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STABILITY ANALYSIS AND DESIGN OF SIMPLE T-S FUZZY CONTROL SYSTEM

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Abstract: Stability and design issues of simple T-S fuzzy control system with simplified linear rule consequent (TSS) are investigated. A systematic approach to find a common matrix $\boldsymbol{P}$ for TSS fuzzy system is presented first, where system matrix $\boldsymbol{A}_i$ is decomposed into proportional part $\tilde{\boldsymbol{A}}_i$ and the remainder $\Delta\boldsymbol{A}_i$. Hence an iterative approach to find a common matrix $\boldsymbol{P}$ for pairwise commutative $\tilde{\boldsymbol{A}}_i$'s can be used. And the stability of the global system is guaranteed if $\Delta\boldsymbol{A}_i$ satisfies certain conditions. Secondly, qualitative instruction for system design can be obtained, by which systematic steps for controller design of TSS are formed. *Copyright © 2002 IFAC*

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1. INTRODUCTION

T-S fuzzy model (Takagi and Sugeno, 1985) is essentially a nonlinear model. It can describe dynamical characteristics of a complex nonlinear system. Since its consequent is usually a linear model, linear control theory can be used to design proper controller. Buckley proved that this kind of controller is a universal controller (Buckley, 1993). Tanaka and Sugeno (1992) proposed sufficient condition for the stability of global fuzzy system. These works formed a framework for modeling, control design, and stability analysis of T-S systems.

Stability analysis of a T-S fuzzy system is mostly based on the result obtained by Tanaka and Sugeno (1992), i.e., to find a common positive definite matrix $\boldsymbol{P}$ for each subsystem $\boldsymbol{A}_i$. A series of research to solve $\boldsymbol{P}$ have been developed, among which LMI method is an effective one (Tanaka, *et al*., 1996; Park, *et al*., 2001). However, this stability condition is sufficient and can only be used after system design. In other words, a common matrix $\boldsymbol{P}$ can be found but after controller design. If $\boldsymbol{P}$ exists, the stability of global control system is guaranteed. However, few literatures investigated the case that the sufficient condition does not hold. At this rate, it is commonly accepted that new controllers for subsystems should be redesigned and $\boldsymbol{P}$ should be searched again. Since the results of last design have few instructions for redesign, design procedure seems non-systematic.

Narendra and Balakrishnan (1994) studied the stability of linear time-invariant multi-model system with pairwise commutative matrix $\boldsymbol{A}_i$, and presented an iterative method of finding a common matrix $\boldsymbol{P}$. Using this conclusion, Joh, *et al.* (1998) did research on stability of T-S fuzzy systems, whose subsystems were also under the pairwise commutative assumption. The robustness issue on uncertainty in each subsystem was also considered. Since it is difficult to satisfy the above assumption, the discussion seems short of pertinency. The design of control system was also not discussed. T-S system with simplified linear rule consequent (TSS) proposed by Ying (1998) is a simplification of T-S systems. It decreases the number of parameters to be identified but still ensures universal approximation. In this paper, system matrices $\boldsymbol{A}_i$'s of TSS are decomposed into proportional parts and their residuum. Stability conditions of a TSS control system are then investigated based on conclusions presented by Joh, *et al.* (1998). An iterative method of finding its common matrix $\boldsymbol{P}_N$ is also presented. Moreover, a simple and systematic design method of TSS control system is presented, which can guarantee system stability.

2. PROBLEM DESCRIPTION

2.1 T-S Fuzzy Model

A T-S fuzzy system with *N* rules can be written as,

$$
\begin{aligned}
&R^i: \ if \ \ x_1(k) \ \ is \ \ L_1^i \ \ and \ \ x_2(k) \ \ is \ \ L_2^i \ \ and \\
&\qquad \cdots and \ \ x_n(k) \ \ is \ \ L_n^i \\
&\quad then \ \ \boldsymbol{X}^i(k+1) = \boldsymbol{A}_i\boldsymbol{X}(k) + \boldsymbol{B}_i\boldsymbol{U}(k) \\
&\qquad\qquad i = 1,2\cdots,N
\end{aligned} \tag{1}
$$

where L_j^i is the fuzzy set of premise variables, x_j is the j th state variable, $\boldsymbol{X}(k) = [x_1(k),\cdots,x_n(k)]^T$, and $\boldsymbol{U}(k) = [u_1(k),\cdots,u_m(k)]^T$ is the input vector. For current state $\boldsymbol{X}(k)$ and input $\boldsymbol{U}(k)$, the T-S fuzzy model infers $\boldsymbol{X}(k+1)$ as its output,

$$\boldsymbol{X}(k+1) = \sum_{i=1}^{N} \lambda_i(k)\left[\boldsymbol{A}_i\boldsymbol{X}(k) + \boldsymbol{B}_i\boldsymbol{U}(k)\right] \tag{2}$$

where $\lambda_i(k) = \dfrac{w_i(k)}{\sum_{j=1}^{N} w_j(k)}$, $w_i(k) = \left(\prod_{j=1}^{n} L_j^i\right)$.

