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ZERO DYNAMICS OF UNDERACTUATED PLANAR BIPED WALKERS

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Abstract: The zero dynamics of a hybrid model of bipedal walking are introduced and studied for a class of N-link, planar robots with one degree of underactuation and outputs that depend only on the configuration variables. Asymptotically stable solutions of the zero dynamics correspond to asymptotically stabilizable orbits of the full hybrid model of the walker. The Poincaré map of the zero dynamics is computed and proven to be diffeomorphic to a scalar, linear, time-invariant system, thereby rendering transparent the existence and stability properties of periodic orbits.

Keywords: limit cycles, geometric approaches, hybrid modes, nonlinear control, robot control

1. INTRODUCTION

A planar biped walker is a robot that locomotes via alternation of two legs in the sagittal plane (see Figure 1). For this paper, it is assumed that there is no open interval of time during which two legs are in simultaneous contact with the ground. The models for such robots are necessarily hybrid, consisting of ordinary differential equations to describe the motion of the robot when only one leg is in contact with the ground (single support or swing phase of the walking motion), and a discrete map to model the impact when the second leg touches the ground (double support phase). The degree of complexity of controlling such a system is a function, among other things, of the number of degrees of freedom of the model as well as the degree of actuation or, more precisely, *underactuation* of the system.

For planar, biped walkers with a torso and *one degree of underactuation*, it was shown for the first time in (Grizzle *et al.*, 2001) for a 3-link model, and in (Plestan *et al.*, 2001) for a 5-link model, that these systems admit feedback control designs that induce walking motions with provable stability properties. The control designs involved the judicious choice of a set of holonomic constraints that were imposed on the robot via feedback control. This was accomplished by interpreting the constraints as output functions depending only on the configuration variables of the robot, and then combining ideas from finite-time stabilization and computed torque. The desired posture of the robot was encoded into the set of outputs in a such a way that the nulling of the outputs was equivalent to achieving the desired posture.

In general, the *maximal internal dynamics of a system that are compatible with the output being identically zero* is called the *zero dynamics* (Isidori, 1995). The zero dynamics of the swing phase were briefly analyzed in (Grizzle *et*

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al., 2001). However, since they are not in general invariant under the impact map, their stability properties could not be related directly to the stability of the orbits of the closed-loop, hybrid system. Here, the required invariance conditions will be analyzed in order to formulate a proper definition of the *hybrid zero dynamics,* that is, the zero dynamics of the full hybrid model of the biped. This will be carried out for a class of N-link, planar biped walkers with one degree of underactuation, and for outputs that depend only on the configuration variables. Under quite general conditions, this yields a hybrid system on the plane that can be analyzed in great detail. In particular, the associated Poincaré return map can be explicitly computed and shown to be diffeomorphic to a scalar, linear time-invariant system, thereby rendering transparent the existence and stability properties of periodic orbits of the hybrid zero dynamics.

When the hybrid zero dynamics admit an asymptotically stable orbit, the general feedback approach developed in (Grizzle *et al.*, 2001; Plestan *et al.*, 2001) can be immediately applied to create a provably, asymptotically stable orbit in the full hybrid model. In (Westervelt and Grizzle, 2002) a convenient parameterization of the hybrid zero dynamics is introduced. Parameter optimization can then be used to tune the hybrid zero dynamics in order to achieve closed-loop, asymptotically stable walking with low energy consumption, while meeting natural kinematic and dynamic constraints. This is similar to (Chevallereau and Aoustin, 2001), but with the additional property that the optimization is essentially being performed in closed loop, so the existence of a controller that asymptotically stabilizes the closed-loop system is immediate. The hybrid zero dynamics also make it possible to transition between feedback controllers that achieve walking at various fixed rates, and still analytically verify that stability is never lost.

