15th Triennial World Congress, Barcelona, Spain

CONTROLLER DESIGN FOR TIME DOMAIN SPECIFICATIONS

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Abstract: The Coefficient Diagram Method (CDM) is a parametric approach to design a controller based on coefficient shaping. It involves two steps; first, characterization of the desired time response in terms of the two specific parameters denoted by (α_i, τ), in this paper, that are functions of a characteristic polynomial. Once such a target characteristic polynomial is obtained, the controller parameters are determined accordingly. During the process of the design, both stability and performance of the closed loop system can be considered together. This paper first summarizes the recent results on basic properties of the parameter set (α_i, τ) and shows a scheme to construct a target characteristic polynomial that meets a desired transient response characteristic. Finally, we show how to achieve a fixed order controller.

Keywords: Model matching, Linear control design, Target transfer function

1. INTRODUCTION

The performance of most control systems is typically judged by its time domain responses characterized by overshoot, settling time, speed of response, etc. Yet, there are few direct design methods available for this purpose. Moreover, if additional constraints such as fixed order, minimum-phase, bandwidth, or maximum magnitude of control input are imposed to a controller, the problem becomes even more difficult. A possible solution to this problem is based on *model matching*. In this framework, the design problem is reduced to the problem of determining either a proper target transfer function or a target characteristic polynomial that meets the given specifications (Keel and Bhattacharyya, 1999). However, it is difficult to determine a target characteristic polynomial (or target transfer function) if the design requirements are described by time response specifications. This is primarily due to a lack of knowledge of how the transient response is related to the coefficient of a fixed order model. In fact, it has been a long-standing question.

In the 1960s, Naslin (1965) extensively studied this problem in the context of so-called *Characteristic Ratio* (CR) and *Characteristic Pulsatance* that are certain ratios of coefficients of a characteristic polynomial. He empirically observed that this set of parameters was strongly connected to the damping of a system. However, Naslin's finding has largely been ignored due to the lack of theoretical justification and stability considerations. Recently, Manabe (1997) proposed *Coefficient Diagram Method* (CDM) that utilizes Naslin's findings in conjunction with the stability criterion given by Lipatov and Sokolov (1979). The method is insightful because damping, stability, and parametric sensitivity can be observed from a single Coefficient Diagram (CD) and demonstrated through several examples (Manabe, 1997; Kim *et al.*, 2000). Despite its usefulness, the method still lacks theoretical justification. An in-depth of study of the relationships between some characteristic parameters and the transient response for the case of an all pole system is given by (Kim *et al.*, 2002).

The purpose of the present paper is to summarize the recent development in CDM results including some basic properties of the parameter set (α_i, τ) and to show a design method based on target polynomial selection. From a theoretical viewpoints, the research in this area is only a beginning and much more work is needed to answer a number of critical issues.

2. OVERVIEW OF CDM

The CDM is a parametric method applied to a controller for a given LTI system. This section gives a brief introduction of the method.

2.1 *Two-Parameter Configuration*

In CDM design, two-parameter configuration is typically used to implement an overall transfer function as shown in Figure 1 where $A_p(s)$m $B_p(s)$ are denominator and numerator polynomials of the plant, respectively. $A_c(s)$, $B_c(s)$, $B_a(s)$ are all polynomials that represent a controller.

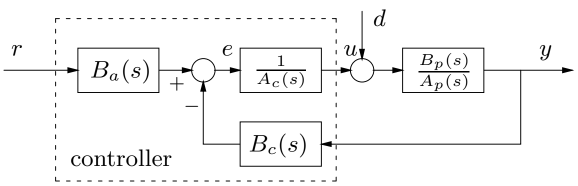

Fig. 1. Two-parameter configuration.

The characteristic polynomial becomes

$$\begin{aligned} P(s) &:= A_c(s)A_p(s) + B_c(s)B_p(s) \\ &:= a_n s^n + \cdots + a_1 s + a_0 \\ &= \sum_{i=0}^{n} a_i s^i, \quad a_i > 0. \end{aligned} \tag{1}$$

As shown in (Kim *et al.*, 2000; Kim *et al.*, 2002; Manabe, 1997; Naslin, 1965), we define the characteristic ratio (CR) or called by the stability index in (Manabe, 1997), the generalized time constant, and stability limit as follows.

