15th Triennial World Congress, Barcelona, Spain

# INTEGRAL ISS FOR SAMPLED-DATA NONLINEAR SYSTEMS

**D.Nešić** *

* *Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3010, Victoria, Australia.*

Abstract: We present sufficient conditions that guarantee that if a controller achieves integral input-to-state stability (iISS) of an approximate discrete-time model of a nonlinear sampled-data system, then the same controller will achieve semiglobal practical iISS of the exact discrete-time model by reducing the sampling period. Results are presented for arbitrary dynamic controllers that can be discontinuous in general.
Keywords: Discrete-time, integral input-to-state stability, nonlinear, Lyapunov method.

## 1. INTRODUCTION

Controllers are nowadays usually implemented digitally using computers which are connected to the continuous-time plant via digital to analog and analog to digital converters. Whenever we are interested in a non-local behaviour of such systems or when we control them using nonlinear controllers, we need to consider nonlinear sampled-data systems. However, the theory needed to analyze and design controllers for this class of systems is still not complete. The main stumbling block in the controller design for nonlinear sampled-data systems appears to be the absence of a good model for controller design even in the cases when the continuous-time plant model is known.

An approach for stabilization of sampled-data nonlinear system via their approximate discrete-time models has been proposed in (Nešić *et al.*, 1999*b*). These results were further extended in (Nešić and Teel, 2000) to cover plants modeled as differential inclusions, dynamic controllers and stability with respect to arbitrary non-compact sets. These papers provide a framework for controller design but they do not present recipes for controller design. An example of control design within this framework can be found in (Nešić and Teel, 2001) where backstepping controllers were developed based on the Euler approximate model of strict feedback systems. Simulation studies presented in (Nešić and Teel, 2001) indicate that this approach may yield much better behaviour than the controller design based on the continuous-time model followed by a discretization of the controller.

Since plants with disturbances are prevalent in control theory, there is a strong motivation to extend the approach of (Nešić *et al.*, 1999*b*; Nešić and Teel, 2000) to this class of plants. The first step in this direction was (Nešić and D.S.Laila, 2001) where a framework for input-to-state stabilization (ISS) of sampled-data nonlinear systems via their approximate discrete-time models was presented. Input-to-state stability (see (Sontag, 1989)) has found a widespread use in control theory but it is just one of the possible types of stability for systems with disturbances that may be of interest. A more general property of integral input-to-state stability (iISS) (see (Angeli *et al.*, 2000*b*; Angeli *et al.*, 2000*a*; Angeli, 2001; Sontag, 1998)) is proving to be as useful as ISS.

It is the main purpose of this paper to present a framework for design of controllers based on approximate discrete-time models that achieve iISS. Note that iISS was investigated in (Angeli, 2001) in the case when the exact discrete-time model of the plant is known. Our results are different since we do not assume existence of the exact discrete-time model, which was a standing assumption in (Angeli, 2001). We consider dynamic control laws that can be discontinuous in general and present sufficient conditions that guarantee that if a controller achieves iISS for an approximate discrete-time plant model, then the same controller will achieve semiglobal-practical iISS of the exact discrete-time plant model. We emphasize that the semiglobal part of our definition is different from the one used in (Angeli *et al.*, 2000*a*), whereas the "practical" iISS that we consider appears to be new and we are not aware of related results. Our approach benefits from the results in numerical analysis literature (Stuart and Humphries, 1996) and in particular (Ferretti, 1997; Grune and Kloeden, 2001).

The paper is organized as follows. In Section 2 we present preliminaries and definitions needed in the sequel. Section 3 contains main results with proofs.

## 2. PRELIMINARIES

Sets of real and natural numbers are denoted respectively as $\mathbb{R}$ and $\mathbb{N}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. $\gamma$ is of class $\mathcal{L}$ if it is continuous and $\gamma(s)$ decreases to zero as $s \to +\infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$ for each $s > 0$. For a given function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, we use the following notation: $w_f[k] = w(t)$ for $t \in [kT, (k+1)T]$ and $w_f[k] = 0$ otherwise, where $k \in \mathbb{N}$ and $T > 0$; and $w(k)$ is the value of the function $w(\cdot)$ at $t = kT, k \in \mathbb{N}$. We denote the norms $\|w_f[k]\|_\infty = \sup_{\tau \in [kT,(k+1)T]} |w(\tau)|$ and $\|w\|_\infty := \sup_{\tau \geq 0} |w(\tau)|$ and in the case when $w(\cdot)$ is a measurable function (in the Lebesgue sense) we use the essential supremum in the definitions. If there exists $r > 0$ such that $\|w\|_\infty \leq r$ or $\int_0^\infty \gamma(|w(s)|)ds \leq r$, with $\gamma \in \mathcal{K}_\infty$, then we write respectively $w \in \mathcal{L}_\infty(r)$ and $w \in \mathcal{L}_\gamma(r)$. Consider a continuous-time nonlinear plant with disturbances:

$$\dot{x}(t) = f(x(t), u(t), w(t)) \ , \tag{1}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^p$ are respectively the state, control input and exogenous disturbance. It is assumed that $f$ is locally Lipschitz and $f(0,0,0) = 0$. The control is taken to be a piecewise constant signal $u(t) = u(kT) =: u(k)$, $\forall t \in [kT, (k+1)T)$, $k \in \mathbb{N}$, where $T > 0$ is the sampling period. Also, we assume that some combination (output) or all of the states $(x(k) := x(kT))$ are available at sampling instant $kT, k \in \mathbb{N}$. The exact discrete-time model for the plant (1), which describes the plant behavior at sampling instants $kT$, is obtained by integrating the initial value problem

$$\dot{x}(t) = f(x(t), u(k), w(t)) \ , \tag{2}$$

with given $w_f[k]$, $u(k)$ and $x_0 = x(k)$, over the sampling interval $[kT, (k+1)T]$. If we denote by $x(t)$ the solution of the initial value problem (2) at time $t$ with given $x_0 = x(k)$, $u(k)$ and $w_f[k]$ and $t_k := kT$, then the exact discrete-time model of (1) can be written as:

$$\begin{aligned} x(k+1) &= x(k) + \int_{t_k}^{t_{k+1}} f(x(\tau), u(k), w(\tau))d\tau \\ &=: F_T^e(x(k), u(k), w_f[k]) \ . \end{aligned} \tag{3}$$

We refer to (3) as a *functional difference equation* since it depends on $w_f[k]$. We emphasize that $F_T^e$ is not known in most cases. Indeed, in order to compute $F_T^e$ we have to solve the initial value problem (2) analytically and this is usually impossible since $f$ in (1) is nonlinear. Hence, we will use an approximate discrete-time model of the plant to design a controller.

Different approximate discrete-time models can be obtained using different methods. Recently, numerical integration schemes for systems with measurable disturbances were considered in (Grune and Kloeden, 2001; Ferretti, 1997). Using these numerical integration techniques we can obtain an approximate discrete-time model

$$x(k+1) = F_T^a(x(k), u(k), w_f[k]) \ , \tag{4}$$

which is in general a functional difference equation. For instance, the simplest such approximate discrete-time model, which is analogous to Euler model, has the following form $x(k+1) = x(k) + \int_{kT}^{(k+1)T} f(x(k), u(k), w(s))ds$ (see (Grune and Kloeden, 2001)). Since we will consider semiglobal stability properties (see Definition 4), we will think of $F_T^e$ and $F_T^a$ as being defined globally for all small $T$, even though the initial value problem (2) may exhibit finite escape times (see discussion on pg. 261 in (Nešić *et al.*, 1999*b*)).

The sampling period $T$ is assumed to be a design parameter which can be arbitrarily assigned. Since we are dealing with a family of approximate discrete-time models $F_T^a$, parameterized by $T$, in order to achieve a certain objective we need in general to obtain a family of controllers, parameterized by $T$. We consider a family of dynamic feedback controllers

$$\begin{aligned} z(k+1) &= G_T(x(k), z(k)) \\ u(k) &= u_T(x(k), z(k)) \ , \end{aligned} \tag{5}$$

where $z \in \mathbb{R}^{n_z}$. To shorten notation, we introduce $\tilde{x} := (x^T\ z^T)^T$, $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$, where $n_{\tilde{x}} := n_x + n_z$ and

$$\mathcal{F}_T^i(\tilde{x}, w_f) := \begin{pmatrix} F_T^i(x, u_T(x,z), w_f) \\ G_T(x,z) \end{pmatrix} . \quad (6)$$

The superscript $i$ may be either $e$ or $a$, where $e$ stands for *exact* model, $a$ for *approximate* model. We omit the superscript if we refer to a general model. We use the following:

*Definition 1.* $u_T$ is said to be locally uniformly bounded if for any $\Delta_{\tilde{x}} > 0$ there exist strictly positive numbers $T^*$ and $\Delta_u$ such that for all $T \in (0, T^*)$, $|\tilde{x}| \le \Delta_{\tilde{x}}$ we have $|u_T(\tilde{x})| \le \Delta_u$.

