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CONSTRUCTIVE THEORY OF OPTIMAL CONTROL AND CLASSICAL PROBLEMS OF REGULATION

Rafail Gabasov * **Faina M.Kirillova** **

* *Byelorussian State University, Minsk, Belarus*
** *Institute of Mathematics, National Academy of Science, Minsk, Belarus*

Abstract: Constructive methods of optimization is used to solve classical problems of regulation. It is shown that fast algorithms of positional optimization of dynamical systems elaborated in Minsk (Belarus) represent the effective tool for solving such problems as 1) transfering an object from one position to the other and stabilizing a new regime by bounded feedbacks, 2) tracking problems (problems of realizing given motions). Results are illustrated by classical benchmarks.

Keywords: optimal control, fast algorithms, regulator, closed-loop control, stabilization, tracking systems

The final test of a theory is its capacity to solve the problems which originated it.

G.Dantzig

1. INTRODUCTION

Mathematical theory of optimal control arose in the middle 50s of the 20th century after the first results on synthesis of optimal systems obtained by engineers on automatic regulation. The advancement of automatic control theory resulted partially in the optimal synthesis problem. As is known, basic problems such as regulation and tracking problems occupied a highly important place among the others. The regulation problem consists (in informal statement) in the following. A control object is given and it has several regimes of function. The problem is to design a feedback under which the control system passes from an initial state to a new regime and is stably functioning in that regime. One can select two parts in the problem: 1) the realization of a transient passing on from the initial state to a new regime; 2) the stabilization of a new regime. A long time scientists paid attention mostly to the second part of the problem, although the solution of the first part was of great importance for applications.

The second fundamental problem of classical regulation theory is known in different variants (a problem of realization of motions, tracking problem etc.). For instance, a problem of realization of motions may be formulated as follows. A control system and a family of motions are given (more often it is a set of periodic motions). The problem is to design a feedback under which a chosen element of the family becomes an asymptotically stable trajectory for the closed-loop system. At solving mentioned classical problems of regulation engineers had to solve important for applications problems such as invariance problems, robustness problems, damping, amortization problems etc. In early 50s in the USSR and the USA engineers stated a problem of creating the theory of synthesis of optimal systems. The time-optimal systems of one freedom degree were the first to be synthesized. So the pioneer optimal systems solved in a best way the first part of the classical problem of regulation. Generalization of these results on more complex systems came across serious difficulties and outstanding mathematicians paid attention to the problem (Pontryagin *et al.*, 1962), (Bellman, 1957). Intensive investigations of Soviet and American scientists led in the middle 50s to discovery of two fundamental methods of optimal control theory — the Pontryagin maximum principle and the Isaacs- Bellman dynamic programming. Since then problems and methods of optimal control theory have been generalized and reinforced in various direction and now optimal control theory represents rich and deep mathematical theory. It is a natural question: "What did optimal control theory give for the solution of the mentioned above classical problems of regulation?"

Below we describe results obtained in Belarus (Minsk) which concern this question. The authors cannot analyze papers of another authors due to limited volume of the paper with reference to (Aizerman, 1975), (Bissel, 1992), (Feldbaum, 1963), (Pesch and Bulirsch, 1994). We begin with results obtained in the field of constructing fast algorithms of optimal control (Balashevich *et al.*, 2000; Gabasov and Kirillova, 2001).

2. LINEAR OPTIMAL CONTROL PROBLEM IN THE CLASS OF DISCRETE CONTROLS

Consider a linear optimal control problem

$$J = c'x(t^*) \to \max, \quad \dot{x} = Ax + bu, \ x(0) = x_0,$$
$$Hx(t^*) = g, \ |u(t)| \le 1, \ t \in T = [0, t^*]. \quad (1)$$

This problem is a simplest problem of optimal control theory but up to now there are no sufficiently effective methods of constructing open-loop and closed-loop solutions to it.

For the concrete solution of problem (1) it is necessary to use numerical tools of discrete actions. In this connection it is natural to investigate problem (1) in the class of discrete functions

$$u(t) = u(kh),$$
$$t \in [kh, (k+1)h[, k = \overline{0, N-1}. \quad (2)$$

(h is a quantization period). In this class of admissible controls problem (1) is equivalent to a linear programming problem which however has at small $h > 0$ series specific features due to which the use of linear programming methods is slightly effective. In Minsk special dynamic modifications of linear programming methods have been elaborated that allowed to construct effectively optimal open-loop solutions and then to justify a new approach to the solution of optimal synthesis problem (Balashevich *et al.*, 2000).

Let us illustrate effectiveness of the method by the following problem.

Example 1. Consider two-mass oscillating system which has to be damped with minimal fuel consumption in a finite time

$$J(u) = \int_0^{25} u(t)dt \to \min,$$
$$\dot{x}_1 = x_3, \dot{x}_2 = x_4,$$
$$\dot{x}_3 = -x_1 + x_2 + u, \dot{x}_4 = 0.1x_1 - 1.02x_2,$$
$$x_1(0) = x_2(0) = 0, x_3(0) = 2, x_4(0) = 1,$$
$$x_1(25) = x_2(25) = x_3(25) = x_4(25) = 0,$$
$$0 \le u(t) \le 1, \ t \in [0, 25[, \quad (3)$$

$x_1, \ x_2$: deviations of the masses from the equilibrium state; $x_3, \ x_4$: velocities of the masses; u : fuel consumption per second used for control.