$$\alpha_i := \frac{a_i^2}{a_{i+1}a_{i-1}}, \qquad i = 1, 2, \cdots, n \tag{2}$$

$$\tau := \frac{a_1}{a_0}, \tag{3}$$

$$\alpha_i^* := \frac{1}{\alpha_{i+1}} + \frac{1}{\alpha_{i-1}}, \qquad \begin{array}{l} i = 1, 2, \cdots, n-1; \\ \alpha_0 = \alpha_n = \infty. \end{array} \tag{4}$$

Also, define *the characteristic pulsatance*

$$\beta_i := \frac{a_i}{a_{i+1}}, \qquad i = 0, 1, \cdots, n-1. \tag{5}$$

Then the following relations hold.

$$\alpha_i = \frac{\beta_i}{\beta_{i-1}}, \qquad i = 1, 2, \cdots, n-1 \tag{6}$$

$$a_i = \frac{a_0 \tau^i}{\alpha_{i-1}\alpha_{i-2}^2 \cdots \alpha_2^{i-2}\alpha_1^{i-1}}. \tag{7}$$

Thus, the characteristic polynomial can be written as:

$$P(s) = a_0 \left[\left(\sum_{i=2}^{n} \left(\Pi_{j=1}^{i-1} \frac{1}{\alpha_{i-j}^j} \right) (\tau s)^i \right) + \tau s + 1 \right].$$

Hereafter, we refer the set (α_i, τ) to *characteristic parameters* (CP). The CR and the characteristic pulsatance were originally introduced by Naslin (1965). It is noted that we call the first characteristic pulsatance the generalized time constant which we elaborate its significance in Section 3.

2.2 *Basic Feature of CDM*

The closed-loop transfer function of Figure 1 is

$$G(s) = \frac{y}{r} = \frac{B_a(s)B_p(s)}{P(s)} \tag{8}$$

and its corresponding Diophantine equation is

$$A_c(s)A_p(s) + B_c(s)B_p(s) = P^*(s) \tag{9}$$

where $P^*(s)$ is the target characteristic polynomial. The CDM based design is similar to the model matching method. Once the $A_c(s)$ and $B_c(s)$ for a given $P^*(s)$ are achieved, then the feedforward part $B_a(s)$ is designed to compensate for the effect of plant zeros and the steady state error. A novelty of the method lies in finding a target characteristic polynomial $P^*(s)$ from the given time domain specifications. As shown above that selection of $P^*(s)$ is equivalent to that of the CP, the task of selecting $P^*(s)$ is accomplished by determining appropriate CP such that the given time domain specifications are met. The relationships between CP and time domain specifications will be addressed in Section 3. In the case of $n-1$th order controllers, it is indeed trivial to construct a controller for a target polynomial (i.e., pole placement problem). However, if a lower and fixed order controller is used, the problem of achieving the target polynomial through controller parameters is no longer possible by a pole placement technique. In this case, the coefficient diagram gives a special advantage through coefficient shaping which we will illustrate by examples in Section 2.4.

2.3 *Role of a Lipatov-Sokolov Stability Condition*
In 1979, Lipatov and Sokolov reported a series of sufficient conditions for Hurwitz stability (Lipatov and Sokolov, 1979). It is an interesting coincident that these conditions are functions of CR introduced by Naslin. The following theorems are found in (Lipatov and Sokolov, 1979).

Theorem 1. $P(s)$ is Hurwitz stable if

$$\sqrt{\alpha_i \alpha_{i+1}} > 1.4656 \qquad \text{for } \; i = 1, 2, \cdots, n-2.$$

Theorem 2. $P(s)$ is Hurwitz stable if

$$\alpha_i \geq 1.12374 \alpha_i^* \qquad \text{for } \; i = 2, 3, \cdots, n-2.$$

Theorem 3. $P(s)$ is unstable if

$$\alpha_i \alpha_{i+1} \leq 1 \qquad \text{for some } \; i = 1, 2, \cdots, n-2.$$

Note that the condition in Theorem 2 is a function of every five consecutive coefficients while the condition in Theorem 1 is a function of every four consecutive coefficients. Hence Theorem 2 gives a tighter condition than that of Theorem 1. It is also sufficient for Theorem 1 that $\alpha_i > 1.4656$ for all i.