In order to prove our main results, we need to guarantee that the mismatch between $F_T^e$ and $F_T^a$ is small in some sense. We define a consistency property, which will be used to limit the mismatch. Similar definitions can be found in numerical analysis literature (see Definition 3.4.2 in (Stuart and Humphries, 1996)) and recently in the context of sampled-data systems with disturbances (for instance, see (Nešić and D.S.Laila, 2001)). In the sequel we use the notation $x = x(k)$, $u = u(k)$, $w_f = w_f[k]$.

*Definition 2.* The family $F_T^a$ is said to be one-step consistent with $F_T^e$ if given any strictly positive real numbers $(\Delta_x, \Delta_u, \Delta_w)$, there exist a function $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that, for all $T \in (0, T^*)$, all $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^m, w \in \mathcal{L}_\infty$ with $|x| \le \Delta_x$, $|u| \le \Delta_u$, $\|w_f\|_\infty \le \Delta_w$, we have $|F_T^e - F_T^a| \le T\rho(T)$.

Sufficient checkable conditions for one-step consistency are given next (for the proof of this result see (Nešić and D.S.Laila, 2001)).

*Lemma 1.* $F_T^a$ is one-step consistent with $F_T^e$ if the following conditions hold: 1. $F_T^a$ is one-step consistent with $\tilde{F}_T^{Euler}(x, u, w_f) := x + \int_{kT}^{(k+1)T} f(x, u, w(s))ds$; 2. given any strictly positive real numbers $(\Delta_x, \Delta_u, \Delta_w)$, there exist $\rho_1 \in \mathcal{K}_\infty$, $T^* > 0$, such that, for all $T \in (0, T^*)$ and all $x_1, x_2 \in \mathbb{R}^{n_x}$ with $\max\{|x_1|, |x_2|\} \le \Delta_x$, all $u \in \mathbb{R}^m$ with $|u| \le \Delta_u$ and all $w \in R^p$ with $|w| \le \Delta_w$, the following holds $|f(x_1, u, w) - f(x_2, u, w)| \le \rho_1(|x_1 - x_2|)$.

## 3. INTEGRAL INPUT TO STATE STABILITY

In this section we state and prove the main results of this paper. The main result (Theorem 1) presents sufficient conditions on the continuous-time plant model, the controller and the approximate discrete-time plant model that guarantee that if the controller achieves semiglobal practical Lyapunov iISS for the approximate model (see Definition 3), then the same controller would yield a semiglobal practical iISS bound on the solutions of the exact discrete-time plant model (see Definition 4). We emphasize that it was shown in (Nešić *et al.*, 1999*b*) that if some of these conditions do not hold, then the controller may not achieve iISS for the exact discrete-time plant model.

In order to state the following two definitions, we consider the family of systems:

$$\tilde{x}(k+1) = \mathcal{F}_T(\tilde{x}(k), w_f[k]) . \quad (7)$$

*Definition 3.* (Lyapunov-SP-iISS). The family of systems (7) is Lyapunov semiglobally practically integral input-to-state stable (Lyapunov-SP-iISS) if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\tilde{\gamma} \in \mathcal{K}$ and a continuous positive definite function $\alpha_3$, and for any strictly positive real numbers $(\Delta_1, \Delta_2, \Delta_3, \delta_1)$ there exist strictly positive real numbers $T^*$ and $L$ such that for all $T \in (0, T^*)$ there exists a function $V_T : \mathbb{R}^{n_{\tilde{x}}} \to \mathbb{R}_{\ge 0}$ such that for all $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$ with $|\tilde{x}| \le \Delta_1$ and all $w \in \mathcal{L}_\infty(\Delta_2) \cap \mathcal{L}_{\tilde{\gamma}}(\Delta_3)$ the following holds:

$$\begin{aligned} &\alpha_1(|\tilde{x}|) \le V_T(\tilde{x}) \le \alpha_2(|\tilde{x}|) \\ &\frac{\Delta V_T}{T} \le -\alpha_3(|\tilde{x}|) + \frac{1}{T}\int_{kT}^{(k+1)T} \tilde{\gamma}(|w(s)|)ds \quad (8) \\ &\qquad + \delta_1 \ , \end{aligned}$$

where $\Delta V_T := V_T(\mathcal{F}_T(\tilde{x}, w_f)) - V_T(\tilde{x})$ and, moreover, for all $x_1, x_2, z$ with $|(x_1^T\ z^T)^T|, |(x_2^T\ z^T)^T| \in [0, \Delta_1]$ and all $T \in (0, T^*)$, we have

$$|V_T(x_1, z) - V_T(x_2, z)| \le L\,|x_1 - x_2| \ .$$

The function $V_T$ is called an iISS-Lyapunov function for the family $\mathcal{F}_T$.