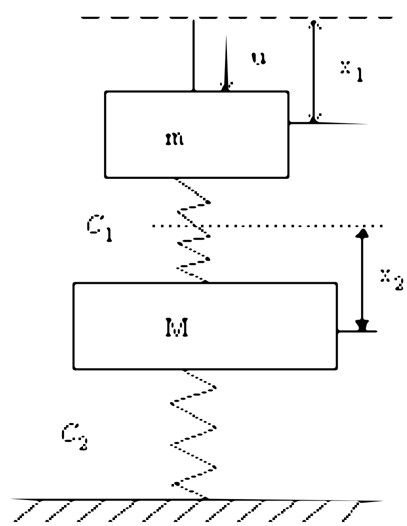

Fig. 1. Two-mass oscillating system

Let a function $u(t), \ t \ge 0$, belong to (2). In this case, problem (3) is reduced to (1), (2) in which the dimension of the state vector equals 5. Following (Fedorenko, 1978) we estimate the complexity of methods according to a number of complete integrations of original or adjoint systems which are necessary to construct an optimal control. With the help of the algorithms from (Balashevich *et al.*, 2000) the following results of computer experiments were obtained.

Table 1 Optimal open-loop control, complexity of calculation

N	h	J^0	C
100	0.25	6.353339	2.41
1000	0.025	6.331252	2.239
10000	0.0025	6.330941	2.2018
25000	0.001	6.330938	2.3564

Table 1 contains results of solving problem (3) for different quantization periods. The last column contains data on complexity of the method. Two complete integrations were spent for preparatory work connected with testing an initial admissible control on optimality. From Table 1 one can observe that a number of complete integrations for constructing the optimal open-loop control depends slightly on the quantization period and does not exceed 2.5. The known methods of constructing optimal open-loop controls based on the maximum principle and other approaches do not possess the mentioned efficiency (Balashevich *et al.*, 2000).

3. POSITIONAL SOLUTIONS

Optimal open-loop solutions play an important part in control theory but they are connected a little with classical problems of regulation as the form of solutions used in these problems is controls of feedback type. Let us imbed problem (1) into the family

$$c'x(t^*) \to \max, \quad \dot{x} = Ax + bu,\ x(\tau) = z,$$
$$Hx(t^*) = g,\ |u(t)| \le 1,\ t \in T(\tau) = [\tau, t^*] \quad (4)$$

depending on a scalar τ and an n-vector z. Let $u^0(t|\tau, z),\ t \in T(\tau)$, be an optimal open-loop control to problem (4) for the position (τ, z), X_τ be a set of all vectors $z \in R^n$ for which problem (4) has a solution with τ fixed. The function

$$u^0(\tau, z) = u^0(\tau|\tau, z),$$
$$z \in X_\tau,\ \tau \in T_h = 0, h, \dots, t^* - h, \qquad (5)$$

is called a positional solution to problem (1) or an optimal control of feedback type. Analytical construction of (5) is as a rule impossible if the dimension of problem (1) exceeds 3. The possibilities of dynamic programming at constructing positional solutions are limited due to the known "curse of dimensionality". A roundabout way to the problem in question based on using fast algorithms of optimal control and modern computer technology was suggested and elaborated (Gabasov *et al.*, 2001), (Gabasov *et al.*, 1995). It consists in the following.

Let the optimal feedback be constructed. We close system (1) by this feedback and consider the behavior of the closed-loop system under constantly acting disturbances

$$\dot{x} = Ax + bu^0(t, x) + w(t),\ x(0) = x_0. \quad (6)$$

Let $w^*(t),\ t \in T$, be a realizing (unknown) disturbance. A transient $x^*(t),\ t \in T$, of closed-loop system (6) satisfies

$$\dot{x}^* = Ax^*(t) + bu^0(t, x^*(t)) + w^*(t),\ t \in T. (7)$$

From (7) one can see that in this concrete process the optimal feedback is used only along the isolated curve $x^*(t),\ t \in T$.

Definition 1. A device which for any current position $(\tau, x^*(\tau))$ able to calculate a value $u^*(\tau)$ of the realization of the optimal feedback $u^*(t) = u^0(t, x^*(t)),\ t \in T$, for the time not exceeding h, is said to be Optimal Controller.

Thus, the optimal synthesis problem is reduced to constructing an algorithm for Optimal Controller. The algorithm for Optimal Controller is based on one dual method of linear programming modified to dynamical structure of problem (1) (Balashevich *et al.*, 2000), (Gabasov *et al.*, 2001).

Let us illustrate the results of functioning of Optimal Controller using the previous example:

$$\dot{x}_1 = x_3,\ \dot{x}_2 = x_4,\ \dot{x}_3 = -x_1 + x_2 + u,$$
$$\dot{x}_4 = 0.1x_1 - 1.02x_2 + w^*(t),$$

where $w^*(t) = 0.3 \sin 4t,\ t \in [0, 9.75[;\ w^*(t) \equiv 0,\ t \ge 9.75$.

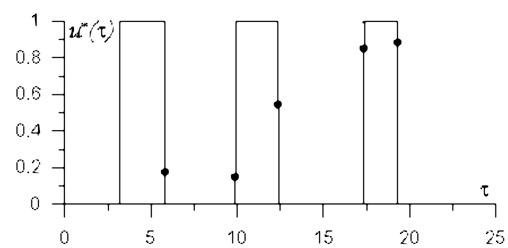

Fig. 2. Optimal Controller. Realization of the optimal feedback

The realization of the optimal feedback is given on Fig 2. The values of complexity $C(\tau)$ of calculation of current values of realization $u^*(\tau),\ \tau \in T$, is presented on Fig 3.

