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SLOW PERIODIC MOTIONS WITH INTERNAL SLIDING MODES IN VARIABLE STR UCTURE SYSTEMS

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Abstract: Singularly perturbed relay systems (SPRS) in which the reduced systems have the stable periodic motions with internal sliding modes are studied. The slow motion integral manifold of such systems consists of the parts which correspond to the different values of relay control and the solutions may contain the jumps from one part of the slow manifold to another. For such systems a theorem about existence and stability of the periodic solutions is proved. An algorithm of asymptotic representation for this periodic solutions using boundary layer method is presented. It is shown that in the neighbourhood of the break away point the asymptotic representation starts with the first order boundary layer function.

Keywords: Periodic motion, Variable structure systems, Sliding mode, Singular perturbations, Stability

1. INTRODUCTION

There are a wide class of relay control systems which work in periodic regimes. For example, such regimes arise every time in relay control systems with time delays because a time delay does not allow an ideal sliding mode to be realized, but results in periodic oscillations (Fridman *et al*, 2000). In controllers of exhaust gases for fuel injector automotive control systems (Choi and Hedrick, 1996) the sensors can measure only the sign of the controlled variable with a delay. In such systems only oscillations around zero value can occur. In the controllers for stabilization of underwater manipulators it is possible to realize only oscillations because of the manipulators properties (Bartolini *et al*, 1997).

Some relay systems work in periodic regimes with internal sliding modes. As the simplest modeling example of the periodic oscillations with the internal sliding modes we will consider the pendulum which has dry friction contact with an inclined uniformly rotating disk (Rumpel, 1996). First this pendulum is moving together with disk until returned point and returning back. In real relay control systems every time we have some unmodeled dynamics which can correspond, for example, to the presence in system of fast actuators or inertial sensors. Usually such dynamics destroy the qualitative behavior of control systems. The complicated model of sliding mode control systems taking into account the presence of fast and inertial sensors is described by singularly perturbed relay systems (SPRS).

SPRS describe the complete model of fuel injector systems taking into account the influence of the additional dynamics (the car motor). The knowledge of properties of SPRS it is necessary in the controllers for stabilization of the underwater manipulator fingers to take into account the influence of the elasticity of these fingers. In the simplest pendulum systems SPRS describe the influence of the second small pendulum on the oscillation of first one.

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For *smooth* singularly perturbed systems there are two main classes of *slow* periodic solutions. Slow periodic solutions of the smooth singularly perturbed systems "without jumps" are situated on slow motion integral manifold. The other important class of such solutions are the relaxation solutions (Mishchenko and Rosov, 1980), which contain the "jumps" from the neighbourhood of the one stable branch of slow motion manifold to the neighbourhood of another one.

The slow motion integral manifold of relay systems is discontinuous and consists at least of parts which correspond to the different values of control. This means that the desired periodic solution of SPRS should have the jumps from the small neighbourhood of the one sheet of integral manifold to the neighbourhood of another one. From this viewpoint the qualitative behavior of this periodic solution will be nearer to the relaxation solution. The main specific feature of systems with relaxation oscillations is the following: at the moment of time corresponding to the jump from the neighbourhood of one branch of the stable integral manifold to the neighbourhood of another one, the value of the right hand side is small. That is why in order to find the asymptotic representation of the relaxation solution it was necessary to make special asymptotic representations. The situation with SPRS is different. The right hand side of a SPRS is switches immediately after the switching moment and the right hand side of fast equations in SPRS after this moment is very big. It allows to use the standard boundary layer functions method (Vasil'eva *et al*, 1995) for asymptotic representation of slow periodic solution of SPRS.

This paper is devoted to the investigation of the influence of additional dynamics on the periodic motion of the relay systems with internal sliding modes. We will consider the SPRS, for which the reduced systems have the periodic solution with internal sliding mode. A theorem about existence and stability of slow periodic solutions for singularly perturbed relay systems with internal sliding mode is proved. The algorithm for asymptotic representation of the periodic solution is suggested and it is proved that there is no zero order boundary layer function in the asymptotic approximation of periodic solutions at break away point.

