. The origin is A_1, and the X axis is the same as the line A_1A_2; the origin is B_1, and the X_1 axis is parallel to line B_2B_3. The side length of mobile platform is $\sqrt{3}r$, where r is circumscribe radius; the side length of the base is R. The transform matrix mobile platform relative to the base is P.

$$P = \begin{bmatrix} u_1 & v_1 & u_2v_3 - u_3v_2 \\ u_2 & v_2 & u_3v_1 - u_1v_3 \\ u_3 & v_3 & u_1v_2 - u_2v_1 \end{bmatrix}$$

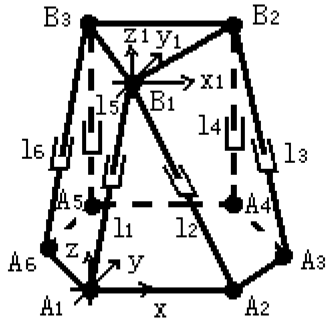

Fig.1 Parallel Manipulator

The unknowns are the coordinates of position vector B_1 in base frame $B_1 = \{x, y, z\}$, and the orientation cosine of mobile platform relative to the base. For convenience, let the set of variables be $V = \{x, y, z, u_1, u_2, u_3, v_1, v_2, v_3\}$.

The sets of kinematic equations can be obtained from six equations of link lengths and three additional constrains due to the orthogonality of transform matrix P:

$|A_1B_1|^2 = l_1^2$, $|A_2B_1|^2 = l_2^2$, $|A_3B_2|^2 = l_3^2$,
$|A_4B_2|^2 = l_4^2$, $|A_5B_3|^2 = l_5^2$, $|A_6B_3|^2 = l_6^2$,
$\sum_{i=1}^{3} u_i^2 = 1$, $\sum_{i=1}^{3} v_i^2 = 1$, $\sum_{i=1}^{3} u_i v_i = 0$.

The tool position is located the center of mobile platform, and its coordinates are $C(rv_1 + x, rv_2 + y, rv_3 + z)$.

The point coordinates A_i, B_i substitute for the above nine equations. Thus, the direct kinematics equations can be derived:

$$\begin{aligned}
&x^2 + y^2 + z^2 = l_1^2 \\
&(x-R)^2 + y^2 + z^2 = l_2^2 \\
&(\sqrt{3}/2\, ru_1 + 3/2\, rv_1 + x - 3/2\, R)^2 + (\sqrt{3}/2\, ru_2 + 3/2\, rv_2 \\
&+ y - \sqrt{3}/2\, R)^2 + (\sqrt{3}/2\, ru_3 + 3/2\, rv_3 + z)^2 = l_3^2 \\
\\
&(\sqrt{3}/2\, ru_1 + 3/2\, rv_1 + x - R)^2 + (\sqrt{3}/2\, ru_2 + 3/2\, rv_2 \\
&+ y - \sqrt{3}R)^2 + (\sqrt{3}/2\, ru_3 + 3/2\, rv_3 + z)^2 = l_4^2 \\
&(-(\sqrt{3}/2)ru_1 + (3/2)rv_1 + x)^2 + (-(\sqrt{3}/2)ru_2 + \\
&(3/2)rv_2 + y - \sqrt{3}R)^2 + (-(\sqrt{3}/2)ru_3 + \\
&(3/2)rv_3 + z)^2 = l_5^2 \\
&(-(\sqrt{3}/2)ru_1 + (3/2)rv_1 + x + (1/2)R)^2 + (-(\sqrt{3}/2)ru_2 \\
&+ (3/2)rv_2 + y - (\sqrt{3}/2)R)^2 + (-(\sqrt{3}/2)ru_3 + \\
&(3/2)rv_3 + z)^2 = l_6^2 \\
&u_1^2 + u_2^2 + u_3^2 = 1 \\
&v_1^2 + v_2^2 + v_3^2 = 1 \\
&u_1v_1 + u_2v_2 + u_3v_3 = 0
\end{aligned} \tag{2}$$

3. REDUCE THE FUNDAMENTAL EQUATIONS

Let the transform $\begin{cases} u = u_2 y + u_3 z \\ v = v_2 y + v_3 z \\ w = y^2 + z^2 \end{cases}$ substitute x, y, w, u_1, u_2, u, v_1, v_2, v for the variables of original equations, and x, w, u, v, u_2, v_2 are expressed as linear functions with respect to y, u_1, v_1.