2. ROBOT MODEL AND MODELING ASSUMPTIONS

The robot is assumed to be planar, consisting of at least a torso and two identical legs, and the legs are connected at a common point called the hips; furthermore, all links have mass, are rigid, are connected in revolute joints, and all kinematic chains formed by the connections of links are assumed to be open. Figure 1 depicts an example of such a robot. All walking cycles will be assumed to take place in the sagittal plane and consist of successive phases of *single support* (meaning the stance leg is touching the walking surface and the swing leg is not) and *double support* (the swing leg and the stance leg are both in contact with the walking surface). During the single support phase, it is assumed that the stance leg acts as a pivot. It is further supposed that the walking gaits of interest are such that, in steady state, successive phases of single support are symmetric with respect to the two legs, involve motion from left to right, and the swing leg is posed in front of the stance leg.

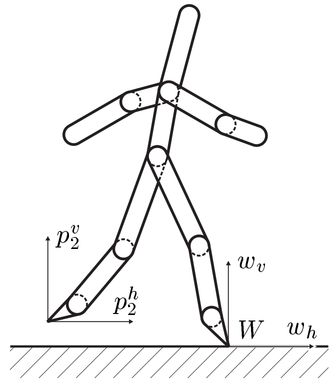

Fig. 1. Schematic illustrating the class of N-link robot models considered here. Note that there is no actuation between the stance leg and the ground, while all other joints are actuated.

The rigid contact model presented in (Hurmuzlu and Marghitu, 1994) is assumed, which collapses the double support phase to an instant in time, and allows a discontinuity in the velocity component of the state, with the position remaining continuous. The biped model is thus *hybrid* in nature, consisting of a continuous dynamics during the swing phase and a re-initialization rule at the contact event. An important source of complexity in a biped system is the degree *underactuation* of the system. It will be assumed that there is no actuation at the end of the stance leg. Thus the system is underactuated during walking, as opposed to fully actuated (a control at each joint and the contact point with the ground).

Swing phase model: Let $N \geq 3$ be the number of links in the robot. The dynamic model of the robot during the swing phase has N-DOF. Let $q = (q_1, \cdots, q_N)'$ be a set of angular coordinates describing the configuration of the robot with respect to a world reference frame W. Since only symmetric gaits are of interest, the same model can be used irrespective of which leg is the stance leg if the coordinates are relabeled after each phase of double support. Using the method of Lagrange, the model is written in the form

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = Bu. \qquad (1)$$

Torques $u_i,\ i = 1$ to $N-1$, are applied between each connection of two links, *but not between the stance leg and ground.* The model is written in state space form by

$$\dot{x} = \begin{bmatrix} \dot{q} \\ D^{-1}(q)\left[-C(q,\dot{q})\dot{q} - G(q) + Bu\right] \end{bmatrix}$$
$$=: f(x) + g(x)u. \quad (2)$$

where $x := (q', \dot{q}')'$. The state space of the model is taken as $TQ := \{x := (q', \dot{q}')' \mid q \in \mathcal{Q},\ \dot{q} \in I\!R^N\}$, where $\mathcal{Q}$ is a simply-connected, open subset of $[0, 2\pi)^N$ corresponding to physically reasonable configurations of the robot (for example, with the exception of the end of the stance leg, all points of the robot being above the walking surface; one could also impose that the knees are not bent backward, etc.).

Impact model: An impact occurs when the swing leg touches the walking surface,

$$S := \{(q, \dot{q}) \in TQ \mid p_2^v = 0, \quad p_2^h > 0\}, \quad (3)$$

also called the ground, where p_2^h and p_2^v are the horizontal and vertical positions of the swing leg end, respectively. The impact between the swing leg and the ground is modeled as a contact between two rigid bodies. In addition to modeling the change in state of the robot, the impact model accounts for the relabeling of the robot's coordinates that occurs after each phase of double support. Let R be a constant matrix such that $R\,q$ accounts for relabeling of the coordinates when the swing leg becomes the new stance leg. Then the impact model of (Hurmuzlu and Marghitu, 1994) under standard hypotheses (see (Grizzle *et al.*, 2001), for example), results in a smooth map $\Delta : S \to TQ$,

$$x^+ = \Delta(x^-), \quad (4)$$

where $x^+ := (q^+, \dot{q}^+)$ (resp. $x^- := (q^-, \dot{q}^-)$) is the state value just after (resp. just before) impact. For later convenience, Δ is expressed as

$$\Delta(x^-) := \begin{bmatrix} \Delta_q\, q^- \\ \Delta_{\dot{q}}(q^-)\,\dot{q}^- \end{bmatrix} \quad (5)$$

where $\Delta_q := R$ and $\Delta_{\dot{q}}(q)$ is an $N \times N$ matrix of smooth functions of q.