2. PROBLEM STATEMENT

Consider the SPRS in the form

$$\mu dz/dt = g(z,\sigma,s,x,u), \quad ds/dt = h_1(z,s,\sigma,x,u), \tag{1}$$
$$d\sigma/dt = h_2(z,s,\sigma,x,u), \quad dx/dt = h_3(z,s,\sigma,x,u),$$

where $z \in R^m, s,\sigma \in R, x \in R^n, u(s) = sign\,(s), g, h_i$ $(i = 1,2,3)$ are the smooth functions of their arguments, μ is the small parameter. Denote by Z, Σ, S, X the domains in which the variables $(z,s,\sigma,x), (s,\sigma,x), (s,x)$ and x are defined. Suppose that $h_1, h_2, h_3, g \in C^2[\bar{Z} \times [-1,1]]$. Then putting $\mu = 0$ and expressing z from the equation

$$g(z_0,s,\sigma,x,u(s)) = 0, \tag{2}$$

we have the reduced system

$$\begin{aligned} z_0 &= \varphi(s,\sigma,x,u),\\ d\bar{s}_0/dt &= H_1(\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u),\\ d\bar{\sigma}_0/dt &= H_2(\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u),\\ d\bar{x}_0/dt &= H_3(\bar{s}_0,\bar{\sigma}_0,\bar{x}_0,u). \end{aligned} \tag{3}$$

Suppose that the measure of the sliding domain $\mathcal{S} = \{(z,\sigma,x)$:

$$h_1(z,0,\sigma,x,1) < 0, \; h_1(z,0,\sigma,x,-1) > 0\}$$

on the surface $s = 0$ in system (3) is nonzero in $\Sigma \times \{0\}$ and Γ is the border of $\mathcal{S}$ is described by equations $s = 0 \cap \big(\sigma = 0 \iff u_{eq}(z,0,x) \equiv 1 \iff h_1(z,0,0,x,1) \equiv 0\big)$, and moreover for all $(z,x) \in \Gamma \in \mathbf{R}^m \times \mathbf{R}^n$

$$h_1(z,0,0,x,-1) > 0, \quad h_2(z,0,0,x,1) > 0.$$

Suppose that the solution of system (1) in the sliding domain $\mathcal{S}$ is uniquely described by the equivalent control method (see for example Utkin, 1992)

$$\mu dz/dt = g(z,\sigma,0,x,u_{eq}(z,\sigma,x)), \tag{1*}$$
$$d\sigma/dt = h_2(z,0,\sigma,x,u_{eq}(z,\sigma,x)),$$
$$dx/dt = h_3(z,0,\sigma,x,u_{eq}(z,\sigma,x)),$$

where the equivalent control $u = u_{eq}(z,\sigma,x)$ at all $(z,\sigma,x) \in \mathcal{S}$ is determined by equation $h_1(z,0,\sigma,x,u_{eq}) = 0$ and everywhere in $\mathcal{S}$ the inequality $|u_{eq}(z,\sigma,x)| < 1$ is true.

The main specific feature of system (1) is the following: the zero approximation of the slow motion integral manifold for system (1) consists of three sheets $\bar{z}_0^{\pm} = \varphi^{\pm}(s,\sigma,x) = \varphi(s,\sigma,x,\pm 1)$, and

$$\bar{z}_0^{*} = \varphi^*(\sigma,x) = \varphi(0,\sigma,x,\bar{u}_{eq}(\sigma,x)),$$

corresponding to the value of relay control $u = \pm 1$ and $u = \bar{u}_{eq}(\sigma,x)$, where $\bar{u}_{eq}(\sigma,x)$ is the value of equivalent control determined by equation

$$H_1(s,\sigma,x,\bar{u}_{eq}(\sigma,x)) = 0.$$

It is obvious, that

$$u_{eq}(\varphi(0,\sigma,x,\bar{u}_{eq}(\sigma,x)),\sigma,x) = \bar{u}_{eq}(\sigma,x).$$

For the description of periodic solution in the reduced system consider two auxiliary systems. The system

$$\begin{aligned} d\bar{s}_0^+/dt &= H_1(\bar{s}_0^+,\bar{\sigma}_0^+,\bar{x}_0^+,1),\\ d\bar{\sigma}_0^+/dt &= H_2(\bar{s}_0^+,\bar{\sigma}_0^+,\bar{x}_0^+,1),\\ d\bar{x}_0^+/dt &= H_3(\bar{s}_0^+,\bar{\sigma}_0^+,\bar{x}_0^+,1) \end{aligned} \tag{3+}$$

describes the motions in (3) for $u = 1$. Consider the system

$$\begin{aligned} d\bar{\sigma}_0^*/dt &= H_2(0,\bar{\sigma}_0^*,\bar{x}_0^*,\bar{u}_{eq}),\\ d\bar{x}_0^*/dt &= H_3(0,\bar{\sigma}_0^*,\bar{x}_0^*,\bar{u}_{eq}), \end{aligned} \tag{3*}$$

corresponding to the motions in (3) in sliding mode on $s = 0$.