We use the following version of semiglobal-practical iISS property (note that it is different from the definition in (Angeli *et al.*, 2000*a*)).

*Definition 4.* (SP-iISS). The family of systems (7) is semiglobally practically integral input-to-state stable (SP-iISS) if there exist $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$ such that for any strictly positive real numbers $(\Delta_{\tilde{x}}, \Delta_{w1}, \Delta_{w2}, \delta)$ there exists $T^* > 0$ such that for all $T \in (0, T^*)$, $|\tilde{x}(0)| \le \Delta_{\tilde{x}}$ and $w \in \mathcal{L}_\infty(\Delta_{w1}) \cap \mathcal{L}_\gamma(\Delta_{w2})$, the solutions of the system satisfy $\alpha(|\tilde{x}(k)|) \le \beta(|\tilde{x}(0)|, kT) + \int_0^{kT} \gamma(|w(s)|)ds + \delta$, $\forall k \in \mathbb{N}$.

The following theorem contains the main result of this paper. It gives checkable conditions on the approximate model, controller and the continuous-time plant model that guarantee that if a controller achieves Lyapunov-SP-iISS of the approximate discrete-time plant model, the same controller would achieve SP-iISS of the exact discrete-time plant model.

*Theorem 1.* Suppose that: (i) The family of approximate discrete-time models $\mathcal{F}_T^a$ is Lyapunov-SP-iISS; (ii) $F_T^a$ is one-step consistent with $F_T^e$; (iii) $u_T$ is uniformly locally bounded. Then, the family of exact discrete-time models $\mathcal{F}_T^e$ is SP-iISS.

We note that our results allows the family of controllers to depend discontinuously on states.

*Remark 1.* Under mild conditions (see for instance results in (Nešić *et al.*, 1999*a*)) it is possible to over-bound also inter-sample bahaviour and to conclude from Theorem 1 that: there exist $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$ such that for any strictly positive real numbers $(\Delta_{\tilde{x}}, \Delta_{w1}, \Delta_{w2}, \delta)$ there exists $T^* > 0$ such that for all $T \in (0, T^*)$, $|\tilde{x}(t_\circ)| \le \Delta_{\tilde{x}}$ and $w \in \mathcal{L}_\infty(\Delta_{w1}) \cap \mathcal{L}_\gamma(\Delta_{w2})$, the solutions of the system satisfy $\alpha(|\tilde{x}(t)|) \le \beta(|\tilde{x}(t_\circ)|, t - t_\circ) + \int_{t_\circ}^t \gamma(|w(s)|)ds + \delta$, $\forall t \ge t_\circ \ge 0$.

*Remark 2.* Similarly to results presented in (Nešić and D.S.Laila, 2001), we may also start with an approximate discrete-time model of the plant for which we assumed that disturbances are constant during sampling intervals $w(t) = w(kT) = const., \forall t \in [kT, (k+1)T), k \in \mathbb{N}$. In this case, the approximate and exact models will depend on $w(kT)$ (not on $w_f[k]$) which means that they are difference equations (not functional difference equations). It was shown in (Nešić and D.S.Laila, 2001) that a "weak" form of consistency property can be stated in this case and it can be used in a very similar manner to state a result similar to Theorem 1 except that the bound in Definition 4 would hold for a smaller class of disturbances whose derivatives also need to be bounded. We did not pursue this direction for space reasons.

## 4. PROOFS OF MAIN RESULTS

**Proof of Theorem 1:** Let $\alpha_3$ come from item (i) of Theorem and let $\tilde{\rho}_1 \in \mathcal{K}_\infty$ and $\tilde{\rho}_2 \in \mathcal{L}$ be generated using Lemma 4 such that $\alpha_3(s) \ge \tilde{\rho}_1(s) \cdot \tilde{\rho}_2(s), \forall s \ge 0$. Let $\rho_1(s) := \tilde{\rho}_1 \circ \alpha_2^{-1}(s)$ and $\rho_2(s) := \tilde{\rho}_2 \circ \alpha_1^{-1}(s)$, $\rho_1'(s) := \frac{1}{2}\rho_1(s)$. Let $\beta$ be generated via Lemma 3 using $\rho_1'$ and $\rho_2$. Let $\gamma(s) := 2\tilde{\gamma}(s)$ and $\alpha(s) := \alpha_1(s)$.