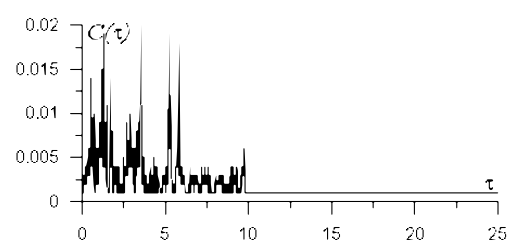

Fig. 3. Complexity of calculation on iterations

So, the time of calculation spent to obtain current values of $u^*(t),\ t \in T$, does not exceed 2% with respect to the time of the whole integration the initial or adjoint systems. It is clear that such volume of work at optimizing dynamic systems many modern microprocessors are able to do.

4. STABILIZATION PROBLEM

Methods of synthesis of optimal systems can be used to solve the mentioned classical problems of regulation. We begin with the stabilization problem. Consider system (1) which at $u(t) \equiv 0,\ t \geq 0$, is not asymptotically stable. Let G be a vicinity of the equilibrium state $x = 0$.

Definition 2. A function $u = u(x),\ x \in G$, is a bounded stabilizing feedback in G if 1) $u(0) \equiv 0$; 2) $|u(t)| \leq L,\ x \in G$; 3) the zero solution of the closed system $\dot{x} = Ax + bu(x)$ is asymptotically stable in G.

As is known, classical methods of construction of stabilizing feedbacks are based on sufficient conditions of asymptotic stability (parametric, frequency conditions, methods of the Lyapunov functions etc.). Optimal control theory with a view of stabilization was first used by R.Kalman and A.M.Lyotov in 60s of the last century. They proved that a positional solution $u^0(x) = k'x$ of a linear-quadratic optimal control problem with an infinite horizon possessed stabilizing property. Later on to construct stabilizing feedbacks also linear-quadratic problems with a finite horizon were widely used. But in these papers geometric constraints on control functions, important for applications, were seldom taken into account. Even the Kalman-Lyotov problem with control constraints cannot give simple optimal positional solutions. In the middle 90s the authors of the paper suggested to use the method of optimal synthesis in combination with the moving horizon principle (Kwon and Pearson, 1977) for the construction of bounded stabilizing feedbacks.

Choose a number $\Theta,\ 0 \leq \Theta < +\infty$ (parameter of the method) and introduce auxiliary (accompanying) optimal control problem

$$\begin{gathered} B_\Theta(z) = \min \int_0^\Theta |u(t)|dt, \\ \dot{x} = Ax + bu,\ x(0) = z, \\ x(\Theta) = 0,\ |u(t)| \leq L,\ t \in T = [0, \Theta], \\ rank\{b, Ab, \ldots, A^{n-1}b\} = n. \end{gathered} \tag{8}$$

Let $u^0(t|z),\ t \in T$, be an optimal open-loop control for z, $G(\Theta)$ be a set of all states z for which problem (8) has a solution. It can be proved that function

$$u(x) = u^0(0|x),\ x \in G(\Theta), \tag{9}$$

is a bounded stabilizing feedback. This feedback possesses very important for applications properties: 1) by choosing Θ, the domain $G(\Theta)$ can be made as close as possible to the maximal domain of stability; 2) transients of the system closed by (9) possess the following extremal property

$$\int_0^\infty |u^*(t)|dt \leq \int_0^\Theta |u^0(t|x_0)dt, \tag{10}$$

i.e. fuel consumption on the whole stabilization process does not exceed its value necessary for optimal damping system (8) for the time Θ. At forming accompanying optimal control problems another criteria can be used, for instance

$$B_\Theta(z) = \min_u \max_t |u(t)|, \quad B_\Theta(z) = \int_0^\Theta u^2(t)dt,$$

that allows to obtain transients with different extremal properties.

Example 2. Consider a stabilization problem for oscillating system (Sussmann *et al.*, 1994)

$$\begin{gathered} \dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + x_3, \\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + u, \end{gathered} \tag{11}$$

where (x_1, x_2, x_3, x_4) are as usual coordinates of state, u is a control. For system (11) the bounded stabilizing feedback constructed in (Sussmann *et al.*, 1994) has the form

$$u = -sat\left(x_4 + \frac{sat(29(-x_1 + x_3 + x_4))}{29}\right) \tag{12}$$

where $|u(t)| \leq 1,\ sat(s) = sgn(s)\min\{|s|, 1\}$.

Introduce the accompanying optimal control problem

$$\begin{gathered} \int_0^\Theta |u(t)|dt \to \min, \\ \dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + x_3, \\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + u, \\ x_1(0) = x_1^*(\tau),\ x_2(0) = x_2^*(\tau), \\ x_3(0) = x_3^*(\tau),\ x_4(0) = x_4^*(\tau), \\ x_i(\Theta) = 0,\ i = \overline{1,4}, \\ |u(t)| \leq 1,\ t \in T = [0, \Theta], \end{gathered} \tag{13}$$

here $x^*(\tau) = (x_1^*(\tau), x_2^*(\tau), x_3^*(\tau), x_4^*(\tau))$ is a state of (13) at a current instant τ.

On Fig. 4 one can observe and compare both the transient in (11) closed by (12) (the curve 1) and the transient constructed by the use of the feedback solution of accompanying optimal control problem (13)(the curve 2).