Let us denote

$$\Delta = \{x \, : \, H_1(0,0,x,1) = 0;\, H_1(0,0,x,-1) > 0\}$$

as the border of sliding domain of system (3). Then the points $(0,0,x) \in \Delta$ are the points in which solutions of (3) are leaving the sliding domain. Suppose that for solution of system (3) with the initial conditions

$$s_0(0) = 0, \quad \sigma_0(0) = 0, \quad x_0(0) = x^0,\, x^0 \in \Delta$$

the following conditions are true

(i) there exists $t = \theta(x^0)$ the smallest root of equation $\bar{s}_0^+(\theta) = 0$, such that
$h_1(\varphi^+(0,\bar{\sigma}_0^+(\theta),\bar{x}_0^+(\theta)),0,\bar{\sigma}_0^+(\theta),\bar{x}_0^+(\theta),1) < 0$,
$h_1(\varphi^+(0,\bar{\sigma}_0^+(\theta),\bar{x}_0^+(\theta)),0,\bar{\sigma}_0^+(\theta),\bar{x}_0^+(\theta),-1) > 0$,

$$\bar{\sigma}_0^+(\theta) < 0;$$

(ii) for solution of system (3*) with initial conditions

$$\bar{\sigma}_0^*(\theta) = \bar{\sigma}_0^+(\theta), \quad \bar{x}_0^*(\theta) = \bar{x}_0^+(\theta),$$

there exists $t = T(x^0)$ the smallest root of equation $\bar{\sigma}_0^*(T) = 0$ such that

- for all $t \in [\theta, T)$
  $h_1(\varphi^*(0,\bar{\sigma}_0^*(t),\bar{x}_0^*(t)),0,\bar{\sigma}_0^*(t),\bar{x}_0^*(t),1) < 0$,
  $h_1(\varphi^*(0,\bar{\sigma}_0^*(t),\bar{x}_0^*(t)),0,\bar{\sigma}_0^*(t),\bar{x}_0^*(t),-1) > 0$;
- $H_2(0,0,\bar{x}_0^*(T),-1) > 0$.

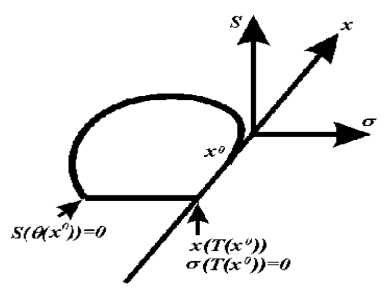

Fig. 1. The Poincare map $\Psi(x^0)$.

Now we can define the Poincare map $\Psi \, : \, x^0 \to \bar{x}_0^*(T(x^0))$ of the border of the sliding domain Δ generated by system (3) into itself (see fig. 1).

The systems (1) and (3) are discontinuous, and consequently for investigation of stability for their periodic solution it is impossible to use equation in variations. That is why we will write down the conditions of existence and stability of periodic solutions for systems (1) and (3) in the form of the Poincare map properties.

Suppose that for the system (3) the following hypotheses are true:

(iii) there exists an isolated fixed point of the Paincare map $\Psi(x) \, : \, \Psi(x_0^*) = x_0^*,\, x_0^* \in \Delta$, corresponding to the periodic solution of (3), such that $det\frac{\partial \Psi}{\partial x}(x_0^*) \neq 0$;

(iv) $\|\frac{\partial \Psi}{\partial x}(x_0^*)\| < q < 1$.