2.4 *Coefficient Diagram (CD)*
The CD is a semi-log diagram of the coefficients of polynomials in logarithmic scale versus the degree of s in linear scale. Information regarding stability and response can be observed from the relative slopes between each pair of coefficients in CD Here We take a simple example to illustrate this diagram. Consider a fourth-order plant with PID controller as follows.

$$\begin{aligned} A_p(s) &= 0.25s^4 + s^3 + 2s^2 + 1.5s, \\ B_p(s) &= 1, \\ A_c(s) &= l_1 s, \\ B_c(s) &= k_2 s^2 + k_1 s + k_0, \end{aligned}$$

where $l_1 = 1$, $k_2 = 0.5$, $k_1 = 1$, $k_0 = 0.2$. The Characteristic polynomial is

$$P(s) = 0.25s^5 + s^4 + 2s^3 + 2s^2 + s + 0.2.$$

We now compute the corresponding CP.

$$\begin{aligned} \{a_i\} &= [a_4 \; \cdots \; a_1 \; a_0] = [0.25 \; 1 \; 2 \; 2 \; 1 \; 0.2], \\ \{\alpha_i\} &= [\alpha_4 \; \cdots \; \alpha_2 \; \alpha_1] = [2 \; 2 \; 2 \; 2.5], \\ \tau &= 5. \\ \{\alpha_i^*\} &= [\alpha_4^* \; \cdots \; \alpha_2^* \; \alpha_1^*] = [0.5 \; 1 \; 0.9 \; 0.5]. \end{aligned}$$

The corresponding CD is shown in Figure 2.

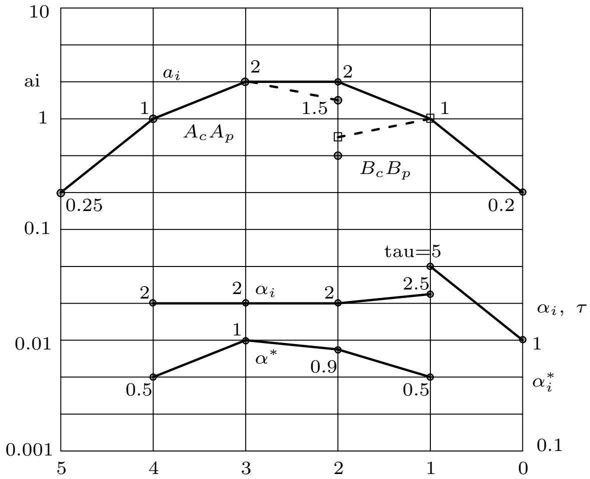

Fig. 2. A coefficient diagram.

As shown in Figure 2, a coefficient diagram contains four curves of a_i, α_i, α_i^*, and τ. Coefficients a_i are read on the left side scale and characteristic ratio α_i, generalized time constant τ, and stability limit α_i^* are all read on the right side scale. The τ is expressed by a line connecting 1 to τ.

To proceed, let us consider the following.

$$\log \alpha_i = 2 \log a_i - \log a_{i+1} - \log a_{i-1}. \quad (10)$$

Clearly, for a larger value of a_i and smaller values of (a_{i-1}, a_{i+1}), we obtain a larger value of α_i in logarithmic scale. If it happens for all a_i, the curvature of the top plot in Figure 2 will be greater. In other words, larger α_is lead greater curvature of the a_i plot in the coefficient diagram. In the next section, we will elaborate on the relationship between (α_i, τ) and the transient response.

Recall Theorem 1 and eq. (2), we have

$$\sqrt{a_{i+1} a_i} > 1.4656 \sqrt{a_{i+2} a_{i-1}} \qquad (11)$$

or

$$\begin{aligned} &\frac{1}{2} \left(\log a_{i+1} + \log a_i \right) \\ &> \log 1.4656 + \frac{1}{2} \left(\log a_{i+2} + \log a_{i-1} \right). \end{aligned}$$

It shows that the stability is guaranteed if the mid-point of the segment (a_{i+1}, a_i) is greater in $\log 1.4656$ than that of (a_{i+2}, a_{i-1}). Similarly, Theorem 2 can also be used visually to determine stability.

A parametric sensitivity may also be observed from the coefficient diagram. Note that

$$P(s) = A_p(s) A_c(s) + B_p(s) B_c(s).$$

The complementary sensitivity function is

$$\frac{B_p(s) B_c(s)}{P(s)}.$$

In Figure 2, the coefficients of $A_p(s)A_c(s)$ are shown as "o" and the coefficients of $B_P(s)B_c(s)$ are shown as "□" with dashed lines. From this, a designer can visually assess the deformation of the coefficient diagram due to the parameter change of k_2, k_1 and k_0. This allows a designer to estimate the shape of the curvature $P(s)$.