Let $(\Delta_x, \Delta_{w1}, \Delta_{w2}, \delta)$ be given. Define

$$\Delta_1 := \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_{w2} + \delta) + 1 \qquad (9)$$

$$\Delta_2 := \Delta_{w1} \qquad (10)$$

$$\Delta_3 := \Delta_{w2} \ . \qquad (11)$$

Let $\delta_1 := \frac{\rho_1(\delta/2)\cdot\rho_2(\Delta_1)}{4}$. Let $(\Delta_1, \Delta_2, \Delta_3, \delta_1)$ generate $T_1^*$ and $L$ via item (i) of Theorem, where without loss of generality we can assume that $L \ge 1$. Let $\Delta_1$ generate $\Delta_u$ and $T_2^*$ via item (iii) of Theorem. Let $(\Delta_1, \Delta_u, \Delta_2)$ generate $\rho$ and $T_3^*$ via item (ii) of Theorem. Let $T_4^* > 0$ be such that

$$LT_4^*\rho(T_4^*) \le \min\left\{\frac{1}{2}, \frac{\delta}{4}\right\}$$
$$L\rho(T_4^*) \le \delta_1 \qquad (12)$$

Let $\tilde{\delta} > 0$ be such that

$$\alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta + \tilde{\delta}) \le \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta) + \frac{1}{2} \ , \qquad (13)$$

and denote $T_5^* := \frac{\tilde{\delta}}{\tilde{\gamma}(\Delta_2)+\delta_1}$. Denote $T_6^* := \frac{\delta}{4(\tilde{\gamma}(\Delta_2)+\delta_1)}$. Finally, we introduce

$$T^* := \min\{T_1^*, T_2^*, T_3^*, T_4^*, T_5^*, T_6^*\} \ .$$

To shorten notation we denote $V_k^e := V_T(\mathcal{F}_T^e(\tilde{x}(k), w[k]))$, $V_k^a := V_T(\mathcal{F}_T^a(\tilde{x}(k), w[k]))$ and $V_k := V(\tilde{x}(k))$.

Consider now an arbitrary $\tilde{x}_k$ such that $V_k \le \alpha_2(\Delta_x) + \Delta_{w2} + \delta$ (this implies $|\tilde{x}_k| \le \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_{w2} + \delta) < \Delta_1$), $w \in \mathcal{L}_\infty(\Delta_2) \cap \mathcal{L}_{\tilde{\gamma}}(\Delta_3)$ and $T \in (0, T^*)$. Using item (ii) and our choice of $T_1^*$, we can write that:

$$V_k^e - V_k \le -T\alpha_3(|\tilde{x}_k|) + \int_{kT}^{(k+1)T} \tilde{\gamma}(|w(s)|)ds + |V_k^e - V_k^a| + T\delta_1 \ . \qquad (14)$$

From our choice of $T_5^*$ we can write using item (i) of Theorem:

$$\alpha_1(|(F_T^a, G_T)|) \le V_k^a \ \le \ V_k + T\tilde{\gamma}(\Delta_2) + T\delta_1 \le \alpha_2(\Delta_x) + \Delta_3 + \delta + \tilde{\delta},$$

which implies from the definition of $\tilde{\delta}$ in (13) that

$$|(F_T^a, G_T)| \le \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta + \tilde{\delta}) \le \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta) + 1/2 < \Delta_1$$

and from our choice of $T_4^*$ in (12) and the fact that $L \ge 1$ we have:

$$|(F_T^e, G_T)| \le |(F_T^a, G_T)| + |(F_T^e, G_T) - (F_T^a, G_T)| \le \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta) + 1/2 + |F_T^e - F_T^a| \le \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta) + 1 = \Delta_1 \ .$$

Hence, using local Lipschitz condition of $V_T$ in item (i), item (ii) and our definition of $T_2^*$, $T_3^*$ and $T_4^*$ in (12) we can write that:

$$|V_k^e - V_k^a| \le LT\rho(T) \ \le \ T\delta_1 \ . \qquad (15)$$

From (14) and (15) and our definitions of $\rho_1, \rho_2$ we can write:

$$V_k^e - V_k \le -T\rho_1(V_k)\rho_2(V_k) + \int_{kT}^{(k+1)T} \tilde{\gamma}(|w(s)|)ds + T2\delta_1 \ , \quad (16)$$

and using the fact that $\rho_1(s)\rho_2(s) \ge 4\delta_1$ for all $s \in [\delta/2, \Delta_1]$, we can write:

$$V_k \ge \frac{\delta}{2} \Rightarrow V_k^e - V_k \le -\frac{T}{2}\rho_1(V_k)\rho_2(V_k) + \int_{kT}^{(k+1)T} \tilde{\gamma}(|w(s)|)ds \ . \quad (17)$$