The first six equations of (2) can be written as:

$$\begin{cases} x = k_1 \\ w = k_2 \\ u_2 = v_1 + k_3 \\ v_2 = 1/3\, u_1 - 2/(3r)\, y + k_4 \\ u = (1/2\, R - d_1)u_1 + ((3\sqrt{3})/2)rv_1 + k_5 \\ v = (\sqrt{3}/2)Ru_1 + (1/2\, R - d_1)v_1 + k_6 \end{cases} \tag{3}$$

Where $k_1 = l_1^2 - l_2^2 + R^2/(2R)$, $k_2 = l_1^2 - k_1^2$,
$k_3 = (l_1^2 + l_3^2 + l_5^2 - l_2^2 - l_4^2 - l_6^2)/(3Rr)$,
$k_4 = (l_6^2 + l_3^2 - l_5^2 - l_4^2 + 3R^2)/(3\sqrt{3}Rr)$,
$k_5 = (2(l_3^2 - l_6^2) + (l_5^2 - l_4^2) + 3(l_1^2 - l_2^2))/(2\sqrt{3}r)$
$k_6 = (2l_3^2 + 2l_6^2 - l_1^2 - l_2^2 - l_4^2 - l_5^2 - 6r^2)/(6r)$

Because of $\begin{cases} u_3^2 = 1 - u_1^2 - u_2^2 \\ v_3^2 = 1 - v_1^2 - v_2^2 \\ u_3 v_3 = -u_1 v_1 - u_2 v_2 \end{cases}$ and

$$\begin{cases} u_3 z = u - u_2 y \\ v_3 z = v - v_2 y \ , \\ z^2 = w - y^2 \end{cases}$$

Three equations f_1, f_2, f_3 can be derived (Huang and Wu, 1991; Wu and Huang, 1994):

$$\begin{aligned} f_1 &= (u_1^2 + u_2^2 - 1)(k_2 - y^2) + (u - u_2 y)^2 \\ f_2 &= (v_1^2 + v_2^2 - 1)(k_2 - y^2) + (v - v_2 y)^2 \\ f_3 &= (u_1 v_1 + u_2 v_2)(k_2 - y^2) + (u - u_2 y)(v - v_2 y) \end{aligned} \quad (4)$$

Replace f_1, f_2, f_3 with formula (3). f_1, f_2, f_3 are expressed with respect to u_1, v_1 and y. Equations f_1, f_2, f_3 with one unknown y as suppressed are rewritten. The three equations by use of the modified Dialytic elimination method are solved, which homogeneous equations taken derivative have the same zeros as original equations.

For the general quadratics[8]:

$$\begin{aligned} &a_i u_1^2 + b_i v_1^2 + c_i y^2 + d_i u_1 v_1 + e_i u_1 y \\ &+ f_i v_1 y + g_i u_1 + h_i v_1 + i_i y + j_i = 0 \end{aligned} \quad i = 1,2,3 \quad (5)$$

(5) was obtained by suppressing y:

$$a_i u_1^2 + b_i v_1^2 + d_i u_1 v_1 + l_i u_1 + m_i v_1 + n_i = 0 \quad i = 1,2,3 \quad (6)$$

where $l_i = e_i y + g_i$, $m_i = f_i y + h_i$,
$n_i = c_i y^2 + i_i y + j_i$
Write homogeneous equations by substituting $u_1 = U_1/W$, $v_1 = V_1/W$, and multiplying W^2

$$a_i U_1^2 + b_i V_1^2 + d_i U_1 V_1 + l_i U_1 W + m_i V_1 W + n_i W^2 = 0 \quad i = 1,2,3 \quad (7)$$