Nonlinear system with impulse effects: The overall biped robot model can be expressed as a nonlinear system with impulse effects (Ye *et al.*, 1998)

$$\begin{array}{llll} \dot{x} & = f(x) + g(x)u & x^- \notin S \\ x^+ & = \Delta(x^-) & x^- \in S, \end{array} \quad (6)$$

where, $x^-(t) := \lim_{\tau \nearrow t} x(\tau)$. Solutions are taken to be right continuous and must have finite left and right limits at each impact event; see (Grizzle *et al.*, 2001) for details.

Informally, a *half-step* of the robot is a solution of (6) that starts with the robot in double support, ends in double support with the positions of the legs swapped, and contains no other impact event. This is more precisely defined as follows. Let $\varphi(t, x_0)$ be a maximal solution of the swing phase dynamics (2) with initial condition x_0 at time $t_0 = 0$, and define the *time to impact* function, $T_I : TQ \to I\!R \cup \{\infty\}$ as the first time that a solution of the swing phase dynamics intersects S; see (Grizzle *et al.*, 2001). Let $x_0 \in S$ be such that $T_I \circ \Delta(x_0) < \infty$. A *half-step* of the robot is the solution of (6) defined on the half-open interval $[0, T_I \circ \Delta(x_0))$ with initial point x_0.

3. SWING PHASE ZERO DYNAMICS

Note that if an output $y = h(q)$ depends only on the configuration variables, then, due to the second order nature of the robot model, the derivative of the output along solutions of (2) does not depend directly on the inputs. Hence its relative degree is at least two. Differentiating the output once again computes the accelerations, resulting in

$$\frac{d^2 y}{dt^2} = L_f^2 h(q, \dot{q}) + L_g L_f h(q) u, \quad (7)$$

where the matrix $L_g L_f h(q)$ is called the decoupling matrix and depends only on the configuration variables. A consequence of the general results in (Isidori, 1995) is that the invertibility of this matrix at a given point assures the existence and uniqueness of the zero dynamics in the neighborhood of that point. With a few extra hypotheses given in the following lemma, these properties can be assured on a given open set.

Lemma 1. **(Swing phase zero dynamics)** Suppose that a smooth function h is selected so that

HH1) h is a function of only the position coordinates;

HH2) there exists an open set $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ such that for each point $q \in \tilde{\mathcal{Q}}$, the decoupling matrix $L_g L_f h(q)$ is square and invertible (i.e., h has vector relative degree $(2, \ldots, 2)'$);

HH3) there exists a smooth real valued function $\theta(q)$ such that $\Phi : \tilde{\mathcal{Q}} \to I\!R^N$ by $\Phi(q) := (h(q)', \theta(q))'$ is a diffeomorphism onto its image;

HH4) there exists at least one point in $\tilde{\mathcal{Q}}$ where h vanishes.

Then,

(1) $Z := \{x \in T\tilde{\mathcal{Q}} \mid h(x) = 0, L_f h(x) = 0\}$ is a smooth two dimensional embedded submanifold of TQ; and

(2) the feedback control

$$u^*(x) = -(L_g L_f h(x))^{-1} L_f^2 h(x)$$

renders Z invariant under the swing phase dynamics; that is, for every $z \in Z$, $f_{\rm zero}(z) := f(z) + g(z)u^*(z) \in T_z Z$.

Z is called the *zero dynamics manifold* and

$$\dot{z} = f_{\rm zero}(z) \tag{8}$$

is called the (swing phase) *zero dynamics*.