For a free system (i.e. $u(k) \equiv 0$), (2) can be written as,

$$\boldsymbol{X}(k+1) = \sum_{i=1}^{N} \lambda_i(k)\boldsymbol{A}_i\boldsymbol{X}(k) \tag{3}$$

Hence, the stability problem of (3) can be transferred to be one with *N* simultaneous linear systems:

$$\boldsymbol{X}(k+1) = \boldsymbol{A}_i\boldsymbol{X}(k), \ \ i = 1,2,\cdots,N \tag{4}$$

Proposition 1 follows then,

Proposition 1 (Tanaka and Sugeno, 1992): The equilibrium state of fuzzy system (3) (namely, $\boldsymbol{X} = \boldsymbol{0}$) is globally asymptotically stable if there exists a common positive definite matrix $\boldsymbol{P}$ such that,

$$\boldsymbol{A}_i^T\boldsymbol{P}\boldsymbol{A}_i - \boldsymbol{P} < 0, \ \ i = 1,2,\cdots,N \tag{5}$$

2.2 State Space Description of TSS Model

Ying (1998) presented a T-S system with simplified linear rule consequent (TSS). He also proved that *the general MISO fuzzy systems with the simplified linear T-S rule consequent could uniformly approximate any multivariate continuous function in closed domain to any degree of accuracy*. TSS system with *N* rules is,

$$
\begin{aligned}
&R^i: \ if \ \ y(k) \ \ is \ \ M_1^i \ \ and\cdots and \ \ y(k-n+1) \ \ is \ \ M_n^i \\
&\qquad and \ \ u(k) \ \ is \ \ N_1^i \ \ and\cdots and \ \ u(k-m+1) \ \ is \ \ N_m^i \\
&then \ \ y^i(k+1) = k_i\left[\sum_{l=1}^{n} a_l y(k-l+1) + \sum_{j=1}^{m} b_j u(k-j+1)\right] \\
&\qquad\qquad i = 1,2\cdots,N
\end{aligned} \tag{6}
$$

where M_l^i, N_j^i are fuzzy sets of model inputs, $y(k-l+1), u(k-j+1)$ are model input and output, a_l, b_j are parameters to be identified. For different rules a_l, b_j are constants. Clearly, the difference between different rule consequents is just the proportion coefficient k_i. Let

$$\boldsymbol{X}(k+1) = [y(k),\cdots y(k-n+1)]^T = [x_1(k),\cdots x_n(k)]^T$$

The consequent of *i*th rule is,

$$\boldsymbol{X}(k+1) = k_i\left[\sum_{l=1}^{n} a_l x_l(k) + \sum_{j=1}^{m} b_j u(k-j+1)\right] \tag{7}$$

For simplicity, in this paper assume $b_j = 0, \forall j > 1$ (Zhao, *et al.*, 1997). It may be shown that this assumption would not essentially change the main conclusions of this paper. Therefore (7) can be written as,

$$
\begin{aligned}
&x_1(k+1) = k_i\left[a_1x_1(k) + a_2x_2(k) + \cdots + a_nx_n(k) + b_1u(k)\right] \\
&x_2(k+1) = x_1(k) \\
&\vdots \\
&x_n(k+1) = x_{n-1}(k)
\end{aligned} \tag{8}
$$

i.e.

$$
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} k_ia_1 & k_ia_2 & \cdots & k_ia_n \\ 1 & 0 & \cdots & \boldsymbol{0} \\ & \ddots & \ddots & \vdots \\ \boldsymbol{0} & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} k_ib_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k) \tag{9}
$$

Then the consequent of *i*th rule is,

$$\boldsymbol{X}(k+1) = \boldsymbol{A}_i\boldsymbol{X}(k) + \boldsymbol{B}_iu(k) \tag{10}$$

where $\boldsymbol{A}_i = \begin{bmatrix} k_ia_1 & k_ia_2 & \cdots & k_ia_n \\ 1 & 0 & & \boldsymbol{0} \\ & \ddots & \ddots & \\ \boldsymbol{0} & & 1 & 0 \end{bmatrix}$, $\boldsymbol{B}_i = \begin{bmatrix} k_ib_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

For current state $\boldsymbol{X}(k)$ and input $u(k)$, $\boldsymbol{X}(k+1)$ of the TSS system is,

$$\boldsymbol{X}(k+1) = \sum_{i=1}^{N} \lambda_i(k)\left[\boldsymbol{A}_i\boldsymbol{X}(k) + \boldsymbol{B}_iu(k)\right] \tag{11}$$

where $\lambda_i(k) = \dfrac{w_i(k)}{\sum_{j=1}^{N} w_j(k)}$, $w_i(k) = \left(\prod_{l=1}^{n} M_l^i\right) \cdot N_1^i$.