Denote by $\theta_0 = \theta(x_0^*)$, $T_0 = T(x_0^*)$. Consider the broken line $\mathcal{L}_0(t) =$

$$\begin{cases} \varphi^+(s_0^+(t),\sigma_0^+(t),x_0^+(t)) \quad \text{for} \quad t \in (0,\theta_0), \\ \varphi^*(\sigma_0^*(t),x_0^*(t)) \quad \text{for} \quad t \in (\theta_0,T_0), \\ (1-\lambda)\varphi^+(0,\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0)) + \\ +\lambda\varphi^*(\bar{\sigma}_0^+(\theta_0),\bar{x}_0^+(\theta_0)), \\ \lambda \in [0,1] \, \text{for} \, t = \theta_0. \end{cases}$$

In this paper the sufficient conditions are found for existence of the isolated orbitally asymptotically stable periodic solution of system (1) with internal sliding modes near to the broken line

$$(\mathcal{L}_0(t), s_0(t), \sigma_0(t), x_0(t)).$$

An algorithm for the asymptotic representation of this periodic solution is suggested. This solution consists of boundary layers at the break away point $t = 0$ and at the point $t = \theta_0$ and it is proved there is no zero order boundary layer function in the asymptotic representation of periodic solution at break away point.

3. EXISTENCE AND STABILITY OF THE SLOW PERIODIC SOLUTION

We will consider only situations in which the fast motions in (1) are uniformly asymptotically stable. This means that for systems

$$\frac{dz}{d\tau} = g(z,s,\sigma,x,1), \tag{4+}$$

$$\frac{dz}{d\tau} = g(z,0,\sigma,x,u_{eq}(z,\sigma,x)), \tag{4*}$$

which describe the fast motions in (1) for $u = 1$ and (1*) respectively, for some $\alpha > 0$, $\delta > 0$ the following conditions are true:

(v) the matrix $\frac{\partial g}{\partial z}(z,s,\sigma,x,1)$ is stable on the set

$$Z^+ = \{(z,s,\sigma,x) \, : \, (z,s,\sigma,x) \in Z\,,\, s > 0,$$
$$\|(\varphi^+(\bar{s}_0^+(t),\bar{\sigma}_0^+(t),\bar{x}_0^+(t)),\bar{s}_0^+(t),\bar{\sigma}_0^+(t),\bar{x}_0^+(t))$$
$$-(z,s,\sigma,x)\| < \delta,\, t \in [0,\theta_0]\}$$

and

$$Re\, Spec\, \frac{\partial g}{\partial z}(z,s,\sigma,x) < -\alpha < 0;$$

(vi) the matrix $\frac{\partial g}{\partial z}(z,0,\sigma,x,u_{eq}(z,0,\sigma,x))$ is stable on the set

$$Z^* = \{(z,0,\sigma,x) \, : \, (z,0,\sigma,x) \in Z,$$
$$\|(\varphi^*(\bar{\sigma}_0^*(t),\bar{x}_0^*(t),\bar{u}_{eq}(\bar{\sigma}_0^*(t),\bar{x}_0^*(t))),\bar{\sigma}_0^*(t),\bar{x}_0^*(t))-$$
$$-(z,0,\sigma,x)\| < \delta,\, t \in [\theta_0,T_0]\}$$

and moreover

$$Re\, Spec\, \frac{\partial g}{\partial z}(z,0,\sigma,x) < -\alpha < 0.$$

It is natural to suppose that at the time moment of input into the sliding mode the corresponding point of system (1) solution is situated in the interior of attractivity domain for slow motion integral manifold of system (1*). Suppose that

(vii) point $\varphi^+(0,\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0))$ is an internal point of attractivity domain of $\varphi^*(\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0))$ which is equilibrium point of system $dz/d\tau =$

$$g(z,0,\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0),u_{eq}(z,\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0))$$

and at all points of segment connected the points $\varphi^+(0,\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0))$ and $\varphi^*(\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0))$ the sufficient conditions for sliding mode existence are true, which means that for all point of segments

$$\Lambda = \lambda\varphi^*(\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0))$$
$$+(1-\lambda)\varphi^+(0,\bar\sigma_0^+(\theta_0),\lambda\in[0,1]$$
$$h_1(\Lambda,\bar x_0^+(\theta_0)),0,\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0),1)<0,$$
$$h_1(\Lambda,\bar x_0^+(\theta_0)),0,\bar\sigma_0^+(\theta_0),\bar x_0^+(\theta_0),-1)>0.$$

For the proof of existence and stability of system (1) periodic solution consider the properties of the Poincare map $\Phi(z,x,\mu)$ of the border of sliding domain Γ into itself, generated by (1).