At constructing stabilizing feedbacks engineers do not restrict themselves by obtaining asymptoti-

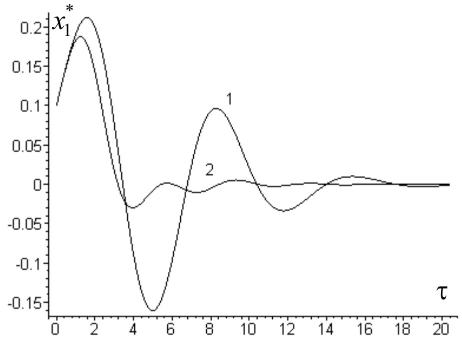

Fig. 4. Behaviour of x_1^* at using feedback (12) and the feedback for (13)

cally stable closed systems. They take into consideration also quality of transients. Except of integral criteria of the estimate of quality very popular among engineers are: degree of stability, degree of oscillations, degree of overcontrol, monotonicity etc. The suggested approach to the solution of optimal synthesis problems proves to be rather effective at obtaining the mentioned indices of quality if the accompanying problem is chosen in an appropriate way.

Example 3. Let the output signals $y(t) = x_1(t),\ t \geq 0$, of system (11) have to possess the degree of stability $\alpha > 0$:

$$|x_1(t)| \leq a \exp(-\alpha t) \geq 0.$$

While solving the accompanying optimal control problem to stabilize system (11), the following parameters were chosen $\Theta = 8,\ h = 0.4,\ x_0^* = (0.1.0.1.0.1.0.1)$.

On Fig. 5 lines 4, 5 stand for the restrictions on the output signal. Curves 1, 2 correspond to the cases 1) $\alpha = 0.1,\ a = 0.2$; 2) $\alpha = 0.5,\ a = 0.4$, curve 3 denotes $y(t) = x_1(t),\ t \geq 0$, when the restrictions on outputs are omitted.

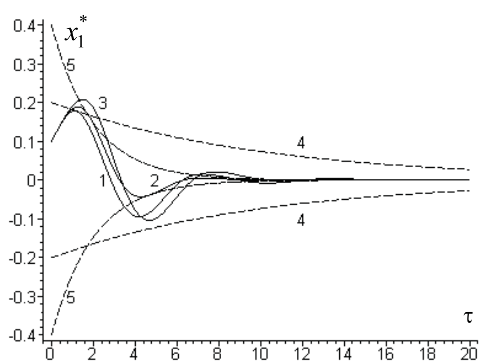

Fig. 5. Behaviour of x_1^* at 1) $\alpha = 0.1$, $a = 0.2$ (curve 1); 2) $\alpha = 0.5$, $a = 0.4$ (curve 2); 3) no restrictions on $y(t)$ (curve 3)

The results given above were prolongated to non-stationary systems, multidimensional controls, systems with delay (Gabasov *et al.*, 1999*b*) , distributed parameter systems, nonlinear control systems. The algorithms were tested by computer on the stabilization problems for the inverted pendulum (Furuta *et al.*, 1999), the problem of damping oscillations of a string (Gabasov *et al.*, 1999*a*).

5. CLASSICAL PROBLEM OF REGULATION

Let a control system be described by (1) where controls $u(t),\ t \in T$, satisfy the inequality $|u(t)| \leq L,\ t \geq 0$. Introduce a set $X_0 = \{x \in R^n : Ax + bu_x = 0,\ |u_x| \leq L\}$. Elements of X_0 are said to be admissible equilibrium states of (1).

Definition 3. For given $L,\ 0 < L < +\infty$, a vector $z \in intX_0$ and a domain $G \in R^n$, a function

$$u = u_z(x),\ x \in G, \tag{14}$$

is said to be a feedback solving the classical problem of regulation for (1) in G if 1) $u_z(z) = u_z$; 2) $|u_z(x)| \leq L,\ x \in G$; 3) the closed system

$$\dot{x} = Ax + bu_z(x),\ x(0) = x_0 \in G, \tag{15}$$

has a solution $x(t) \in G,\ t \geq 0$; 4) the equilibrium state $x(t) \equiv z,\ t \geq 0$, of (15) is asymptotically stable in G.

From the point of view of applications it is important that in addition to the mentioned 5) domain of attraction G of z would be sufficiently large; 6) transients of (15) possess in some sense a high quality.

Example 4. Consider a crane that transfers the load hanging on cable from one equilibrium state to the vicinity of another (Fig. 6). The linearized model has the form

$$\begin{gathered}(M+m)\ddot{x} - mH\ddot{\varphi} = u,\\ I\ddot{\varphi} + mgH\varphi = mH\ddot{x},\\ x(0) = \dot{x}(0) = 0,\ \varphi(0) = \dot{\varphi}(0) = 0,\\ x(t) \to z,\ \dot{x}(t) \to 0, \quad \varphi(t) \to 0,\ \dot{\varphi}(t) \to 0,\end{gathered} \tag{16}$$

where x: deviation of the crane from the equilibrium state, φ: deviation of the cable from the vertical, M: mass of the crane, m: mass of the load, H: the distance from the crane to the center of inertia of the load, I: moment of inertia of the load relatively to the point of the suspension. Choose parameters of (16) putting $M = 7,\ m = 3,\ H = 3,\ g = 10,\ I = mH^2 = 27$. Then system (16) takes the form

$$\begin{gathered}\dot{x}_1 = x_2,\ \dot{x}_2 = -30/7x_1 + 1/7u,\\ \dot{x}_3 = x_4,\ \dot{x}_4 = -100/21x_1 + 1/21u\end{gathered} \tag{17}$$