3. STABILITY ANALYSIS OF TSS SYSTEM

Before discussing the stability of TSS system (10), some important results in Joh, *et al.* (1998) will be introduced at first.

Proposition 2 (Joh, *et al.*, 1998): Consider system (3). Suppose that $\boldsymbol{A}_i$'s are Schur and satisfy,

$$\boldsymbol{A}_i\boldsymbol{A}_{i+1} = \boldsymbol{A}_{i+1}\boldsymbol{A}_i, \ \ i = 1,2,\cdots,N \tag{12}$$

Consider the following *N Lyapunov* equations,

$$\begin{aligned} &\boldsymbol{A}_1^T \boldsymbol{P}_1 \boldsymbol{A}_1 - \boldsymbol{P}_1 = -\boldsymbol{Q} \\ &\boldsymbol{A}_2^T \boldsymbol{P}_2 \boldsymbol{A}_2 - \boldsymbol{P}_2 = -\boldsymbol{P}_1 \\ &\vdots \\ &\boldsymbol{A}_N^T \boldsymbol{P}_N \boldsymbol{A}_N - \boldsymbol{P}_N = -\boldsymbol{P}_{N-1} \end{aligned} \quad (13)$$

where $\boldsymbol{Q} > 0$ and $\boldsymbol{P}_i\,(i = 1, 2, \cdots, N)$ is the unique positive definite symmetric solution of each equation. Then,

$$\boldsymbol{A}_i^T \boldsymbol{P}_N \boldsymbol{A}_i - \boldsymbol{P}_N < 0, \quad i = 1, 2, \cdots, N \quad (14)$$

Proposition 3 (Joh, *et al.*, 1998): A T-S fuzzy model with uncertain items,

$$\boldsymbol{X}_i(k+1) = \left[\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(k)\right]\boldsymbol{X}_i(k), \quad i = 1, 2, \cdots, N \quad (15)$$

is quadratically stable if

$$\max_{\Delta\boldsymbol{A}_i \in \Omega_i} \lambda_{\max}\left[(\boldsymbol{A}_i + \Delta\boldsymbol{A}_i)^T \boldsymbol{P}_N (\boldsymbol{A}_i + \Delta\boldsymbol{A}_i) - \boldsymbol{P}_N\right] < 0 \quad (16)$$

where $\boldsymbol{A}_i$'s are Hurwitz and pairwise commutative and $\boldsymbol{P}_N$ is defined by (13), Ω_i is a known compact set. $\lambda_{\max}[\cdot]$ denotes the maximum eigenvalue of the designated symmetric matrix, and it always appears at the protruded point in Ω_i.

Conclusion 1 (Joh, *et al.*, 1998): Consider a T-S fuzzy system (15), and $\boldsymbol{A}_i$'s are Schur and pairwise commutative, $\Delta\boldsymbol{A}_i(k) \in \Omega_{\delta_i}$, $\Omega_{\delta_i} = \left\{\delta_i \boldsymbol{E}_{fixed_i} \middle| \boldsymbol{E}_{fixed_i} \in \Omega_{fixed_i}\right\}$, Ω_{fixed_i} are compact and convex hyperpolyhedron with fixed shape and alterable size, $\boldsymbol{E}_{fixed_i}$ is the protruded point of Ω_{fixed_i}. Define the maximum possible bounds of $\Delta\boldsymbol{A}_i(k)$ as Λ_i, which assure quadratically stability of (15),

$$\Lambda_i = \left\{(\delta_i)_{\max} \boldsymbol{E}_{fixed_i} \middle| \boldsymbol{E}_{fixed_i} \in \Omega_{fixed_i}\right\} \quad (17)$$

$$\frac{1}{(\delta_i)_{\max}} = \max_{\boldsymbol{E}_{fixed_i} \in \Omega_{fixed_i}} \frac{1}{\delta_i^*\left(\boldsymbol{E}_{fixed_i}\right)} \quad (18a)$$

$$\frac{1}{\delta_i^*\left(\boldsymbol{E}_{fixed_i}\right)} = \lambda_{\max} \triangleq \max_j \left\{\beta_j^i\right\} \quad (18b)$$

where $i = 1, 2, \cdots, N$, β_j^i can be solved by the following general eigenvalue problem, $j = 1, \cdots, l$, l is the number of eigenvalues.

$$\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{E}_{fixed_i} \\ -\boldsymbol{E}_{fixed_i}^T & \boldsymbol{0} \end{bmatrix} \xi_j^i = \beta_j^i \begin{bmatrix} \boldsymbol{P}_N^{-1} & \boldsymbol{A}_i \\ \boldsymbol{A}_i^T & \boldsymbol{P}_N \end{bmatrix} \xi_j^i \quad (19)$$