Theorem 1. Under conditions $(i)-(vii)$ for sufficiently small μ in the neighborhood of the broken line $(\mathcal{L}_0(t),\bar s_0(t),\bar\sigma_0(t),\bar x_0(t))$ there exists orbitally asymptotically stable periodical solution with period $T(\mu)=T_0+O(\mu)$ and boundary layers at $t=0$ near to the point $t=\theta_0$. The zero order boundary layer function at $t=0$ is equal zero.

The contraction properties of the Poincare map $\Phi(z,x,\mu)$ at the point $(\varphi(x_0^*),0,0,x_0^*)$ one can conclude analogously Fridman, (1997) from :

- the theorems about the smoothness of singularly perturbed systems solutions at the end of the finite time interval (Strygin and Sobolev, 1988);
- the theorems about asymptotic properties of singularly perturbed relay systems solutions (Fridman and Rumpel, 1996).

4. ASYMPTOTIC REPRESENTATION FOR SOLUTION

Suppose that $h_1,h_2,h_3,g\in C^{k+3}[\bar Z\times[-1,1]]$ and conditions $(i)-(vii)$ are true. Denote by $y^T=(z^T,s,\sigma,x^T)$ and $v^T=(s,\sigma,x^T)$. Then the asymptotic representation of the point $\theta(\mu)$ and period the $T(\mu)$ of desired periodic solution of system (1) on interval $[0,\tilde T_{k+1}(\mu)]$ has the form

$$Y_k(t,\mu)=\sum_{i=0}^{k}[\bar y_i(t)+\Pi_i^* y(\tau_{k+1})]\mu^i+\sum_{j=1}^{k}\Pi_j^+ y(\tau)\mu^i, \qquad (AS)$$

$$V_k(t,\mu)=\sum_{i=0}^{k}\bar v_i(t)\mu^i+\sum_{i=2}^{k}\Pi_i^+ v(\tau)\mu^i$$

$$+\sum_{i=1}^{k}\Pi_i^* v(\tau_k)\mu^i,\ \tau=t/\mu,\ \tau_{k+1}=(t-\tilde\theta_{k+1}(\mu)))/\mu,$$

$$\tilde\theta_{k+1}(\mu)=\theta_0+\mu\theta_1+\cdots+\mu^{k+1}\theta_{k+1},$$
$$\tilde\Theta_{k+1}(\mu)=\Theta_0+\mu\Theta_1+\cdots+\mu^{k+1}\Theta_{k+1},$$
$$\tilde T_k(\mu)=T_0+\mu T_1+\cdots+\mu^k T_k,$$
$$\|\Pi_i^* y(\tau)\|<Ce^{-\gamma\tau},C,\gamma>0,\text{for}\quad \tau>0$$
$$\Pi_i^* y(\tau)\equiv 0\quad\text{for}\quad \tau<0,$$
$$\|\Pi_i^+ y(\tau_{k+1})\|<Ce^{-\gamma\tau_{k+1}},\ \text{for}\,\tau_{k+1}>0$$
$$\Pi_i^+ y(\tau_{k+1})\equiv 0\,\text{for}\,\tau_{k+1}<0.$$

Theorem 2. Under conditions $(i)-(vii)$

$$|\tilde T_k(\mu)-T(\mu)|<C\mu^{k+1}$$

and uniformly on $t\in[0,\hat T(\mu)]$, where $\hat T(\mu)=max\{T(\mu);\tilde T_{k+1}(\mu)\}$, the following inequalities hold

$$\|y(t,\mu)-Y_k(t,\mu)\|<C\mu^{k+1};$$
$$\|v(t,\mu)-V_k(t,\mu))\|<C\mu^{k+1}.$$

The proof of this theorem follows from asymptotic properties of singularly perturbed relay systems solutions (Fridman and Rumpel, 1996).

