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A MIN-MAX PREDICTIVE CONTROL ALGORITHM FOR UNCERTAIN NORM-BOUNDED LINEAR SYSTEMS

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Abstract: A novel robust predictive control algorithm for input-saturated uncertain linear discrete-time systems with structured norm-bounded uncertainties is presented. The solution is based on the minimization, at each time instant, of a LMI convex optimization problem obtained by a recursive use of the S-procedure. The general case of N free moves is presented. Stability and feasibility are proved and comparisons with robust multi-model (polytopic) MPC algorithms are also presented via an example.

Keywords: Uncertain linear systems, Predictive control, Constraint satisfaction problems, Convex programming, Minimax techniques.

1. INTRODUCTION

Model predictive control (MPC) has become an attractive feedback strategy, especially for linear plants subject to input and state/output inequality constraints (Rawlings and Muske, 1993). More recently, attempts at extending the basic strategy to uncertain linear systems have also been accomplished in (Kothare *et al.*, 1996). The key idea used there was to explicitly take into account the plant uncertainty by resorting to a min-max cost index (minimizing the worst-case value of the objective function, where the worst case is taken over the set of all admissible plant uncertainty). While most of the robust MPC literature deals with polytopic or multi-model uncertain linear systems (see also (Casavola *et al.*, 2000), (Scuurmans and Rossiter, 2000) and references therein), in this paper, instead, we propose a robust MPC strategy for uncertain norm-bounded linear systems. The main defect of polytopic MPC schemes is in their large numerical burdens. It is well known, in fact, that the number of constraints grow exponentially with the control horizon N. On the contrary, it will be shown that such a growth in only linear in the proposed Norm-Bounded (NB) MPC scheme while the control performance remains essentially the same. On this subject, Kothare et al. (Kothare *et al.*, 1996) gave the first constructive solution for the case $N = 0$. More recently, Primbs and Nevistic (Primbs and Nevistić., 2000) developed robustness analysis tools for optimization-based control strategies, postulating the existence of robust MPC schemes for NB uncertainty. However, no algorithms were there presented. Therefore, at the best of authors' knowledge, this is the first algorithm that solve the problem for arbitrary control horizons N. The method is based on the minimization, at each time step, of an upper bound of the worst-case infinite horizon quadratic cost under LMI constraints derived by a recursive use of the S-procedure (Yakubovich, 1992). It is found that the number of LMI to be considered grows linearly with the control horizon N. Finally, closed-loop stability and feasibility properties are proved and comparisons with polytopic MPC schemes are provided via an example.

2. PROBLEM FORMULATION

Consider the following discrete-time uncertain system with uncertainties or perturbations appearing in the feedback loop

$$\Sigma : \begin{cases} x(t+1) & = \Phi x(t) + G u(t) + B_p p(t) \\ y(t) & = C x(t) \\ q(t) & = C_q x(t) + D_q u(t) \\ p(t) & = (\Delta q)(t) \end{cases} \tag{1}$$

with $x \in \mathbb{R}^{n_x}$ denoting the state, $u \in \mathbb{R}^{n_u}$ the control input, $y \in \mathbb{R}^{n_u}$ the output, $p, q \in \mathbb{R}^{n_p}$ additional variables accounting for the uncertainty. The uncertain operator Δ is block-diagonal $\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \ldots, \Delta_r)$, with $\Delta_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $i = 1, \ldots, r$. Note that Δ can represent either a memoryless time-varying matrix with $\|\Delta(t)\|_2 \leq 1$, $\forall t$, or a convolution operator with the operator norm induced by the truncated ℓ_2-norm less than 1, viz.

$$\sum_{j=0}^{t} p_i(j)^T p_i(j) \leq \sum_{j=0}^{t} q_i(j)^T q_i(j), \forall t \geq 0. \tag{2}$$

$i = 1, \ldots, r$. It is assumed that the plant input is subject to the following saturation-type constraints

$$|u_j(k)| \leq u_{j,\max}, k \geq 0, j = 1, 2, \ldots, n_u. \tag{3}$$

The objective is to determine a state-feedback control law

$$u(t) = g(x(t)), \tag{4}$$

such that the system (1) under the input constraint (3) is asymptotically stable. We recall that the system Σ is robustly stabilizable by a constant state-feedback gain K if all the closed loop trajectories of (1) converge to zero as $t \to \infty$. From the literature (Boyd *et al.*, 1994), it is well known that, when a linear state-feedback control law is chosen, viz. $u = Kx$, the system (1) is quadratically stabilizable if there exist a matrix $P = P^T > 0$ and a set of scalars $\lambda_1, \lambda_2, \ldots, \lambda_r > 0$ such that the following matrix inequality is satisfied

$$\begin{bmatrix} \begin{pmatrix} \Phi_K^T P \Phi_K - P \\ + C_{q,K}^T \Lambda C_{q,K} \end{pmatrix} & \Phi_K^T P B_p \\ B_p^T P \Phi_K & B_p^T P B_p - \Lambda \end{bmatrix} \leq 0 \tag{5}$$