Now consider free sub-systems of the TSS model (11). The *i*th subsystem

$$\boldsymbol{X}(k+1) = \boldsymbol{A}_i \boldsymbol{X}(k) = \begin{bmatrix} k_i a_1 & k_i a_2 & \cdots & k_i a_n \\ 1 & 0 & \cdots & \boldsymbol{0} \\ \vdots & \ddots & & \\ \boldsymbol{0} & & 1 & 0 \end{bmatrix} \boldsymbol{X}(k) \quad (20)$$

can be rewritten as,

$$\boldsymbol{X}(k+1) = \left(\tilde{\boldsymbol{A}}_i + \Delta\boldsymbol{A}_i\right)\boldsymbol{X}(k) \quad (21)$$

where

$$\tilde{\boldsymbol{A}}_i = \begin{bmatrix} k_i a_1 & k_i a_2 & \cdots & k_i a_n \\ k_i & 0 & \cdots & \boldsymbol{0} \\ \vdots & \ddots & & \\ \boldsymbol{0} & & k_i & 0 \end{bmatrix}, \Delta\boldsymbol{A}_i = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1-k_i & 0 & \cdots & \boldsymbol{0} \\ \vdots & \ddots & & \\ \boldsymbol{0} & & 1-k_i & 0 \end{bmatrix}.$$

Different from the dynamical uncertainty item in Conclusion 1, $\Delta\boldsymbol{A}_i$ can be regarded as constant "uncertainty". Therefore Theorem 1 follows,

Theorem 1: For a TSS free system with sub-system (21), find a common positive definite matrix $\boldsymbol{P}_N$ for $\tilde{\boldsymbol{A}}_i$ $(i = 1, 2, \cdots, N)$ according to (13), and solve general eigenvalue problem (19) by Conclusion 1, if k_i's make $\Delta\boldsymbol{A}_i \in \Lambda_i$ (see (17) for Λ_i), then TSS fuzzy system is quadratically stable.

Example 1: Consider a TSS system with three rules.

R^i: *if* $y(k-1)$ *is* M_i

then $y^i(k+1) = k_i\left[\sum_{j=1}^{2} a_j y(k-l+1) + b_1 u(k)\right]$

$i = 1, 2, 3$

where $a_1 = 0.25$, $a_2 = 0.1$, $b_1 = 0.2$, $k_1 = 1$, $k_2 = 1.25$, $k_3 = 1.5$. Therefore the consequent of TSS system can be rewritten as (10), where

$\boldsymbol{A}_1 = \begin{bmatrix} 0.2500 & 0.1000 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{A}_2 = \begin{bmatrix} 0.3125 & 0.1250 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{A}_3 = \begin{bmatrix} 0.3750 & 0.1500 \\ 1 & 0 \end{bmatrix}$. The membership functions are shown in Fig.1.

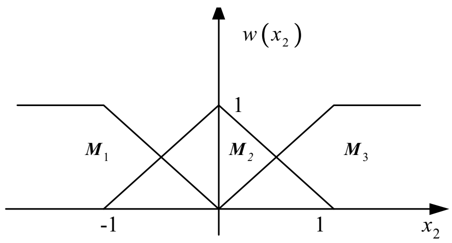

Fig.1 Membership functions in Example1

Consider the given TSS free system. Choose the uncertainty compact set $\Omega_{fixed_i} = \begin{bmatrix} \{\vartheta_1 \| \vartheta_1| \le 0.5\} & \{\vartheta_2 \| \vartheta_2| \le 2\} \\ \{\vartheta_2 \| \vartheta_2| \le 2\} & \{\vartheta_1 \| \vartheta_1| \le 0.5\} \end{bmatrix}$. Therefore its protruded points are $\boldsymbol{E}_{fixedi}^1 = \begin{bmatrix} -0.5 & -2 \\ -2 & -0.5 \end{bmatrix}$, $\boldsymbol{E}_{fixedi}^2 = \begin{bmatrix} -0.5 & 2 \\ 2 & -0.5 \end{bmatrix}$, $\boldsymbol{E}_{fixedi}^3 = \begin{bmatrix} 0.5 & -2 \\ -2 & 0.5 \end{bmatrix}$, $\boldsymbol{E}_{fixedi}^4 = \begin{bmatrix} 0.5 & 2 \\ 2 & 0.5 \end{bmatrix}$. Using (21), rewrite $\boldsymbol{A}_i$ as

$\boldsymbol{A}_i = \tilde{\boldsymbol{A}}_i + \Delta\boldsymbol{A}_i$. $\Delta\boldsymbol{A}_1 = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \end{bmatrix}$, $\Delta\boldsymbol{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\Delta\boldsymbol{A}_3 = \begin{bmatrix} 0 & 0 \\ -0.2 & 0 \end{bmatrix}$. Calculate $\boldsymbol{P}_3 = \begin{bmatrix} 6.6157 & 0.7010 \\ 0.7010 & 1.2183 \end{bmatrix}$ by (13). So $\tilde{\boldsymbol{A}}_i \boldsymbol{P}_3 \tilde{\boldsymbol{A}}_i - \boldsymbol{P}_3 < 0$ $(i = 1,2,3)$. Solve the coefficients $(\delta_i)_{\max}$'s of maximum uncertainty bounds using Conclusion 1. The results obtained from (19) are listed in Table1, where $\lambda_{\max}$ is the maximum eigenvalue.