The zero dynamics are now developed in a convenient set of local coordinates. Since the columns of g in (2) are involutive, by (Isidori, 1995), page 222, in a neighborhood of any point where the decoupling matrix is invertible, there exists a smooth scalar function γ such that

$$\begin{aligned} \eta_1 &= h(q),\ \eta_2 = L_f h(q,\dot{q}) \\ \xi_1 &= \theta(q),\ \xi_2 = \gamma(q,\dot{q}) \end{aligned} \tag{9}$$

is a valid coordinate transformation and $L_g\gamma = 0$. Moreover, by applying the constructive proof of the Frobenius theorem of (Isidori, 1995), page 23, (see also, (Grizzle *et al.*, 2001)) one obtains that $\gamma(q,\dot{q})$ has the form $\gamma_0(q)\,\dot{q}$ and (9) can be shown to be a valid coordinate change on all of $T\tilde{\mathcal{Q}}$.

In the coordinates (9), the zero dynamics become

$$\begin{aligned} \dot{\xi}_1 &= L_f\theta \\ \dot{\xi}_2 &= L_f\gamma \end{aligned} \tag{10}$$

where the right hand side is evaluated at

$$q = \Phi^{-1}(0,\xi_1) \tag{11}$$

$$\dot{q} = \begin{bmatrix} \dfrac{\partial h}{\partial q} \\ \gamma_0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \xi_2 \end{bmatrix}. \tag{12}$$

Theorem 2. Under the hypotheses of Lemma 1, $(\xi_1,\xi_2) = (\theta(q),\gamma_0(q)\,\dot{q})$ is a valid set of coordinates on Z, and in these coordinates the zero dynamics take the form

$$\dot{\xi}_1 = \kappa_1(\xi_1)\xi_2 \tag{13}$$

$$\dot{\xi}_2 = \kappa_2(\xi_1). \tag{14}$$

Moreover, if the model (2) is expressed in $N-1$ *relative* angular coordinates, $(q_1,\cdots,q_{N-1})$, plus one *absolute* angular coordinate, q_N, the following interpretations can be given for the various functions appearing in the zero dynamics:

$$\xi_1 = \theta|_Z \tag{15}$$

$$\xi_2 = \left.\frac{\partial K}{\partial \dot{q}_N}\right|_Z \tag{16}$$

$$\kappa_1(\xi_1) = \left.\frac{\partial\theta}{\partial q}\begin{bmatrix} \dfrac{\partial h}{\partial q} \\ \gamma_0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right|_Z \tag{17}$$

$$\kappa_2(\xi_1) = -\left.\frac{\partial V}{\partial q_N}\right|_Z, \tag{18}$$

where $K(q,\dot{q}) = \frac{1}{2}\,\dot{q}'D(q)\dot{q}$ is the kinetic energy of the robot, $V(q)$ is its potential energy, and γ_0 is the last row of D, the mass-inertia matrix.

PROOF. The form of (13) is immediate from (10) and (12) since both h and γ_0 are functions of only q, and hence when restricted to Z, are functions of ξ_1 only. Suppose now that the model (2) is expressed in $N-1$ relative angular coordinates and one absolute coordinate. Since the kinetic energy of the robot, $K(q,\dot{q})$, is independent of the choice of world coordinate frame (Spong and Vidyasagar, 1989, page 140), and since q_N fixes this choice, $K(q,\dot{q})$ is independent of q_N. Since $D := \partial\left[(\partial K/\partial\dot{q})'\right]/\partial\dot{q}$, $\partial D/\partial q_N = 0$. Let D_N, C_N, and G_N be the last rows of D, C, and G, respectively. Then $\xi_2 = \gamma_0(q)\,\dot{q}$ is equal to $D_N(q)\,\dot{q}$ (Grizzle *et al.*, 2001), and thus is equal to $\partial K/\partial\dot{q}_N$ since $K = \frac{1}{2}\,\dot{q}'D\dot{q}$. Continuing, $\dot{\xi}_2 := L_f\gamma$ becomes

$$\begin{aligned} L_f\gamma &= \begin{bmatrix} \dot{q}'\,\dfrac{\partial D_N'}{\partial q} & D_N \end{bmatrix} \begin{bmatrix} \dot{q} \\ -D^{-1}\left[C\dot{q} + G\right] \end{bmatrix} \\ &= \dot{q}'\,\frac{\partial D_N'}{\partial q}\,\dot{q} - C_N\dot{q} - G_N. \end{aligned} \tag{19}$$