The Jacobian matrix J of (7) with respect to homogeneous coordinates is:

$$J = \begin{bmatrix} 2a_1 U_1 + d_1 V_1 + l_1 W & 2b_1 V_1 + d_1 U_1 + m_1 W & l_1 U_1 + m_1 V_1 + 2n_1 W \\ 2a_2 U_1 + d_2 V_1 + l_2 W & 2b_2 V_1 + d_2 U_1 + m_2 W & l_2 U_1 + m_2 V_1 + 2n_2 W \\ 2a_3 U_1 + d_3 V_1 + l_3 W & 2b_3 V_1 + d_3 U_1 + m_3 W & l_3 U_1 + m_3 V_1 + 2n_3 W \end{bmatrix} \quad (8)$$

The determinant of this matrix was formed. Then a cubic polynomial is as follows:

$$|J| =$$

$$\begin{aligned} &A(y)U_1^3 + B(y)U_1^2 V_1 + C(y)U_1^2 W + D(y)U_1 V_1^2 \\ &+ E(y)U_1 W^2 + F(y)U_1 V_1 W + G(y)V_1^3 \\ &+ H(y)V_1^2 W + I(y)V_1 W^2 + K(y)W^3 \end{aligned} \quad (9)$$

Derivatives of this equation are taken with respect to homogeneous coordinates:

$$\begin{bmatrix} a_1(y^2) & b_1(y) & d_1(y) & l_1(y) & m_1(y) & n_1(y^2) \\ a_2(y) & b_2(y^2) & d_2(y) & l_2(y^2) & m_2(y^2) & n_2(y^2) \\ a_3(y) & b_3(y) & d_3(y^2) & l_3(y^2) & m_3(y^2) & n_3(y^2) \\ 3A(y^6) & D(y^6) & 2B(y^6) & 2C(y^6) & F(y^6) & E(y^6) \\ B(y^6) & 3G(y^5) & 2D(y^6) & F(y^6) & 2H(y^6) & I(y^6) \\ C(y^6) & H(y^6) & F(y^6) & 2E(y^6) & 2I(y^6) & 3K(y^6) \end{bmatrix} \begin{bmatrix} U_1^2 \\ V_1^2 \\ U_1 V_1 \\ U_1 W \\ V_1 W \\ W^2 \end{bmatrix} = 0 \quad (10)$$

The condition that equation (10) has a solution is the determinant of the coefficient matrix equals to zero. Set the determinant of coefficient matrix equal to zero. Expand and derive a 24th degree polynomial equation in unknown y. Where $a_1[y^2]$ express a_1 is the function with respect to unknown y, and the most exponent is 2, so the determinant of the coefficient matrix is function in terms of y.

$$g(y) = 0 \quad (11)$$

Set $W = 1$, and obtain

$$\begin{bmatrix} a_1 & b_1 & d_1 & l_1 & m_1 \\ a_2 & b_2 & d_2 & l_2 & m_2 \\ a_3 & b_3 & d_3 & l_3 & m_3 \\ 3A & D & 2B & 2C & F \\ B & 3G & 2D & F & 2H \end{bmatrix} \begin{bmatrix} u_1^2 \\ v_1^2 \\ u_1 v_1 \\ u_1 \\ v_1 \end{bmatrix} = -\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ E \\ I \end{bmatrix} \quad (12)$$

u_1, v_1 can be solved from formula (12), x, v_2, u_2 as well as mid-variables u, v, w can be solved from formula (3), and u_3, v_3, z can be solved from following formula:

$$\begin{aligned} z &= \sqrt{w - y^2} \\ u_3 &= (u - u_2 y)/z \\ v_3 &= (v - v_2 y)/z \end{aligned} \quad (13)$$

In a word, the degree of equation $g(y) = 0$ is 24th, which can be solved and 24 solutions can be obtained. Once given the value of unknown y, other unknowns are uniquely determinate.