5. EXAMPLE

Let us show the existence and stability and design the asymptotic representation for slow periodic solution with internal sliding mode for SPRS in form

$$\mu dz/dt=-z+u;\ ds/dt=2s+\sigma+5-5u;$$
$$d\sigma/dt=-6s-\sigma+x+4z,\ dx/dt=-x+\mu z, \qquad (5)$$

where $u=sign\,[s(t)]$, $z,s,\sigma,x\in R,\mu$ is the small parameter. Let us show that for system (5) the conditions of theorem 1 and 2 are true. For $\mu=0$ system (5) takes the form

$$\bar z_0=u,\ d\bar s_0/dt=2\bar s_0+\bar\sigma_0+5-5u, \qquad (6)$$
$$d\bar\sigma_0/dt=-6\bar s_0-\bar\sigma_0+\bar x_0+4u,\ d\bar x_0/dt=-\bar x_0.$$

Than for system $s>0$ instead of (6) one has

$$d\bar s_0^+/dt=2\bar s_0^++\bar\sigma_0^+,\ d\bar\sigma_0^+/dt=-6\bar s_0^+-\bar\sigma_0^++\bar x_0^++4,$$
$$d\bar x_0^+/dt=-\bar x_0^+. \qquad (6+)$$

The set

$$\mathcal{S}=\{\,\sigma\,:\,-10<\sigma<0\,\}$$

is a stable sliding mode domain for system (6). The motions into $\mathcal{S}$, are described

$$d\sigma_0^*/dt=-\frac{\bar\sigma_0^*}{5}+\bar x_0^*+4,\ d\bar x_0^*/dt=-\bar x_0^*. \qquad (6*)$$

Then for the solution of the system (6+) with initial conditions

$$\bar s_0^+(0)=0,\ \bar\sigma_0^+(0)=0,\ \bar x_0^+(0)=\xi$$

we have

$$\bar s_0^+(t,\xi)=1+\frac{e^{-t}\xi}{6}+\left(\frac{\sqrt{15}}{15}+\xi\sqrt{15}\right)$$
$$\times e^{\frac{t}{2}}sin\frac{\sqrt{15}t}{2}-\left(\frac{\xi}{6}+1\right)e^{\frac{t}{2}}cos\frac{\sqrt{15}t}{2};$$
$$\bar\sigma_0^+(t,\xi)=-2-\frac{e^{-t}\xi}{2}+\left(\frac{2\sqrt{15}}{5}+\xi\frac{\sqrt{15}}{30}\right)e^{\frac{t}{2}}sin\frac{\sqrt{15}t}{2}$$
$$+\left(2+\frac{\xi}{2}\right)e^{\frac{t}{2}}cos\frac{\sqrt{15}t}{2};$$

$$\bar{x}_0{}^{+}(t,\xi)=e^{-t}\xi.$$

The last equation of (6) is independent and only the solution of the equation $\bar{x}_0(t)\equiv 0$ can correspond to the periodic solution of (6). Then to find θ_0 as the input moment into the sliding mode we have the equation

$$\bar{s}_0^{+}(t,0)=1+\frac{\sqrt{15}}{15}e^{\frac{t}{2}}sin\frac{\sqrt{15}t}{2}-e^{\frac{t}{2}}cos\frac{\sqrt{15}t}{2}=0.$$

Then $\theta_0\approx 2,45$, $\bar{\sigma}_0^{+}(\theta_0,0)\approx -7,03$.

The solution of system (6) on the switching surface takes the form

$$\bar{\sigma}_0^{*}(t,\bar{\sigma}_0^{+}(\theta_0,0))=20-(20-\bar{\sigma}_0^{+}(\theta_0,0)-$$
$$-\frac{5}{4}\bar{x}_0^{+}(\theta_0,0))e^{\frac{-(t-\theta_0)}{5}}-\frac{5}{4}\bar{x}_0^{+}(\theta_0,0)e^{-(t-\theta_0)};$$
$$\bar{x}_0^{*}(t,\bar{\sigma}_0^{+}(\theta_0,0))=\bar{x}_0^{+}(\theta_0,0)e^{-(t-\theta_0)}.$$

Now the period of system (6) periodic solution is defined by equation $\bar{\sigma}_0^{*}(T_0,\bar{\sigma}_0^{+}(\theta_0,0))=$

$$=20-(20-\bar{\sigma}_0^{+}(\theta_0,0))e^{\frac{-(T_0-\theta_0)}{5}}=0.$$

And consequently

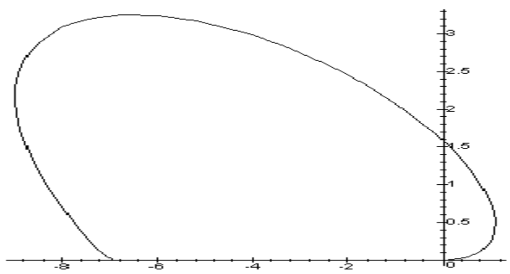

Fig. 2. Periodic solution of the reduced system.