Noting that (see (Spong and Vidyasagar, 1989, page 142))

$$C_N = \dot{q}'\,\frac{\partial D_N'}{\partial q} - \frac{1}{2}\,\dot{q}'\,\frac{\partial D}{\partial q_N},$$

(19) becomes $L_f\gamma = -G_N = -\partial V/\partial q_N$, which when evaluated on Z is a function of ξ_1 only and, hence, (14) follows.

4. HYBRID ZERO DYNAMICS

This section incorporates the impact model into the notion of the *maximal internal dynamics compatible with the output being identically zero*, in order to obtain a zero dynamics of the complete model of the biped walker, (6). Towards this goal, let $y = h(q)$ be an output satisfying the hypotheses of Lemma 1 and suppose there exists a trajectory, $x(t)$, of the hybrid model (6) along which the output is identically zero. If the trajectory contains no impacts with S, then $x(t)$ is a solution of the swing phase dynamics and also of its zero dynamics. If the trajectory does contain impact events, then let (t_0,t_f) be an open interval of time containing exactly one impact at t_e. By definition, on the intervals (t_0,t_e) and (t_e,t_f), $x(t)$

is a solution of the swing phase dynamics and hence also of its zero dynamics, so $x(t) \in Z$; since also by definition of a solution, $x^-(t_e) := \lim_{t \nearrow t_e} x(t)$ exists, is finite, and lies in S, it follows that $x^-(t_e) \in S \cap Z$. Moreover, by definition of a solution of (6), $x(t_e) := x^+(t_e) := \Delta(x^-(t_e))$, from which it follows that $\Delta(x^-(t_e)) \in Z$. On the other hand, if $\Delta(S \cap Z) \subset Z$, then from solutions of the swing phase zero dynamics it is clearly possible to construct solutions to the complete model of the biped walker along which the output $y = h(q)$ is identically zero. This leads to the following definition.

Definition 3. Let $y = h(q)$ be an output satisfying the hypotheses of Lemma 1, and let Z and $\dot{z} = f_{\text{zero}}(z)$ be the associated zero dynamics manifold and zero dynamics of the swing phase model. Suppose that $S \cap Z$ is a smooth, one-dimensional, embedded submanifold of TQ. If $\Delta(S \cap Z) \subset Z$, then the nonlinear system with impulse effects,

$$\begin{aligned} \dot{z} &= f_{\text{zero}}(z) & z^- &\notin S \cap Z \\ z^+ &= \Delta(z^-) & z^- &\in S \cap Z, \end{aligned} \tag{20}$$

with $z \in Z$, is the *hybrid zero dynamics* of the model (6).

Remark 4. From standard results in (Boothby, 1975), $S \cap Z$ will be a smooth one-dimensional embedded submanifold if $S \cap Z \neq \emptyset$ and the map $[h' \ (L_f h)' \ p_2^v]'$ has constant rank equal to $2N-1$ on $S \cap Z$. A simple argument shows that this rank condition is equivalent to rank of $[h' \ p_2^v]' = N$, and under this rank condition, $S \cap Z \cap \tilde{Q}$ consists of the isolated zeros of $[h' \ p_2^v]'$. Let q^- be a solution of $(h(q), p_2^v(q)) = (0,0)$, $p_2^h(q) > 0$. Then the connected component of $S \cap Z$ containing $(q^-, 0)$ is diffeomorphic to $I\!R$ per $\sigma : I\!R \to S \cap Z$, where

$$\sigma(\omega) := \begin{bmatrix} \sigma_q \\ \sigma_{\dot{q}}\, \omega \end{bmatrix} \tag{21}$$

$\sigma_q := q^-$, and

$$\sigma_{\dot{q}} = \begin{bmatrix} \dfrac{\partial h}{\partial q}(q^-) \\ \\ \gamma_0(q^-) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{22}$$

In view of this, the following additional assumption is made about the output h and the open set $\tilde{Q}$

HH5) there exists a unique point $q^- \in \tilde{Q}$ such that $(h(q^-), p_2^v(q^-)) = (0,0)$, $p_2^h(q^-) > 0$ and the rank of $[h', \ p_2^v]'$ at q^- equals N.