Moreover, using (14), (15) and the definitions of $T_4^*$ and $T_6^*$ we can write:

$$V_k^e \le V_k^a + |V_k^e - V_k^a| \le V_k + \frac{\delta}{4} + \frac{\delta}{4} \ . \quad (18)$$

Introduce $w_k := \int_0^{kT} \tilde{\gamma}(|w(s)|)ds$ and define $y_k := V_k - w_k$. Note that $w_k$ is nondecreasing, $w_0 = 0$ and $y_\circ = V_0$. Then we have from (17) and (18), with $\tilde{\rho}(s) := \rho_1'(s)\rho_2(s)$ that

$$y_{k+1} \le y_k + \frac{\delta}{2} \quad (19)$$

$$y_k \ge \frac{\delta}{2} \Rightarrow y_{k+1} - y_k \le -T\tilde{\rho}(\max\{y_k + w_k, 0\}) \quad (20)$$

whenever $y_k \le \alpha_2(\Delta_x) + \Delta_3 + \delta - \Delta_3$. Note that since $V_k \ge 0$ and $w_k \le \Delta_{w2}$ for all $k \ge 0$, we have that $y_k \ge -\Delta_3, \forall k \ge 0$. Moreover, we show now by induction that $y_\circ \in [0, \alpha_2(\Delta_x)+\delta]$ implies that $y_k \le \alpha_2(\Delta_x) + \delta, \forall k \ge 0$. For $k = 0$ we have that either $y_\circ \in [\delta/2, \alpha_2(\Delta_x)+\delta]$ in which case we have from (19) that $y_1 \le y_\circ \le \alpha_2(\Delta_x) + \delta$ or we have that $y_\circ \in [0, \delta/2)$ in which case from (19) we have that $y_1 \le y_\circ + \delta/2 < \delta < \alpha_2(\Delta_x) + \delta$. Suppose now that $y_k \in [-\Delta_3, \alpha_2(\Delta_x) + \delta]$. Then we have that either $y_k \in [\delta/2, \alpha_2(\Delta_x)+\delta]$, in which case we have from (20) that $y_{k+1} \le y_k \le \alpha_2(\Delta_x)+\delta$ or we have that $y_k \in [-\Delta_3, \delta/2)$, in which case we have from (19) that $y_{k+1} \le y_k + \delta/2 < \delta < \alpha_2(\Delta_x) + \delta$. Hence, for any $y_\circ \in [0, \alpha_2(\Delta_x) + \delta]$ we have that $y_k \in [-\Delta_3, \alpha_2(\Delta_x) + \delta], \forall k \ge 0$ and therefore all conditions of Lemma 2 hold with $k^* = \infty$. We conclude from Lemma 2 with $\Delta_y = \alpha_2(\Delta_x) + \delta$, $c_1 = c_2 = \delta/2$ and $\tilde{\rho}(s) = \rho_1'(s)\rho_2(s)$ that

$$y_k \le \beta(y_\circ, kT) + w_k + \frac{\delta}{2} + \frac{\delta}{2} \ , \ \forall k \ge 0,$$

which implies (using the definition of $y_k$ and the fact that $y_\circ = V_\circ$) that

$$V_k \le \beta(V_\circ, kT) + 2w_k + \delta \ , \ \forall k \ge 0$$

and consequently

$$\alpha_1(|\tilde{x}(k)|) \le \beta(\alpha_2(|\tilde{x}_\circ|), kT) + 2\int_0^{kT} \tilde{\gamma}(|w(s)|)ds + \delta, \forall k \ge 0 \ ,$$

which completes the proof.

*Lemma 2.* Given any continuous positive definite function $\tilde{\rho} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, there exists a $\mathcal{KL}$ function with the following property. Suppose that $y : \mathbb{N} \to \mathbb{R}$ and a nondecreasing function $w : \mathbb{N} \to \mathbb{R}_{\ge 0}$ satisfy the following

$$y_{k+1} \le y_k + c_1 \quad (21)$$

and

$$y_k \ge c_2 \ \Rightarrow \ y_{k+1} - y_k \le -T\tilde{\rho}(\max\{y_k + w_k, 0\}) \quad (22)$$