Table1 $\lambda_{\max}$ of (19) for Each Rule — $\boldsymbol{A}_i$

$\boldsymbol{A}_1$	$\lambda_{\max}$	$\boldsymbol{A}_2$	$\lambda_{\max}$	$\boldsymbol{A}_3$	$\lambda_{\max}$
$\boldsymbol{E}^1_{fixed_1}$	4.225	$\boldsymbol{E}^1_{fixed_2}$	4.188	$\boldsymbol{E}^1_{fixed_3}$	4.214
$\boldsymbol{E}^2_{fixed_1}$	5.942	$\boldsymbol{E}^2_{fixed_2}$	6.484	$\boldsymbol{E}^2_{fixed_3}$	7.359
$\boldsymbol{E}^3_{fixed_1}$	4.437	$\boldsymbol{E}^3_{fixed_2}$	4.457	$\boldsymbol{E}^3_{fixed_3}$	4.568
$\boldsymbol{E}^4_{fixed_1}$	6.592	$\boldsymbol{E}^4_{fixed_2}$	7.583	$\boldsymbol{E}^4_{fixed_3}$	9.308

From Table1, the maximum eigenvalues corresponding to $\tilde{\boldsymbol{A}}_1$, $\tilde{\boldsymbol{A}}_2$, and $\tilde{\boldsymbol{A}}_3$ are 6.592, 7.583, and 9.308, respectively. By (18a) and (18b), the maximum uncertainty bound for each rule is the reciprocal of maximum eigenvalue, i.e. 0.152, 0.132, and 0.107. Therefore uncertainty sets of plant rules are,

$$\Lambda_1 = \begin{bmatrix} \{\Delta a_1^1 \| \Delta a_1^1 | \le 0.0759\} & \{\Delta a_2^1 \| \Delta a_2^1 | \le 0.3034\} \\ \{\Delta a_3^1 \| \Delta a_3^1 | \le 0.3034\} & \{\Delta a_4^1 \| \Delta a_4^1 | \le 0.0759\} \end{bmatrix},$$

$$\Lambda_2 = \begin{bmatrix} \{\Delta a_1^2 \| \Delta a_1^2 | \le 0.0659\} & \{\Delta a_2^2 \| \Delta a_2^2 | \le 0.2636\} \\ \{\Delta a_3^2 \| \Delta a_3^2 | \le 0.2636\} & \{\Delta a_4^2 \| \Delta a_4^2 | \le 0.0659\} \end{bmatrix},$$

$$\Lambda_3 = \begin{bmatrix} \{\Delta a_1^3 \| \Delta a_1^3 | \le 0.0537\} & \{\Delta a_2^3 \| \Delta a_2^3 | \le 0.2148\} \\ \{\Delta a_3^3 \| \Delta a_3^3 | \le 0.2148\} & \{\Delta a_4^3 \| \Delta a_4^3 | \le 0.0537\} \end{bmatrix}.$$

Clearly, $\Delta\boldsymbol{A}_1 = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \end{bmatrix} \in \Lambda_1$, $\Delta\boldsymbol{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \Lambda_2$, $\Delta\boldsymbol{A}_3 = \begin{bmatrix} 0 & 0 \\ -0.2 & 0 \end{bmatrix} \in \Lambda_3$. Thus by Theorem 1, TSS free system with the given $\boldsymbol{A}_i$'s is quadratically stable. This can also be checked since $\boldsymbol{P}_3$ satisfies Proposition 1 for $\boldsymbol{A}_i$'s.

4. STABILITY ANALYSIS AND DESIGN OF TSS CONTROL SYSTEM

Zhao, *et al*. (1997) considered two kinds of state feedback controllers. One is non-fuzzy state feedback controller, another is fuzzy one, i.e., $u(k) = \boldsymbol{KX}(k)$ and $u(k) = \sum_{i=1}^{N} \lambda_i u_i(k) = \left[\sum_{i=1}^{N} \lambda_i \boldsymbol{K}_i\right] \boldsymbol{X}(k)$. Here, the stability of TSS control system (11) with the above two type controllers will be analyzed.