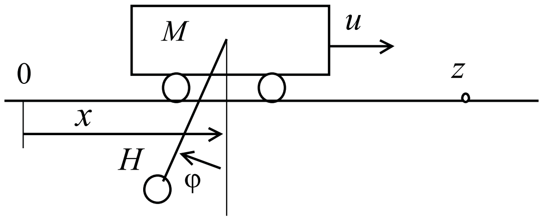

Fig. 6. A crane with its load on the cable. z is a new equilibrium state

Introduce the accompanying optimal control problem

$$\int_0^{\Theta} |u(t) - u_z| dt \to \min,$$
$$\dot{x}_1 = x_2,\ \dot{x}_2 = -30/7x_1 + 1/7u,$$
$$\dot{x}_3 = x_4,\ \dot{x}_4 = -100/21x_1 + 1/21u,$$
$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0,$$
$$x_1(\Theta) = 6,\ x_2(\Theta) = x_3(\Theta) = x_4(\Theta) = 0,$$
$$|u(t)| \le L,\ t \in T = [0, \Theta]. \quad (18)$$

In the given computer experiments two values of Θ were taken: 1) $\Theta = 5$; 2) $\Theta = 10$. The behaviour x is given on Fig. 7, Fig 8 where curves 1 correspond to $\Theta = 5$, curves 2 stand for $\Theta = 10$.

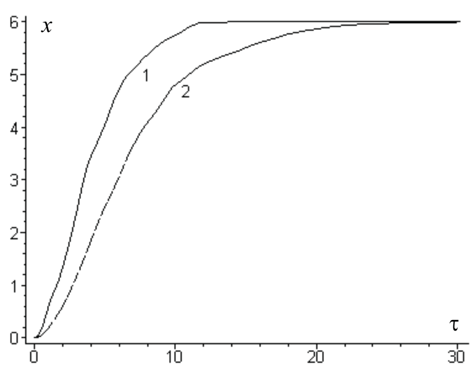

Fig. 7. Behaviour of $x = x_1$ at $\Theta = 5,\ 10$

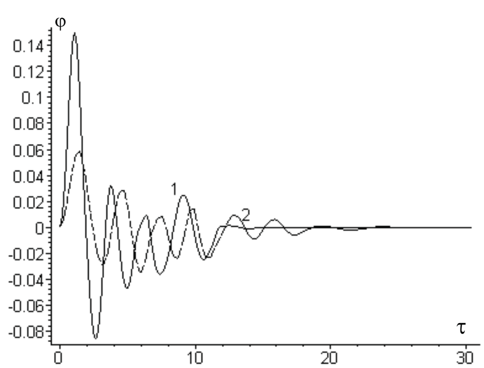

Fig. 8. $\Theta = 5$ (curve 1), $\Theta = 10$ (curve 2)

6. A PROBLEM OF REALIZING GIVEN MOTIONS

Consider dynamical system (1). Let together with system (1) a motion

$$x = x_f(t),\ t \ge 0, \quad (19)$$

be given. Motion (19) is said to be accessible (realizable) if there exists such an admissible control $u_f(t),\ |u_f(t)| \le L,\ t \ge 0$, that $\dot{x}_f(t) = Ax_f(t) + bu_f(t),\ t \ge 0$. Let $G \in R^n$ be a domain of phase space of (1). Suppose $x_f(t) \in intG,\ t \ge 0$.

Definition 4. A function

$$u = u(t, x),\ x \in G,\ t \ge 0 \quad (20)$$

is said to be a bounded feedback which realizes motion (19) if 1) $u(t, x_f(t)) = u_f(t),\ t \ge 0$; 2) $|u(t, x)| \le L,\ x \in G,\ t \ge 0$; 3) the closed system

$$\dot{x} = Ax + bu(t, x),\ x(0) \in G, \quad (21)$$

has a solution $x(t),\ t \ge 0$; 4) the solution $x = x_f(t),\ t \ge 0$, of (21) is asymptotically stable in G.

If the motion $x = \varphi(t),\ t \ge 0$, is a closed curve, then the problem of constructing a bounded stabilizing feedbacks arises at which the closed system has a limiting cycle $\varphi(t),\ t \ge 0$.

Example 5. Consider the control system

$$\dot{x} = y + u, \quad \dot{y} = -x + u, \quad (22)$$

which at $u \equiv 0$ has periodic solutions. Let the closed curve $[x^*(t) - 2]^2 + [y^*(t) - 2]^2 = 1$ be given. A bounded feedback was constructed (Gabasov *et al.*, 1999*a*). After the closure of system (22) this curve became a limiting cycle (Fig. 9).

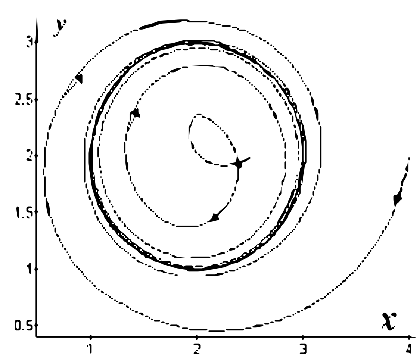

Fig. 9. Closed-loop system (22)

Example 6. Consider the control system

$$\ddot{x} + x = u. \quad (23)$$

At $u(t) \equiv 0,\ t \ge 0$, system (23) has periodic solutions but they are not limiting cycles. Select one curve $x^2(t) + y^2(t) = 1$, $t \geq 0$, $(y = \dot{x})$, and use the method under consideration. Results are given on Fig. 10.