3. CHARACTERIZATION OF TRANSIENT RESPONSE BY CHARACTERISTIC PARAMETER

3.1 *Basic Properties of Characteristic Parameters*

In this section, we discuss the relationship between the CP (α_i, τ) and the transient response.

Consider a real polynomial

$$\begin{aligned} P(s) &= a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \qquad (12)\\ &= a_n (s-s_1)(s-s_2)\cdots(s-s_n), \quad a_i > 0, \end{aligned}$$

where s_i, $i = 1, 2, \cdots, n$ are roots of $P(s)$. From the definitions given in Section 2, we have

$$a_1 = a_0 \tau \tag{13}$$

$$a_k = \frac{a_0 \tau^k}{\alpha_{k-1}\alpha_{k-2}^2 \cdots \alpha_2^{k-2}\alpha_1^{k-1}}, \quad k \geq 2. \tag{14}$$

Then the following is true.

Proposition

$$\tau = \frac{a_1}{a_0} = -\sum_{i=1}^{n} \frac{1}{s_i}. \tag{15}$$

The following theorem states the relationship between the generalized time constant τ and the speed of response with respect to an arbitrary input.

Theorem 4. (Kim *et al.*, 2002) Let two all pole transfer functions $G_1(s)$ and $G_2(s)$ of the same order be

$$G_1(s) = \frac{Y_1(s)}{R(s)} = \frac{a_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

$$G_2(s) = \frac{Y_1(s)}{R(s)} = \frac{b_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}$$

with

$$\tau_1 := \frac{a_1}{a_0} \quad \text{and} \quad \tau_2 := \frac{b_1}{b_0}.$$

Let $y_i(t)$ be the response of the system $G_i(s)$. Then

$$y_1(t) = y_2\left(\frac{\tau_1}{\tau_2}\cdot t\right)$$

if and only if the denominators of $G_1(s)$ and $G_2(s)$ have the same characteristic ratios.

It is interesting to note the following. Let s_i and $\hat{s}_i$ be poles of $G_1(s)$ and $G_2(s)$, respectively. $\mathcal{R}_i$ is the ray drawn from the origin extended to ∞ passing through a pole s_i. Then $\hat{s}_i$ is also found on the ray $\mathcal{R}_i$ under the condition given in Theorem 4. This indicates that τ only changes the speed of the response while it preserves the amount of damping.

We now consider the property of characteristic ratios (CR). We begin with examining the square of the magnitude function below.

$$|G(j\omega)|^2 = \frac{a_0^2}{P(j\omega)P(-j\omega)} = \frac{1}{\bar{Q}^2(\omega)} \tag{16}$$

where $P(s)$ follows the notation in eq. (12). Define

$$\Delta_i^j := \begin{cases} \Pi_{k=i,i<j}^{j}\alpha_k, & \text{if } i<j \\ \alpha_i, & \text{if } i=j. \end{cases} \tag{17}$$

Using eqs. (13), (14), and (17),

$$\begin{aligned} \bar{Q}^2(\omega) &= 1 + \eta_1\tau^2\omega^2 + \frac{\eta_2\tau^4}{(\Delta_1^1)^2}\omega^4 + \frac{\eta_3\tau^6}{(\Delta_1^1\Delta_1^2)^2}\omega^6 + \\ &\cdots + \frac{\eta_n\tau^{2n}}{\left(\Delta_1^1\Delta_1^2\Delta_1^3\cdots\Delta_1^{n-1}\right)^2}\omega^{2n} \qquad (18) \end{aligned}$$

where

$$\begin{aligned} \eta_k &:= 1 - \frac{2}{\alpha_k} + \frac{2}{\alpha_k\Delta_{k-1}^{k+1}} - \frac{2}{\alpha_k\Delta_{k-1}^{k+1}\Delta_{k-2}^{k+2}} + \cdots \\ &\cdots + (-1)^k\frac{2}{\alpha_k\Pi_{j=1}^{k-1}\Delta_{k-j}^{k+j}}. \qquad (19) \end{aligned}$$

The objective of this discussion is to determine α_is such that $|G(j\omega)|$ has no peaks anywhere except $\omega = 0$ and is a monotonically decreasing function over ω. Despite lack of mathematical proof, it is a common belief that such a function has a smaller overshoot when the step input is applied (Chestnut and Mayer, 1959). The following theorem classifies a set of such functions.