4.1 Non-Fuzzy State Feedback Controller

Suppose that the non-fuzzy state feedback controller is adopted, i.e., $u(k) = \boldsymbol{KX}(k) = [K_1 \quad K_2 \quad \cdots \quad K_n] \boldsymbol{X}(k)$, then (11) can be rewritten as,

$$\begin{aligned} \boldsymbol{X}(k+1) &= \sum_{i=1}^{N} \lambda_i(k) \left(\boldsymbol{A}_i \boldsymbol{X}(k) + \boldsymbol{B}_i \boldsymbol{KX}(k) \right) \\ &= \sum_{i=1}^{N} \lambda_i(k) \left(\boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{K} \right) \boldsymbol{X}(k) \triangleq \sum_{i=1}^{N} \lambda_i(k) \underline{\boldsymbol{A}}_i \boldsymbol{X}(k) \\ &\triangleq \sum_{i=1}^{N} \lambda_i(k) \left(\tilde{\underline{\boldsymbol{A}}}_i + \Delta\underline{\boldsymbol{A}}_i \right) \boldsymbol{X}(k) \end{aligned} \quad (22)$$

where $\tilde{\underline{\boldsymbol{A}}}_i$ and $\Delta\underline{\boldsymbol{A}}_i$ are defined as $\tilde{\boldsymbol{A}}_i$ and $\Delta\boldsymbol{A}_i$ in (21). $\Delta\underline{\boldsymbol{A}}_i$ can be regarded as "uncertainty" in free system (22). Similar to free system (21), several conclusions can be easily obtained as follows.

Theorem 2: For system (22), find a common positive definite matrix $\boldsymbol{P}_N$ for $\tilde{\underline{\boldsymbol{A}}}_i$ $(i = 1, 2, \cdots, N)$ by Proposition 2, the TSS control system (22) is quadratically stable if

$$\max_{\Delta\underline{A}_i \in \Omega_i} \lambda_{\max} \left[\left(\tilde{\underline{\boldsymbol{A}}}_i + \Delta\underline{\boldsymbol{A}}_i \right)^T \boldsymbol{P}_N \left(\tilde{\underline{\boldsymbol{A}}}_i + \Delta\underline{\boldsymbol{A}}_i \right) - \boldsymbol{P}_N \right] < 0 \quad (23)$$

Proof: From (22),

$$\begin{aligned} \underline{\boldsymbol{A}}_i &= \boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{K} \\ &= \begin{bmatrix} k_i a_1 + k_i b_1 K_1 & k_i a_2 + k_i b_1 K_2 & \cdots & k_i a_n + k_i b_1 K_n \\ 1 & 0 & \cdots & \boldsymbol{0} \\ \vdots & \ddots & & \\ \boldsymbol{0} & & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} k_i(a_1 + b_1 K_1) & k_i(a_2 + b_1 K_2) & \cdots & k_i(a_n + b_1 K_n) \\ k_i & 0 & \cdots & \boldsymbol{0} \\ \vdots & \ddots & & \\ \boldsymbol{0} & & k_i & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1-k_i & 0 & \cdots & \boldsymbol{0} \\ \vdots & \ddots & & \\ \boldsymbol{0} & & 1-k_i & 0 \end{bmatrix} \triangleq \tilde{\underline{\boldsymbol{A}}}_i + \Delta\underline{\boldsymbol{A}}_i \end{aligned}$$

It can be easily proved that $\tilde{\underline{\boldsymbol{A}}}_i \tilde{\underline{\boldsymbol{A}}}_{i+1} = \tilde{\underline{\boldsymbol{A}}}_{i+1} \tilde{\underline{\boldsymbol{A}}}_i$. Therefore one can use Proposition 2 to find a common positive definite matrix $\boldsymbol{P}_N$. If (23) is satisfied, it can be concluded from Proposition 3 that TSS control system (22) is quadratically stable. □

Corollary 1: For system (22), find a common positive definite matrix $\boldsymbol{P}_N$ for $\tilde{\underline{\boldsymbol{A}}}_i$ $(i = 1, 2, \cdots, N)$ by (13), and solve the general eigenvalue problem (19). If

$\Delta \underline{\underline{A}}_i \in \Lambda_i$ (see (17) for Λ_i), the TSS control system (22) is quadratically stable.

4.2 Fuzzy State Feedback Controller

Suppose that the controller is fuzzy state feedback controller, i.e.,
$u_i(k) = \boldsymbol{K}_i \boldsymbol{X}(k) = [K_{i1} \quad K_{i2} \quad \cdots \quad K_{in}] \boldsymbol{X}(k)$,
$u(k) = \sum_{i=1}^{N} \lambda_i u_i(k) = \left[\sum_{i=1}^{N} \lambda_i \boldsymbol{K}_i\right] \boldsymbol{X}(k)$. (11) can be rewritten as,

$$\boldsymbol{X}(k+1) = \sum_{i=1}^{N} \lambda_i(k)\left(\boldsymbol{A}_i \boldsymbol{X}(k) + \boldsymbol{B}_i\left[\sum_{j=1}^{N} \lambda_j \boldsymbol{K}_j\right]\right)\boldsymbol{X}(k)$$
$$= \sum_{i=1}^{N} \lambda_i(k)\left(\boldsymbol{A}_i + \boldsymbol{B}_i\left[\sum_{j=1}^{N} \lambda_j \boldsymbol{K}_j\right]\right)\boldsymbol{X}(k)$$
$$\triangleq \sum_{i=1}^{N} \lambda_i(k) \underline{\underline{A}}_i(k) \boldsymbol{X}(k) \triangleq \sum_{i=1}^{N} \lambda_i(k)\left(\tilde{\underline{\underline{A}}}_i + \Delta \underline{\underline{A}}_i(k)\right)\boldsymbol{X}(k) \quad (24)$$