4. NUMERICAL EXAMPLE AND ANALYSIS

4.1 Numerical Example

Let the side length of equilateral triangle of the mobile platform be 10, that is $r = 10/\sqrt{3}$; the side length of equilateral hexagon of the base platform be $R = 7$; the six link lengths be $l_1 = 12.1$, $l_2 = 12.3$, $l_3 = 12.3$, $l_4 = 12.5$, $l_5 = 12.3$, $l_6 = 12.2$ respectively. The minimum length of link is 10.2, and the maximum length of link is 13. In computer algebra system Mathematica the three solutions that satisfy formula (13) as well as constraint conditions (14) can be calculated and obtained. They are as follows:

Table 1 Real Solutions Satisfying Constraint Conditions

u_1=0.0117244	u_2=-0.543086	u_3=-0.839595
v_1=-0.482097	v_2=0.73255	v_3=-0.480576
x=3.15143	y=-0.549797	z=11.6695
u_1=0.999432	u_2=-0.0305947	u_3=0.0141561
v_1=0.030394	v_2=0.999437	v_3=0.0141794
x=3.15143	y=-0.010176	z=11.6824
u_1=0.0739381	u_2=0.528439	u_3=0.845745
v_1=0.589428	v_2=0.660926	v_3=-0.46449
x=3.15143	y=0.249588	z=11.6797

$$\begin{cases} |u_i| \le 1 \\ |v_i| \le 1 \\ u_1^2 + u_2^2 + u_3^2 = 1 \\ v_1^2 + v_2^2 + v_3^2 = 1 \\ u_1 v_1 + u_2 v_2 + u_3 v_3 = 0 \\ |y| \le \sqrt{w} \end{cases} \tag{14}$$

4.2 Analysis

When original equations (2) are converted into (4), the sets of solutions are extended, so the solutions from formula (11) should satisfy constraint conditions (14). In addition, because the top platform is above the base, $z > 0$, or $w - y^2 > 0$, that is $|y| < \sqrt{w}$.

The scope of x must satisfy the left boundary

$$\begin{cases} (x-R)^2 + y^2 + z^2 = l_{\max}^2 \\ x^2 + y^2 + z^2 = l_{\min}^2 \end{cases}$$

and the right boundary

$$\begin{cases} x^2 + y^2 + z^2 = l_{\max}^2 \\ (x-R)^2 + y^2 + z^2 = l_{\min}^2 \end{cases}$$ respectively, that is

$$x \in [\frac{l_{\min}^2 + R^2 - l_{\max}^2}{2R}, \frac{l_{\max}^2 + R^2 - l_{\min}^2}{2R}]$$

The scope of z must satisfy:

$$z \in [\sqrt{l_{\min}^2 - (R/2)^2 - ((\sqrt{3}/2)R - r)^2}, \sqrt{l_{\max}^2 - (R/2)^2 - ((\sqrt{3}/2)R - r)^2}\,]$$

From formula (11), the sets of multivariate equations have 24 complex solutions can be known, and 24 complex solutions by formula (11), (12), (3), and (13) can be easily solved. Because the sets of solutions are extended, redundant solutions can be reduced by constraint condition (14). Finally, three real solutions to satisfy practical problem are obtained.

5. CONCLUSION

In this paper, an algebraic method was explored to establish the kinematic model of a class of parallel manipulators. The establishment of the coordinates and the construction of functions made the decrease of the numbers of variables and the most exponents of variables. In the mean while, the kinematic equations could be reduced in terms of a modified Dialytic elimination method and a 24th degree polynomial equation in an unknown would be obtained. Then a close form of direct kinematics was performed. The solutions satisfying practical problems could be easily obtained. Also, this algorithm is very simple comparing with previous methods. Moreover, Extraneous solutions were not introduced in terms of the modified Dialytic elimination method.

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