for all $k \in [0, k^*)$ with $0 < k^* \le \infty$ and all $y_k \le \Delta_y$ where $\Delta_y > c_1 + c_2$. Then there exists $\beta \in \mathcal{KL}$ such that for all $y_\circ \le \Delta_y$ and all $k \in [0, k^*)$ the following holds:

$$y_k \le \beta(y_\circ, kT) + w_k + c_1 + c_2 \ . \quad (23)$$

**Proof:** First we note that for all $y_k \ge c_2$ we have from (22) that $y_{k+1} \le y_k$ and if $y_k \le c_2$ we have from (21) that $y_{k+1} \le y_k + c_1 \le c_1 + c_2$. Moreover, since $\Delta_y > c_1 + c_2$ we conclude that the set

$$\{y : y \le c_1 + c_2\} \quad (24)$$

is forward invariant, that is, $y_\circ \le c_1 + c_2$ implies $y_k \le c_1 + c_2$ for all $k \in [0, k^*)$.

Suppose now that $\Delta \ge y_\circ > c_1 + c_2 > 0$. Define

$$k_1 := \min\{k \ge 0 : y_k \le c_1 + c_2\}$$

(with $k_1 = k^*$ if $y_k > c_1 + c_2$ for all $k \in [0, k^*)$). Hence, for all $k \ge k_1$ (if $k_1 < k^*$) we have that $y_k \le c_1 + c_2$ since the set (24) is forward invariant and so (23) holds. Define now

$$k_0 := \min\{k \ge 0 : y_k \le w_k\}$$

(with $k_0 = k_1$ if $y_k > w_k$ for all $k \in [0, k_1)$). Note that for all $k \in [0, k_1)$ we have from (22) that $y_k$ is non-increasing and also recall $w_k$ is assumed to be a nondecreasing function of time. Hence, for all $k \in [k_0, k_1)$ (if $k_0 < k_1$) we have that $y_k \le w_k$ and so (23) holds. Finally, consider $k \in [0, k_0)$. Note that $y_k > w_k \ge w_i$ for all $i \in [0, k]$ and since $y$ is non-increasing, we have that $y_i \ge y_k > w_i$ for all such $i$. Therefore, $0 \le y_i \le y_i + w_i \le 2y_i$ for all $i \in [0, k]$. From Lemma 4 and (22) we can write that

$$y_{i+1} - y_i \le -T\rho_1(y_i)\rho_2(2y_i), \ \forall i \in [0, k] \ .$$

From Lemma 3 we conclude that

$$y_i \le \beta(y_\circ, iT), \ \forall i \in [0, k] \ ,$$

and hence the bound (23) holds, which completes the proof.

*Lemma 3.* Suppose that $T > 0$ and $y : \mathbb{N} \to \mathbb{R}_{\ge 0}$ satisfy the following inequality for all $k \in [0, k^*)$

$$y_{k+1} - y_k \le -T\rho_1(y_k) \cdot \rho_2(2y_k) \ , \quad (25)$$

where $k^* \in \mathbb{N} \cup \{\infty\}$, $\rho_1 \in \mathcal{K}_\infty$ is locally Lipschitz and $\rho_2 \in \mathcal{L}$. Then, there exists $\beta \in \mathcal{KL}$ such that the following holds $y_k \le \beta(y_\circ, kT), \ \forall k \in [0, k^*)$.

**Proof:** Consider an arbitrary $y_\circ$ and the corresponding sequence $y_k$. We introduce a new continuous and piecewise linear variable $\forall t \in [kT\rho_2(2y_\circ),(k+1)T\rho_2(2y_\circ)), k \in [0,k^*-1)$:

$$\eta(t) = y_k + \left(\frac{t}{T\rho_2(2y_\circ)} - k\right)(y_{k+1} - y_k)\ ,$$

and we let $\eta((k^*-1)T\rho_2(2y_\circ)) = y_{k^*-1}$ if $k^* \neq \infty$. Note that $\eta(kT\rho_2(2y_\circ)) = y_k$ for all $k \in [0,k^*)$. Denote $t^* := k^*T\rho_2(2y_\circ)$. Since $\eta$ is continuous and piecewise linear, it is differentiable for almost all $t \in [0,t^*)$. Hence, we can write that for all $t \in [kT\rho_2(2y_\circ),(k+1)T\rho_2(2y_\circ)), k \in [0,k^*-1)$ we have:

$$\dot{\eta}(t) = \frac{y_{k+1} - y_k}{T\rho_2(2y_\circ)} \ \leq\ -\rho_1(y_k)\frac{\rho_2(2y_k)}{\rho_2(2y_\circ)} \quad (26)$$