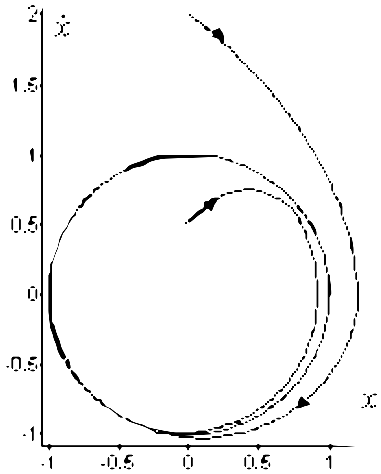

Fig. 10. Closed-loop system (23)

Example 7. Let the system

$$\ddot{x} - \dot{x} + x = u \tag{24}$$

which has no periodic solutions ($u(t) \equiv 0$) and a fixed motion

$$x^2 + \dot{x}^2 = 1 \tag{25}$$

be given. By virtue of using a bounded feedback (Gabasov *et al.*, 1999*a*), motion (25) became the limiting cycle for the system closed by the constructed feedback (Fig. 11).

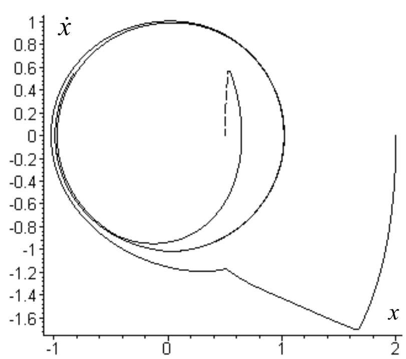

Fig. 11. Closed-loop system (24)

7. OPTIMAL CONTROL FOR NONLINEAR SYSTEMS

Consider the nonlinear optimal control problem

$$c'x(t^*) \to \max,$$

$$\dot{x} = f(x) + bu, \quad x(0) = x_0, \tag{26}$$

$$Hx(t^*) = g, \quad |u(t)| \leq L, \quad t \in T = [0, t^*],$$

where $x = x(t) \in R^n$; $u = u(t) \in R$ is a scalar control from the class of discrete functions.

At the local solution of problem (26) a linear approximation often leads to satisfactory results. In our approach the solution of the problem for this case consists of two procedures: 1) the solution of the linearized optimal control problem, 2) the asymptotic correction of approximate solutions obtained (Gabasov *et al.*, 1998). A peculiarity of the approach is the use as the basis for iterations switching points of the optimal controls. Switching points and the Lagrange multipliers are used at asymptotic expansions of the second procedure. At the global optimization of nonlinear control system (26) the process of solving also consists of two procedures. At first, a set $X \subset R^n$ in which processes of system (26) are studied is represented as a unification of polyhedral sets $X_1, X_2, \ldots, X_p$, such that int $X_i \cap$ int $X_j = \emptyset$, $i \neq j$. The function $f(x)$, $x \in X$, in system (26) is replaced by a continuous function $\hat{f}(x)$, $x \in X$, linear on each set X_j, $j = \overline{1,p}$. A number

$$\delta = \max_{x \in \bar{X}} \|f(x) - \hat{f}(x)\| / \|f(x)\| \tag{27}$$

is said to be an accuracy of approximation. Then the piecewise linear problem of optimal control corresponding to (26) is solved. After that the solution of piecewise linear optimal control problem is corrected by the asymptotic methods elaborated for the piecewise quasilinear control systems (Gabasov *et al.*, 1998). To do this, problem (26) is rewritten in the equivalent form

$$c'x(t^*) \to \max,$$

$$\dot{x} = \hat{f}(x) + \delta g(x) + bu, \quad x(0) = x_0, \tag{28}$$

$$Hx(t^*) = g, \quad |u(t)| \leq 1, \quad t \in T,$$

where $g(x) = (f(x) - \hat{f}(x))/\delta$.

Then problem (28) is imbedded into the family of problems

$$c'x(t^*) \to \max,$$

$$\dot{x} = \hat{f}(x) + \mu g(x) + bu, \quad x(0) = x_0, \tag{29}$$

$$Hx(t^*) = g, \quad |u(t)| \leq 1, \quad t \in T,$$

depending on a small parameter μ.

To construct an asymptotic solution $u^s(t, x, \mu)$, $x \in X_t$, $t \in T$, $\mu \geq 0$, for problem (29) with any degree of accuracy s fast algorithms based on solutions of linear and piecewise linear approximative problems have been elaborated (Gabasov *et al.*, 1998). Computer experiments have shown that even at rather gross approximations of nonlinear elements it is possible to obtain the solution with sufficiently high accuracy.

Example 8. Consider the problem of optimal damping of the pendulum

$$\int_0^{10} u(t)dt \longrightarrow \min,$$
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 + u,$$
$$x_1(0) = 1.5,\ x_2(0) = 0, \quad x_1(10) = x_2(10) = 0, \tag{30}$$
$$0 \le u(t) \le 0.5,\ t \in T = [0, 10],$$

in the domain $X = \{(x_1, x_2):\ |x_1| < \pi/2\}$.

We use two approximations of the nonlinear element $-\sin x_1$:
1) linear approximation $-x_1$, $x \in X$;
2) piecewise linear approximation $(1-4/\pi)x_1 + 1 - \pi/2$, $x \in X_1 = \{(x_1, x_2):\ \pi/4 < x_1 < \pi/2\}$; $-x_1$, $x \in X_2 = \{(x_1, x_2):\ |x_1| < \pi/4\}$. The accuracy (27) of the linear approximation is $\delta_1 = 0.570796$, using the piecewise linear approximation one gets $\delta_2 = 0.110721$.