Theorem 5. (Kim *et al.*, 2002) The frequency magnitude, $|G(j\omega)|$, of an all pole transfer function $G(s)$ is monotonically decreasing over ω with $|G(j0)| = 1$ if

(i) $\alpha_1 > 2$, for all i;
(ii) $\alpha_k = \frac{\sin\left(\frac{k\pi}{n}\right)+\sin\left(\frac{\pi}{n}\right)}{2\sin\left(\frac{k\pi}{n}\right)}\cdot\alpha_1, \quad k > 2.$

The following corollary gives a simpler condition while it is more restrictive than the conditions in Theorem 5.

Corollary 6. $|G(j\omega)|$ is a monotonically decreasing function over ω with $|G(j0)| = 1$ if $\alpha_i > 2$ for all i.

Some results that relate the specific values of characteristic ratios α_i and sectors on which the poles of the system lie are also available and they are summarized below.

Theorem 7. Let $\mathcal{D}$ be the left half plane region bounded by the damping sector of 0.707 and s_i be the poles of the all pole transfer function. Then $\alpha_1, \alpha_{n-1} > 2$ if $s_i \in \mathcal{D}$ for all i.

Theorem 8. (Lipatov and Sokolov, 1979) All roots of $P(s)$ lie on the negative real axis of the complex plane if $\alpha_i > 4$ for all i.

Based on examination, it appears that no all α_i are important. In other words, smaller changes in some α_i lead to greater changes in overshoot. We observe this phenomena as follows. Consider eqs. (16) and (18). The straight-line approximation of the magnitude plot is shown in Figure 3. Suppose that $\eta_i > 0$ for all i. Define the intersection of every adjacent pair of straight-lines in Figure 3 as

$$\omega_i := \sqrt{\frac{\eta_i}{\eta_{i+1}}} \cdot \frac{\Delta_1^i}{\tau}, \quad i = 0, 1, \cdots, n-1 \quad (20)$$

where $\eta_0 = 1$ and $\Delta_1^0 = 1$. Let

$$d_i := \log \omega_i - \log \omega_{i-1},$$

then we write

$$d_0 = -\left(\log \tau + \frac{1}{2} \log \eta_1\right) \quad (21)$$

$$d_i = \log \eta_i - \frac{1}{2} \left(\log \eta_{i-1} + \log \eta_{i+1}\right) + \log \alpha_i, \quad i = 1, 2, \cdots, n-1. \quad (22)$$

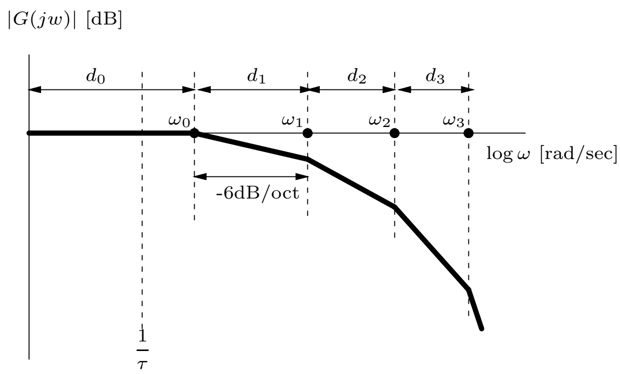

Fig. 3. Pseudo-asymptotic diagram of $|G(j\omega)|$.

From eq. (19), we see that η_i becomes smaller for larger values of α_i. Thus, $d_i \approx \alpha_i$ for larger values of α_i. Moreover $\omega_i < \omega_{i+1}$ for all i. If such a relation holds, it is clear that α_i with lower indices i has much greater influence that those with higher indices i. In fact, α_1 and α_2 are the most dominant factors to dictate overshoot in many cases. On the other hand, it is intuitive from Figure 3 that τ dictates the bandwidth.

3.2 *Composition of a Target Polynomial*

Since CDM is a type of model matching design, it begins with selecting a desired characteristic polynomial which is often called *a target polynomial* that meets the given time response specifications such as overshoot and settling time. When the plant has no zeros or its zeros are located relatively far from the origin, the target model can be chosen as an all pole transfer function. In this section, we give the steps of selecting a target polynomial by using the properties of characteristic parameters discussed in the previous sections.
Step 1: Set $\tau = 1$ and choose a set of α_i based on Theorem 5 or Corollary 6.
Step 2: Calculate the coefficients of $P(s)$ by substituting (α_i, τ) into eq. (7) and $a_0 > 0$ can be chosen arbitrary. Obtain the step response of the first target model $G_1(s)$ and find the settling time t_{s1}.
Step 3: Determine the value of τ by using Theorem 4 so that the desired settling time is satisfied.
Step 4: If overshoot is not satisfactory, adjust α_i until satisfactory overshoot is achieved. In many cases, an appropriate set of α_i may be obtained by adjusting lower indices α_i.
Example: Using the above procedure, we found a 8^{th} order all pole target transfer function that gives its step response with overshoot being less than 1% and the settling time being approximately 5 sec.