Here, $\Delta \underline{\underline{A}}_i(k)$ can also be regarded as "uncertainty" in free system (24).

Theorem 3: For TSS system (24), find a common positive definite matrix $\boldsymbol{P}_N$ for $\tilde{\underline{\underline{A}}}_i$ ($i = 1, 2, \cdots, N$) by Proposition 2, and solve the general eigenvalue problem (19). If $\Delta \hat{\underline{\underline{A}}}_i \in \Lambda_i$ (see (17) for Λ_i), the TSS control system (24) is quadratically stable.

Proof: From (24),

$$\underline{\underline{A}}_i = \boldsymbol{A}_i + \boldsymbol{B}_i\left[\sum_{i=1}^{N} \lambda_i \boldsymbol{K}_i\right]^T = \begin{bmatrix} k_i a_1 & k_i a_2 & \cdots & k_i a_n \\ k_i & 0 & \cdots & \boldsymbol{0} \\ \vdots & \ddots & & \\ \boldsymbol{0} & & k_i & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} k_i b_1 \sum_{i=1}^{N} \lambda_i K_{i1} & k_i b_1 \sum_{i=1}^{N} \lambda_i K_{i2} & \cdots & k_i b_1 \sum_{i=1}^{N} \lambda_i K_{in} \\ 1 - k_i & 0 & \cdots & \boldsymbol{0} \\ \vdots & \ddots & & \\ \boldsymbol{0} & & 1 - k_i & 0 \end{bmatrix}$$
$$\triangleq \tilde{\underline{\underline{A}}}_i + \Delta \underline{\underline{A}}_i(k) \quad (25)$$

It can be easily proved that $\tilde{\underline{\underline{A}}}_i \tilde{\underline{\underline{A}}}_{i+1} = \tilde{\underline{\underline{A}}}_{i+1} \tilde{\underline{\underline{A}}}_i$. In fact, at each instant *k*, the value of $\sum_{i=1}^{N} \lambda_i K_{ij}$ ($j = 1, \cdots, n$) are exact, which forms $\Delta \underline{\underline{A}}_i(k)$. Define $\Delta \hat{\underline{\underline{A}}}_i$ as,

$$\Delta \hat{\underline{\underline{A}}}_i = \begin{bmatrix} \theta_1^i & \theta_2^i & \cdots & \theta_n^i \\ 1 - k_i & 0 & \cdots & \boldsymbol{0} \\ \vdots & \ddots & & \\ \boldsymbol{0} & & 1 - k_i & 0 \end{bmatrix} \quad (26)$$

where $\theta_j^i = \max_{\lambda_i}\left(k_i b_1 \sum_{i=1}^{N} \lambda_i K_{ij}\right)$, $j = 1, 2 \cdots, n$. If $\Delta \hat{\underline{\underline{A}}}_i \in \Lambda_i$, Theorem 3 immediately follows. □

Of course, the stability condition in Theorem 3 is conservative, since it can not be guaranteed that all of $\sum_{i=1}^{N} \lambda_i K_{ij}$ reach their maximum at each instant.

Remark 1: The stability results presented in this section is for TSS control systems. They can also be used to evaluate the stability of general T-S fuzzy control systems, provided that given systems satisfy the conditions required in Proposition 2.

4.3 Systematic Design for Fuzzy Control System with Stability Guarantee

Usually, the approach to stability analysis for T-S fuzzy control systems is to find a common positive definite matrix $\boldsymbol{P}$ for all subsystems. It is only a sufficient condition to assure stability of global system. When users design T-S fuzzy control system, they usually design controllers for each subsystem by PDC method at first, without taking stability of the global system into account. Then they try to find the satisfactory matrix $\boldsymbol{P}$. If they can, the system is stable; otherwise they should redesign sub-controllers. However, the results of last design have few instructions for redesign and users should repeat the "design-evaluate" procedure, which makes the design process blind and rather complicated.

In this section, an easier and more systematic method to design TSS fuzzy control system will be presented, which assures system stability.