Moreover, since $y_{k+1} \leq y_k$ and $\rho_2(2y_k) \geq \rho_2(2y_\circ)$ for all $k \in [0,k^*)$, we have $\eta(t) \leq y_k$ for all $t \in [kT\rho_2(2y_\circ),(k+1)T\rho_2(2y_\circ)), k \in [0,k^*-1)$. We can conclude from (26) that

$$\dot{\eta}(t) \leq -\rho_1(\eta(t)),\ for\ a.a.\ t \in [0,t^*)\ .$$

Using the standard comparison principle (see Proposition 2.5 in (Lin *et al.*, 1996)) and since $\rho_1$ is assumed locally Lipschitz, we conclude that there exists $\beta_1 \in \mathcal{KL}$ such that we have:

$$\eta(t) \leq \beta_1(\eta_\circ,t),\ \forall t \in [0,t^*)\ . \quad (27)$$

We let $t = kT\rho_2(2y_k)$ to obtain

$$y_k \leq \beta_1(y_\circ, kT\rho_2(2y_\circ)) \quad (28)$$

Since $y_{k+1} \leq y_k, k \in [0,k^*-1)$ we conclude that $y_k \leq y_\circ, \forall k \in [0,k^*)$ and we since $\rho_2 \in \mathcal{L}$, we can write:

$$\begin{aligned} y_k &\leq \beta_1(y_\circ, kT\rho_2(2y_\circ)) \\ &=: \beta(y_\circ, kT), \forall k \in [0,k^*), \end{aligned} \quad (29)$$

where it is easy to see that $\beta(s,t) := \beta_1(s,t\rho_2(s)) \in \mathcal{KL}$.

*Lemma 4.* (Angeli *et al.*, 2000*b*) Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous positive define function. Then there exist $\rho_1 \in \mathcal{K}_\infty$ and $\rho_2 \in \mathcal{L}$ such that $\rho(r) \geq \rho_1(r)\rho_2(r), \forall r \geq 0$.

## 5. CONCLUSION

We have provided a framework for integral input-to-state stabilization of nonlinear sampled-data systems via their approximate discrete-time models. Designing controllers for particular classes of systems and approximate models is an interesting topic for further research.

## 6. ACKNOWLEDGMENTS

This work is supported by the Australian Research Council under the Large Grants Scheme.

## 7. REFERENCES

Angeli, D. (2001). Intrinsic robustness of global asymptotic stability. *Systems and Control Letters* p. to appear.

Angeli, D., E.D. Sontag and Y. Wang (2000*a*). Further equivalences and semiglobal versions of integral input to state stability. *Dynamics and Control Journal* **10**, 127–149.

Angeli, D., E.D. Sontag and Y. Wang (2000*b*). A lyapunov characterization of integral input to state stability. *IEEE Transactions on Automatic Control* **45**, 1082–1097.

Ferretti, R. (1997). Higher order approximations of linear control systems via runge kutta schemes. *Computing* **58**, 351–364.

Grune, L. and P.E. Kloeden (2001). Higher order numerical schemes for affinely controlled nonlinear systems. *Numerische Mathematik.*

Lin, Y., E.D. Sontag and Y. Wang (1996). A smooth converse lyapunov theorem for robust stability. *SIAM Journal on Control and Optimization* **34**, 124–160.

Nešić, D. and A.R. Teel (2000). Set stabilization of nonlinear sampled-data differential inclusions via their approximate discrete-time models. In: *Conference on Decision and Control.* IEEE. Sydney. pp. 2112–2117.

Nešić, D. and A.R. Teel (2001). Backstepping on the euler approximate model for stabilization of sampled-data nonlinear systems. In: *Conference on Decision and Control.* IEEE. Orlando. p. to appear.

Nešić, D. and D.S.Laila (2001). A note on input-to-state stabilization for nonlinear sampled-data systems. *IEEE Transactions on Automatic Control.*

Nešić, D., A.R. Teel and E.D. Sontag (1999*a*). Formulas relating $\|l$ stability estimates of discrete-time and sample-data nonlinear systems. *System and Control Letters* **38**, 49–60.

Nešić, D., A.R. Teel and P. V. Kokotović (1999*b*). Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations. *System and Control Letters* **38**, 259–270.

Sontag, E.D. (1989). Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control* **34**, 435–443.

Sontag, E.D. (1998). Comments on integral variants of input-to-state stability. *System and Control Letters* **34**, 93–100.

Stuart, A.M. and A.R. Humphries (1996). *Dynamical systems and numerical analysis.* Cambridge University Press. New York.