The next result characterizes when the swing phase zero dynamics are compatible with the impact model, leading to a non-trivial hybrid zero dynamics.

Theorem 5. **(Hybrid zero dynamics existence)** Consider the robot model (6) along with a smooth output function h satisfying HH1–HH5. Then, the following statements are equivalent:

(a) $\Delta(S \cap Z) \subset Z$;
(b) $h \circ \Delta|_{(S \cap Z)} = 0$ and $L_f h \circ \Delta|_{(S \cap Z)} = 0$;
(c) there exists at least one point $(q^-, \dot{q}^-) \in S \cap Z$ such that $\gamma_0(q^-)\,\dot{q}^- \neq 0$, $h \circ \Delta_q(q^-) = 0$, and $L_f h \circ \Delta(q^-, \dot{q}^-) = 0$.

PROOF. The equivalence of (a) and (b) is immediate from the definition of Z as the zero set of h and $L_f h$. The equivalence of (b) and (c) follows from Remark 4 once it is noted from (5) that $L_f h \circ \Delta$ is linear in $\dot{q}$.

Under the hypotheses of Theorem 5, the hybrid zero dynamics are well-defined. Let $z^- \in S \cap Z$, $z^+ = \Delta(z^-)$ and suppose that $T_I(z^+) < \infty$. Let $\varphi : [0, t_f] \to Z$, $t_f = T_I(z^+)$, be a solution of the zero dynamics, (8), such that $\varphi(0) = z^+$. Define $\hat{\theta}(t) := \theta \circ \varphi(t)$ and $\dot{\hat{\theta}} := d\hat{\theta}(t)/dt$.

Proposition 6. Assume the hypotheses of Theorem 5. Then over any half-step of the hybrid zero dynamics, $\dot{\hat{\theta}} : [0, t_f) \to I\!R$ is never zero. In particular, $\hat{\theta} : [0, t_f) \to I\!R$ is strictly monotonic and thus achieves its maximum and minimum values at the end points.

The proof is omitted for reasons of space. By Remark 4, it follows that $\hat{\theta}(0) = \theta \circ \Delta_q(q^-)$ and $\hat{\theta}(t_f) = \theta(q^-)$, that is, the extrema can be computed *a priori*. Denote these by

$$\theta^- := \theta(q^-) \tag{23}$$
$$\theta^+ := \theta \circ \Delta_q(q^-). \tag{24}$$

Without loss of generality, it is assumed that $\theta^+ < \theta^-$; that is, that along any half-step of the hybrid zero dynamics, θ is *monotonically increasing.*

5. POINCARÉ ANALYSIS OF THE ZERO DYNAMICS

Assume the hypotheses of Theorem 5. Take the Poincaré section to be $S \cap Z$ so that the Poincaré return map is the partial map $\rho : S \cap Z \to S \cap Z$ defined as follows (Grizzle *et al.*, 2001): let $\varphi(t, z_0)$ be a maximal solution of the swing phase zero dynamics, $\dot{z} = f_{\text{zero}}(z)$. Since both $f_{\text{zero}}(z)$ and Z are smooth, a solution of (13)–(14) from a given initial condition, z_0, is unique and depends smoothly on z_0. Then by (Grizzle *et al.*, 2001, Lemma 2), $\tilde{Z} := \{z \in Z \mid 0 < T_I(z) < \infty \text{ and } L_f p_2^v(\varphi(T_I(z), z)) \neq 0\}$ is open.