$$T_0\approx 3,96,\quad \frac{\partial\bar{x}_0^{*}}{\partial\xi}(0)=e^{-T_0}\approx e^{-3,96}\approx 0,019\neq 0.$$

This means that for system (5) the conditions of Theorems 1 and 2 are true.

To finish with zero approximation of desired periodic it is necessary to define

$$\bar{z}_0(t)=\begin{cases}1 \quad \text{for} \quad 0\le t\le\theta_0,\\ \dfrac{\bar{\sigma}_0^{*}(t,\bar{\sigma}_0^{+}(\theta_0,0))+5}{5}, \text{for}\,\theta_0\le t\le T_0.\end{cases}$$

Then

$$d\Pi_0^{+}z/d\tau=-\Pi_0^{+}z;\quad \Pi_0^{+}z(0)=-\frac{\bar{\sigma}_0^{+}(\theta_0,0)}{5};$$
$$\Pi_0^{+}z(\tau)=-\frac{\bar{\sigma}_0^{+}(\theta_0,0)}{5}e^{-\tau}.$$

Let us compute the first approximation of desired periodic solution. Equations for the slow part of first approximation for $u=1$ have the form

$$\bar{z}_1^{+}=0;\quad d\bar{s}_1^{+}/dt=2\bar{s}_1^{+}+\bar{\sigma}_1^{+}, \tag{7}$$
$$d\bar{\sigma}_1^{+}/dt=-6\bar{s}_1^{+}-\bar{\sigma}_1^{+}+\bar{x}_1^{+},\quad d\bar{x}_1^{+}/dt=-\bar{x}_1^{+}+1.$$

Than the solution of (7) with initial conditions

$$\bar{s}_1^{+}(0)=\bar{\sigma}_1^{+}(0)=0,\ \bar{x}_1^{+}(0)=x_1^{*}$$

takes the form

$$\bar{s}_1^{+}(t)=\frac{1}{4}+\frac{1}{6}e^{-t}(x_1^{*}-1)++\left(\frac{-\sqrt{15}}{60}+\frac{\sqrt{15}}{30}x_1^{*}\right)$$
$$\times e^{\frac{t}{2}}sin\frac{\sqrt{15}t}{2}-\left(\frac{x_1^{*}}{6}+\frac{1}{12}\right)e^{\frac{t}{2}}cos\frac{\sqrt{15}t}{2};$$
$$\bar{\sigma}_1^{+}(t,x_1^{*})=-\frac{1}{2}-\frac{1}{2}e^{-t}(x_1^{*}-1)+\frac{\sqrt{15}}{30}\left(x_1^{*}+2\right)$$
$$\times e^{\frac{t}{2}}sin\frac{\sqrt{15}t}{2}+\frac{x_1^{*}}{2}e^{\frac{t}{2}}cos\frac{\sqrt{15}t}{2};$$
$$\bar{x}_1^{+}(t,x_1^{*})=(x_1^{*}-1)e^{-t}+1.$$

Then taking into account that we have $\theta_0\approx 2,45$, $\bar{s}_1^{+}(\theta_0)\approx 0,45-0,45x_1^{*}$;

$\bar{\sigma}_1^{+}(\theta_0)\approx -1,34-0,42x_1^{*}$; $\bar{x}_1^{+}(\theta_0)\approx 0,91+0,09x_1^{*}$.