Table 2 Open-loop controls

Control	Switching points	Cost function	Endpoint state
$u_1^0(t),\ t \in T$	0.723, 2.419	1.696	0.017
	7.006, 8.702		-0.517
$u_1^1(t),\ t \in T$	1.008, 2.517	1.489	-0.012
	7.547, 9.018		-0.056
$u_2^0(t),\ t \in T$	1.078, 2.593	1.509	-0.014
	7.372, 8.877		-0.112
$u_2^1(t),\ t \in T$	1.065, 2.573	1.496	-0.001
	7.553, 9.037		-0.009
$u^0(t, \delta),\ t \in T$	1.065, 2.574	1.496	10^{-8}
	7.566, 9.049		10^{-8}

Table 2 contains results of open-loop solution to problem (30). Trajectories of system (30) have been constructed by the following controls: 1) $u_1^0(t)$, $t \in T$, is the optimal control of the linear base problem (Gabasov *et al.*, 1998); 2) $u_1^1(t)$, $t \in T$, is the realization of asymptotically 1-optimal open-loop control for the fixed value $\mu = \delta_1$ in (29); 3) $u_2^0(t)$, $t \in T$, is the optimal control of the piecewise linear base problem; 4) $u_2^1(t)$, $t \in T$, is the realization of asymptotically 1-optimal open-loop control for the fixed value $\mu = \delta_2$; 5) $u^0(t, \delta)$, $t \in T$, is the sample optimal open-loop control of problem (30).

In each case the control has the form

$$u(t) = \begin{cases} 0, & t \in [0, t_1[\cup [t_2, t_3[\cup [t_4, 10[, \\ 0.5, & t \in [t_1, t_2[\cup [t_3, t_4[. \end{cases} \tag{31}$$

Let us construct the positional solution to problem (30). The realization $u^{1*}(t)$, $t \in T$, was constructed by the 1-optimal controller ($h = 0.01$). Necessary values of auxiliary functions (Gabasov *et al.*, 1998) have been constructed by the Fehlberg fourth-fifth order Runge-Kutta method (Forsythe, 1977). The control $u^{1*}(t)$, $t \in T$, has the form (31) with switching points 1.06, 2.58, 7.567630, 9.039940 and the transition instant between domains of linearity $\Theta_1^* = 1.21$ of the piecewise linear approximative function. At the instant $t^* = 10$ the trajectory of system (30) reaches the state $(-0.000129, -0.000455)$, the value of the cost function is 1.496155.

Now we consider the behaviour of the system under the disturbance $w^*(t) = 0.4 \sin 3t$, $t \in [0, 7[$, $w^*(t) \equiv 0$, $t \ge 7$, unknown for the controller:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 + u + w^*(t). \tag{32}$$

Trajectories of system (32) have been constructed under various realizations of the optimal feedback: 1) $u^{1*}(t)$, $t \in T$, has been constructed by the 1-optimal controller with calculating the values of auxiliary functions by the method (Forsythe, 1977); 2) $u_N^{1*}(t)$, $t \in T$, has been constructed by the 1-optimal controller with the use of the middle box quadrature with N knots for calculating the values of auxiliary functions for $N = 10, 50, 100, 300$. The constructed controls have form (31). For each of them the realized value of the transition instant is $\Theta_1^* = 1.39$.

Table 3 Closed-loop controls

Control	Switching points	Cost function	Endpoint state
$u^{1*}(t)$,	1.3, 2.7	1.449196	0.00002
$t \in T$	7.7894, 9.2877		-0.00119
$u_{10}^{1*}(t)$,	1.3, 2.7	1.469973	-0.00372
$t \in T$	7.7180, 9.2579		-0.00282
$u_{50}^{1*}(t)$,	1.3, 2.7	1.449858	-0.00007
$t \in T$	7.7785, 9.2782		-0.00123
$u_{100}^{1*}(t)$,	1.3, 2.7	1.449955	10^{-6}
$t \in T$	7.7784, 9.2783		-0.00120
$u_{300}^{1*}(t)$,	1.3, 2.7	1.449982	0.00001
$t \in T$	7.7784, 9.2784		-0.00113

Table 3 contains switching points of the realizations, the corresponding values of the cost function and the endpoint states of system (32).

Example 9. Consider the mathematical model of an inverted pendulum which is controlled by horizontal movements of the pin (Åström and Furuta, 2000)

$$\ddot{x} - \sin x + u \cos x = 0. \tag{33}$$

Here x is an angle between the vertical and the pendulum. Let $x = x_1$, $\dot{x} = x_2$. The state $(\pi, 0)$ is the lower stable equilibrium of (33), $(0, 0)$ is the upper unstable one. Denote

$$X = \{(x_1, x_2):\ -\pi \le x_1 \le \pi\} \tag{34}$$

and divide domain (34) into subdomains $X_1 = \{(x_1, x_2):\ -\pi \le x_1 \le -3\pi/4\}$, $X_2 = \{(x_1, x_2):$

$-3\pi/4 \le x_1 \le -\pi/2\}$, $X_3 = \{(x_1,x_2): -\pi/2 \le x_1 \le -\pi/4\}$, $X_4 = \{(x_1,x_2): -\pi/4 \le x_1 \le \pi/4\}$, $X_5 = \{(x_1,x_2): \pi/4 \le x_1 \le \pi/2\}$, $X_6 = \{(x_1,x_2): \pi/2 \le x_1 \le 3\pi/4\}$, $X_7 = \{(x_1,x_2): 3\pi/4 \le x_1 \le \pi\}$. In computer experiments a piecewise linear approximation for $\sin x$ and a piecewise constant approximation for $\cos x$ were used.