$$[\alpha_1 \cdots \alpha_7] = [2.2\ 1.695\ 1.555\ 1.521\ 1.555\ 1.695\ 2.2]$$

$$G(s) = \frac{1}{\begin{array}{c}0.006s^8 + 0.011s^7 + 0.089s^6 + 0.432s^5 \\ +1.348s^4 + 2.763s^3 + 3.64s^2 + 2.83s + 1\end{array}}.$$

The step response and pole locations of this target transfer function are shown in Figures 4 and 5.

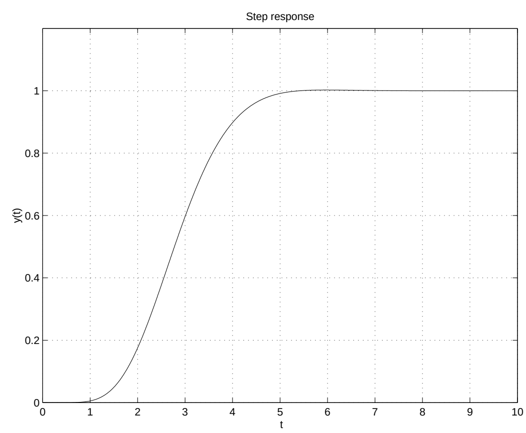

Fig. 4. Step response.

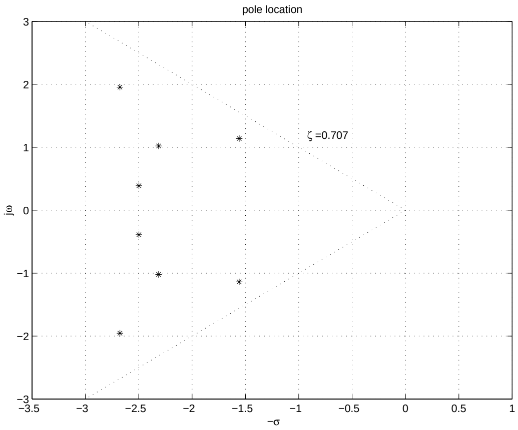

Fig. 5. Closed loop pole locations.

4. DISCUSSION AND FUTURE RESEARCH

Controller design based on coefficients of a characteristic polynomial has always been an attractive solution. For example, the pole placement problem also solves a set of linear equations constructed from the coefficients of a target polynomial. This is due to the fact that controller transfer function coefficients enter linearly into coefficients of the characteristic polynomial. On the other hand, study on the time response characteristics has mainly been centered around poles and zeros of a transfer function. For this reason, designing a controller that meets the given time domain requirements is usually ad-hoc and difficult. The uniqueness of the CDM is that it employs time domain information characterized by coefficients of a polynomial rather than its roots. It turns out that the method is intuitive and easy to use for designing a controller. However, much of the relationships between coefficients and the time domain property are based on empirical observations. The results shown in this paper are only the beginning for providing theoretical justification to the method. Here we discuss some of the problems that need to be investigated to fertilize the method.

(a) A limited aspect of α_i with regard to "overshoot of the step response" has been discussed in this paper. It is still unknown how these α_i are exactly related to the overshoot.

(b) How to shape a coefficient diagram for the case of a lower order controller? For a given plant, there are no systematic steps to determine a lowest order of controller to satisfy the given specifications or to check whether such a controller of the specified order exists or not. Presently, the method relies upon trial and error.

(c) Although the method has successfully been used to design controllers for general systems, available theoretical foundation is restricted to the case of an all pole system and two-parameter configuration. Some theoretical studies are needed to incorporate more general cases of systems.

(d) We have shown that larger α_is lead to greater curvature of the coefficient diagram. We believe that issues on robust stability and/or parameter sensitivity may be studied in this context.

(e) Presently, the proof of stability solely relies upon Lipatov-Sokolov conditions which are known to be conservative. It is not clear whether this stability region contains "good controllers" or not. If not, less conservative stability conditions may have to be employed.

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