For $z \in S \cap \tilde{Z}$, define the Poincaré return map for the hybrid zero dynamics as

$$\rho(z) := \varphi(T_I \circ \Delta(z), \Delta(z)). \tag{25}$$

In a special set of local coordinates, the return map can be explicitly computed. Indeed, express the hybrid zero dynamics in the coordinates of Theorem 2, namely, $(\xi_1, \xi_2) = (\theta, \gamma)$. In these coordinates, $S \cap Z$ and $\Delta : (\xi_1^-, \xi_2^-) \to (\xi_1^+, \xi_2^+)$ simplify to

$$S \cap Z = \left\{ (\xi_1^-, \xi_2^-) \mid \xi_1^- = \theta^-, \xi_2^- \in I\!R \right\} \tag{26}$$

$$\xi_1^+ = \theta^+ \tag{27}$$

$$\xi_2^+ = \delta_{\text{zero}}\, \xi_2^-, \tag{28}$$

where $\delta_{\text{zero}} := \gamma_0(q^+)\, \Delta_{\dot{q}}(q^-)\, \sigma_{\dot{q}}(q^-)$, a constant. The hybrid zero dynamics are thus given by (13)–(14) during the swing phase, and at impact with $S \cap Z$ the re-initialization rules (27) and (28) are applied. By Proposition 6, over any half-step $\dot{\xi}_1$ is non-zero, and thus (13)–(14) are equivalent to

$$\frac{d\xi_2}{d\xi_1} = \frac{\kappa_2(\xi_1)}{\kappa_1(\xi_1)\xi_2}. \tag{29}$$

From (13), $\dot{\xi}_1 \neq 0$ implies $\xi_2 \neq 0$, and thus $\zeta_2 := \frac{1}{2}(\xi_2)^2$ is a valid change of coordinates on (29). In these coordinates, (29) becomes

$$\frac{d\zeta_2}{d\xi_1} = \frac{\kappa_2(\xi_1)}{\kappa_1(\xi_1)}. \tag{30}$$

For $\theta^+ \leq \xi_1 \leq \theta^-$, define

$$V_{\text{zero}}(\xi_1) := \int_{\theta^+}^{\xi_1} \frac{\kappa_2(\xi)}{\kappa_1(\xi)}\, d\xi \tag{31}$$

$$\zeta_2^- := \frac{1}{2}(\xi_2^-)^2 \tag{32}$$

$$\zeta_2^+ := \delta_{\text{zero}}^2\, \zeta_2^-. \tag{33}$$

Then (30) may be integrated over a half-step to obtain

$$\zeta_2^- = \zeta_2^+ + V_{\text{zero}}(\theta^-), \tag{34}$$

as long as[3] $\zeta_2^+ + K > 0$, where,

$$K := \min_{\theta^+ \leq \xi_1 \leq \theta^-} V_{\text{zero}}(\xi_1). \tag{35}$$

These results yield the following theorem.

Theorem 7. **(Poincaré map for hybrid zero dynamics)** Assume the hypotheses of Theorem 5 and let (θ, γ) be as in Theorem 2. Then in the coordinates $(\zeta_1, \zeta_2) = (\theta, \frac{1}{2}\gamma^2)$, the Poincaré return map of the hybrid zero dynamics, $\rho : S \cap Z \to S \cap Z$, is given by

$$\rho(\zeta_2^-) = \delta_{\text{zero}}^2\, \zeta_2^- + V_{\text{zero}}(\theta^-), \tag{36}$$

with domain of definition

$$\left\{ \zeta_2^- > 0 \mid \delta_{\text{zero}}^2\, \zeta_2^- + K \geq 0 \right\}. \tag{37}$$

If $\delta_{\text{zero}}^2 \neq 1$ and

$$\zeta_2^* := \frac{V_{\text{zero}}(\theta^-)}{1 - \delta_{\text{zero}}^2} \tag{38}$$

is in the domain of definition of ρ, then it is a fixed point of ρ. Moreover, if ζ_2^* is a fixed point, then ζ_2^* is an asymptotically stable equilibrium point of

$$\zeta_2(k+1) = \rho(\zeta_2(k)) \tag{39}$$

if, and only if, $\delta_{\text{zero}}^2 < 1$, and in this case, its domain of attraction is (37), the entire domain of definition of ρ.

[3] By definition, $\zeta_2 := \frac{1}{2}(\xi_2)^2$ must be positive along any solution.

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