Let the initial state $x_0 = (\pi, 0)$, values L, Θ, h be given. The problem is to transfer the pendulum to the upper unstable state $(0,0)$ and stabilize it. Consider the auxiliary optimal control problem

$$\int_0^\Theta |u(t)|dt \longrightarrow \min_{u,\Theta_1,\Theta_2,\Theta_3}, \qquad (35)$$

$$\ddot{x}^1 = -x^1 + \pi - u,\ x^1(0) = z_1, \dot{x}^1(0) = z_2,$$
$$t \in [0,\Theta_1];$$
$$\ddot{x}^2 = (1-\pi/4)x^2 + 3/2 - \pi/2 - u(1-4/\pi),$$
$$t \in [\Theta_1,\Theta_2];$$
$$\ddot{x}^3 = (\pi/4-1)x^3 + \pi/2 - 1 + u(4/\pi - 1),$$
$$t \in [\Theta_2,\Theta_3];$$
$$\ddot{x}^4 = x^4 - u,\ t \in [\Theta_3,\Theta];$$
$$x^1(\Theta_1) = \pi/4,\ x^2(\Theta_2) = \pi/2,\ x^3(\Theta_3) = 3\pi/4,$$
$$x^4(\Theta) = 0,\ \dot{x}^4(\Theta) = 0;\ |u(t)| \le L,\ t \in T,$$

where

$$\Theta_1^0 = \Theta_1^0(0) = \Theta_1^0(x(0),\dot{x}(0)),$$
$$\Theta_2^0 = \Theta_2^0(0) = \Theta_2^0(x(0),\dot{x}(0)),$$
$$\Theta_3^0 = \Theta_3^0(0) = \Theta_3^0(x(0),\dot{x}(0))$$

are optimal moments of crossing boundaries of the corresponding domains $I: X_4,\ II: X_5,\ III: X_6,\ IV: X_7,\ I \to II,\ II \to III,\ III \to IV$.

At testing the algorithm (Gabasov *et al.*, 1999*a*) the influence of parameters L, Θ was studied. The results are given in Table 4. The values of minimal cost functions are given in the first line (corresponds to the problem of damping the pendulum to (0, 0)) and in the second line (corresponds to the solution of problem (35) which gives a realization of the feedback on the interval $[0,20]$).

On Fig. 12 the trajectories of the closed-loop system for different L, Θ are pictured. A curve number corresponds to a number given in Table 4. The feedback realized for the piecewise linear approximation of (33) by solving problem (35) manages to damp out oscillations for nonlinear model (33). The corresponding phase trajectories are presented on Fig. 13 for $\Theta = 3,\ L = 4$. Both the piecewise linear system and the original nonlinear system were closed by the optimal feedback for (35). The curve 1 and the curve 2 correspond to them. It can be seen from Table 4 that inequality (10) holds.

Table 4 Influence of $\Theta,\ L$

$\Theta(h)$	$L=4$	$L=2$	$L=1$
3 (0.1)	40.54041		
	21.77021_1		
4 (0.1)	25.44218		
	21.64750_1		
5 (0.2)	16.34493	17.79221	
	15.41292_1	16.57428_1	
6 (0.2)	15.72967	16.70155	
	14.41151_1	16.41495_1	
8 (0.2)	15.45285	16.44656	16.90072
	15.41129_1	16.10531_2	16.45942_2
10 (0.4)	12.24193	13.22911	13.61861
	12.23775_1	13.10527_2	13.45920_2
12 (0.4)	12.23832	12.62847	12.92553
	12.23775_1	12.59836_2	12.89541_2
14 (0.4)	12.23457	12.57533	12.75824
	12.23331_2	12.56470_2	12.74239_2
16 (0.4)	12.23309	12.56614	12.74453
	12.22891_3	12.56647_3	12.742238_3
18 (0.4)	12.22947	12.46490	12.74237
	12.22891_3	12.46470_3	12.742238_3

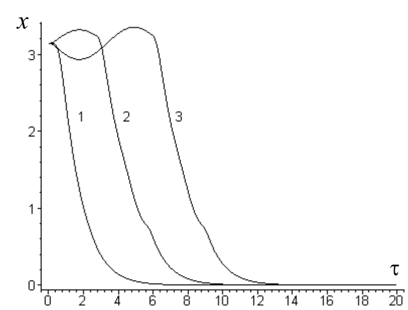

Fig. 12. Trajectories of closed-loop system (33)

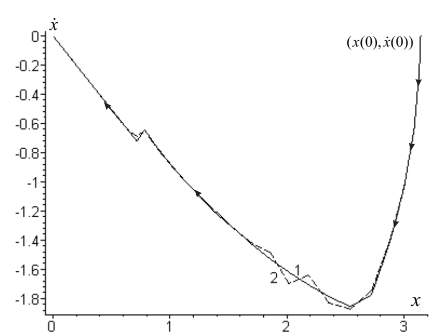

Fig. 13. Phase trajectories of closed-loop system (33) and its piecewise linear approximation

8. CONCLUSION

A method of constructing optimal feedbacks (positional solutions) is applied to the solution of classical problems of regulation. Several control, damping and stabilization problems are considered supplemented by the following examples: damping of two-mass oscillating system, stabilization of oscillating systems, the classical problem of regulation, realization of given motions (or limiting cycles), damping of pendulums. The optimal feedbacks suggested have been tested on invariance and robustness. These feedbacks preserve their quality at the large variations of parameters of control systems and under influence of considerable disturbances.

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