Consider a TSS control system (24). Clearly $\tilde{\underline{\underline{A}}}_i$'s satisfy $\tilde{\underline{\underline{A}}}_i \tilde{\underline{\underline{A}}}_{i+1} = \tilde{\underline{\underline{A}}}_{i+1} \tilde{\underline{\underline{A}}}_i$. Hence a common positive definite matrix $\boldsymbol{P}_N$ for $\tilde{\underline{\underline{A}}}_i$ ($i = 1, 2, \cdots, N$) can be designed by Proposition 2, which satisfy

$$\tilde{\underline{\underline{A}}}_i^T \boldsymbol{P}_N \tilde{\underline{\underline{A}}}_i - \boldsymbol{P}_N < 0 \quad (i = 1, 2, \cdots, N)$$

Design state feedback controllers of subsystems using $\boldsymbol{P}_N$ as follows,

$$u_i = \boldsymbol{K}_i \boldsymbol{X} = -\boldsymbol{R} \boldsymbol{B}_i^T \boldsymbol{P}_N \boldsymbol{X} \quad (27)$$

where $\boldsymbol{R}$ is a weighting positive definite symmetric matrix given by users. Substitute $\boldsymbol{K}_i$ into (25). If $\Delta \hat{\underline{\underline{A}}}_i \in \Lambda_i$, where $\Delta \hat{\underline{\underline{A}}}_i$ is designed by (26), stop controller design. The obtained state feedback matrix may guarantee the stability of TSS control system. Otherwise just qualitatively decrease weight matrix $\boldsymbol{R}$ in certain range and compute new state feedback matrix by (27). Usually Theorem 3 can be satisfied. If not, search a new common matrix $\boldsymbol{P}_N$ using its systematic design method as (13).

Remark 2: To evaluate the stability of a fuzzy control system, the users usually design sub-controllers at first, then search common matrix $\boldsymbol{P}$ and check it using Proposition 1. Whereas the design method of TSS control system given here first searches common matrix $\boldsymbol{P}$ for all nominal sub-systems, then uses $\boldsymbol{P}$ to design sub-controllers. Generally speaking, only the matrix $\boldsymbol{R}$ should be adjusted and the whole search procedure for $\boldsymbol{P}$ would not be repeated.

Example 2: Consider the TSS system given in Example 1, where the input matrix $\boldsymbol{B}_i$'s are $\boldsymbol{B}_1 = [0.2 \quad 0]^T$, $\boldsymbol{B}_2 = 1.25\boldsymbol{B}_1 = [0.25 \quad 0]^T$, $\boldsymbol{B}_3 = 1.5\boldsymbol{B}_1 = [0.3 \quad 0]^T$. Here suppose that the premise variable is x_2 only. First rewrite the system as (24). Compute state feedback matrix $\boldsymbol{K}_i$'s by systematic method, where $\boldsymbol{P}_3 = \begin{bmatrix} 6.5157 & 0.7010 \\ 0.7010 & 1.2183 \end{bmatrix}$, $\boldsymbol{R} = 1$. The satisfactory results can be found until $\boldsymbol{R} = 0.09$ with $\boldsymbol{K}_1 = [-0.1173 \quad -0.0126]$, $\boldsymbol{K}_2 = [-0.1466 \quad -0.0158]$, $\boldsymbol{K}_3 = [-0.1759 \quad -0.0189]$. By (26), $\Delta\hat{\underline{\underline{A}}}_1 = \begin{bmatrix} -0.0352 & -0.0038 \\ 0.2 & 0 \end{bmatrix}$, $\Delta\hat{\underline{\underline{A}}}_2 = \begin{bmatrix} -0.0440 & -0.0048 \\ 0 & 0 \end{bmatrix}$, $\Delta\hat{\underline{\underline{A}}}_3 = \begin{bmatrix} -0.0528 & -0.0057 \\ -0.2 & 0 \end{bmatrix}$. Clearly, $\Delta\hat{\underline{\underline{A}}}_i \in \Lambda_i$, where Λ_i's are the same ones in Example 1. Therefore the designed state feedback matrix $\boldsymbol{K}_i$'s can guarantee global system stability.

Remark 3: In fact, the stability results presented in this section can also be used to evaluate the stability of T-S fuzzy control system, provided that state matrix $\boldsymbol{A}_i$'s satisfy $\boldsymbol{A}_i\boldsymbol{A}_{i+1} = \boldsymbol{A}_{i+1}\boldsymbol{A}_i$.

5. CONCLUSION

Stability and design problem of a TSS system were investigated. Considering the special characteristics of TSS, this paper presented a systematic approach to find the common matrix $\boldsymbol{P}$. System matrix $\boldsymbol{A}_i$ was decomposed into proportional part $\tilde{A}_i$ and the remainder ΔA_i first. And a common matrix $\boldsymbol{P}$ for $\tilde{A}_i$'s could be found by an iterative approach. Then the global system stability was guaranteed with $\boldsymbol{P}$ if ΔA_i was in a certain range. Stability results for both free and control systems were obtained, based on which systematic controller design methods for TSS were also investigated.

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