Then

$$\theta_1=-[\bar{\sigma}_0^{+}(\theta_0)]^{-1}\bar{s}_1^{+}(\theta_0)\rightarrow\theta_1\approx 0.063-0.063x_1^{*}.$$

$$\Pi_1^{*}\sigma(\tau)=4\int_{\infty}^{\tau}\Pi_0^{+}z(\Theta)d\Theta=4\frac{\bar{\sigma}_0^{+}(\theta_0,0)}{5}e^{-\tau}.$$

$$\Pi_1^{*}\sigma(0)=4\frac{\bar{\sigma}_0^{+}(\theta_0,0)}{5},\ \Pi_1^{*}x(\tau)\equiv 0.$$

The initial conditions for the first approximation of slow variables on the sliding surface are defining by equations

$$\bar{\sigma}_1^{*}(\theta_0,x_1^{*})+\Pi_1^{*}\sigma(0)=\bar{\sigma}_1^{+}(\theta_0,x_1^{*})+\theta_1(x_1^{*})\frac{\bar{\sigma}_0^{+}}{dt}(\theta_0,0);$$
$$\bar{\sigma}_1^{*}(\theta_0,x_1^{*})=\bar{\sigma}_1^{+}(\theta_0,x_1^{*})+\theta_1(x_1^{*})(4-\bar{\sigma}_0^{+}(\theta_0,0))-$$
$$4\frac{\bar{\sigma}_0^{+}(\theta_0,0)}{5},\ \bar{x}_1^{*}(\theta_0,x_1^{*})=\bar{x}_1^{+}(\theta_0,x_1^{*})-\theta_1(x_1^{*})\bar{x}_0^{+}(\theta_0),$$
$$\bar{\sigma}_1^{*}(\theta_0,x_1^{*})\approx -1.12x_1^{*}+4.98;$$
$$\bar{x}_1^{*}(\theta_0,x_1^{*})\approx 0.09x_1^{*}+0.91.$$

At the same time the slow coordinates of system (6) periodic solution are describing by equations

$$d\bar{\sigma}_1^{*}/dt=-\frac{\bar{\sigma}_1^{*}}{5}+\bar{x}_1^{*}-4d\bar{z}_0^{*}(t)/dt;$$
$$d\bar{x}_1^{*}/dt=-\bar{x}_1^{*}+\bar{z}_0^{*}.$$

Now

$$\bar{\sigma}_1^{*}(t,x_1^{*})=25-(3.34+0,11x_1^{*})e^{-(t-\theta_0)}$$
$$-(1.01x_1^{*}+16.67+11,08t)e^{-(t-\theta_0)/5},$$
$$\bar{x}_1^{*}(t)=5-6.76e^{-(t-\theta_0)/5}+e^{-(t-\theta_0)}(2.67+0.09x_1^{*}).$$

Taking into account that $t=T_0$ we have

$$\bar{\sigma}_1^{*}(T_0,x_1^{*})\approx 4.38-0.77x_1^{*},$$
$$\bar{x}_1^{*}(T_0,x_1^{*})\approx 0.59+0.0x_1^{*}.$$

The value x_1^{*} is determined by equation $\bar{x}_1^{*}(T_0)=x_1^{*}$, which means that

$$x_1^{*}\approx 0.60,\ \theta_1\approx 0.025,\ \Theta_1\approx 0.22,\ T_1\approx 0.25$$

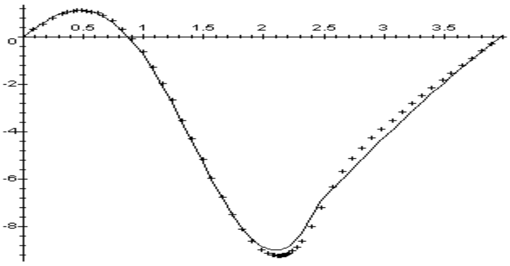

Fig. 3. σ coordinate for the periodic solution of the original system (line) and it's first order asymptotic representation (points) for $\mu = 0.2$.

6. CONCLUSIONS

Singularly perturbed relay systems (SPRS) for which the reduced systems have stable periodic motions with internal sliding modes are studied. For such systems a theorem about existence and stability of the periodic solutions is proved. The algorithm for the asymptotic representation of this periodic solutions using boundary functions method is presented. It is proved that in the asymptotic representation of periodic solutions with internal sliding modes there are two boundary layers:

- the boundary layer at the point of input into the sliding mode which corresponds to the jump of solution to the small neighbourhood of the slow motion integral manifold of singularly perturbed system describing the behavior of original SPRS into the sliding domain;
- the boundary layer at the break away point in which the solution is leaving the sliding domain.

It is proved that the zero order boundary function in the asymptotic representation of the periodic solution at the break away point is equal to zero because the zero approximation of the slow motion integral manifold at this point is continuous.

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