where $\Phi_K \triangleq \Phi + GK$, $C_{q,K} \triangleq C_q + D_q K$ and $\Lambda \triangleq \mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \ldots, \lambda_r I_{n_r})$. In order to solve the design problem (4) in a receding horizon fashion, we will denote as $v(\cdot|t) \triangleq \{v(t+k|t\}_{k=0}^{\infty}$ the k-steps ahead prediction of a generic system variable v when the actual state is $\hat{x}(t|t) \triangleq x(t)$ and the plant input is $\hat{u}(\cdot|t)$. Moreover, let $C(P, \rho) \triangleq \{x \in \mathbb{R}^n \,| x^T P x \leq \rho\}$ denote ellipsoidal sets which will be used in the sequel to define robust positively invariant regions for the predicted states of the uncertain system. The following input strategy will be adopted

$$u(\cdot|t) = \begin{cases} \hat{u}(t+k|t) \triangleq K\hat{x}(t+k|t) + \hat{c}(t+k|t), & k = 0, 1, \ldots, N-1, \\ \hat{u}(t+k|t) \triangleq K\hat{x}(t+k|t), & k \geq N, \end{cases}, \tag{6}$$

where $\hat{c}(\cdot|t)$ denotes N free terms and

$$\hat{x}(\cdot|t) \triangleq \Phi_K^k \hat{x}(t|t) + \sum_{i=0}^{k-1} \Phi_K^{k-1-i} (G\hat{c}(t+i|t) + B_p \hat{p}(t+i|t)) \tag{7}$$

the set-valued state predictions, computed under the condition $\hat{p}(t+i|t) \in S_{\hat{p}(t+i|t)}$ $i = 0, 1, ..., N-1$, $S_{\hat{p}(t+i|t)}$ being the set of all admissible perturbations along the system trajectories corresponding to command sequences (6). It results that

$$S_{\hat{p}(t+i|t)} \triangleq \{\hat{p}(t+k|t) \mid \|\hat{p}(t+k|t)\|_2 \leq \\ \|C_{q,K}\hat{x}(t+k|t) + D_q\hat{c}(t+k|t)\|_2\} \tag{8}$$

Our control strategy will consist in determining, at each instant $t \in \mathbb{Z}_+$, an instance of (6) that satisfies the saturation constraint (3) and minimizes the following minmax quadratic index

$$V(x(t), P, \hat{c}(\cdot|t)) \triangleq \sum_{k=0}^{N-1} \max_{\hat{p}(t+k|t) \in S_{\hat{p}(t+k|t)}} \|\hat{x}(t+k|t)\|_{R_x}^2 + \|\hat{c}(t+k|t)\|_{R_u}^2 \\ + \max_{p(t+N|t) \in S_{\hat{p}(t+N|t)}} \|\hat{x}(t+N|t)\|_{P(t)}^2, \tag{9}$$

where $\|x\|_Q^2 \triangleq x' Qx$ and P is computed at each time $t = 0$ on the basis of the initial state $x(0)$ and satisfies next LMI conditions (11)-(14). When $N = 0$, the problem has been solved in (Kothare *et al.*, 1996): at each time step t solve

$$\min_{\rho, Q, Y, \Lambda} \rho \tag{10}$$

subject to

$$\begin{bmatrix} 1 & x(t)^T \\ x(t) & Q \end{bmatrix} \geq 0, \tag{11}$$

$$\begin{bmatrix} Q & * & * & * & * \\ R_u^{1/2} Y & \rho I_{n_u} & 0 & 0 & 0 \\ R_x^{1/2} Q & 0 & \rho I_{n_x} & 0 & 0 \\ C_q Q + D_{qu} Y & 0 & 0 & \Lambda & 0 \\ \Phi Q + GY & 0 & 0 & 0 & Q - B_p \Lambda B_p^T \end{bmatrix} \geq 0 \tag{12}$$

(asterisk denotes the corresponding transposed element), where

$$\Lambda = \mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \ldots, \lambda_r I_{n_r}) > 0, \tag{13}$$

$$\begin{bmatrix} X & Y \\ Y^T & Q \end{bmatrix} \geq 0, \text{ with } X_{jj} \leq u_{j,\max}^2, \, j = 1, 2, \ldots, n_u, \tag{14}$$

$P \triangleq \rho Q^{-1}$, $K = YQ^{-1}$.

In the sequel, for simplicity, we will consider the case of a single uncertain block, viz. $r = 1$ or $\Delta = \Delta_1$. The extension to the general case is direct and will be presented elsewhere.

3. UPPER BOUND DERIVATION

In order to determine a suitable upper-bound to the cost rewrite temporarily the right-hand term of (9) as

$$J_{(x(0), P, c(\cdot|0))} = x(0)^T R_x x(0) + \sum_{i=0}^{N-2} \left\{ \hat{x}^T(i+1|0) R_x \hat{x}(i+1|0) + \hat{c}(i)^T R_u \hat{c}(i) \right\} \\ + \hat{x}^T(N|0) P \hat{x}(N|0) + \hat{c}(N-1)^T R_u \hat{c}(N-1), \tag{15}$$

where $\hat{c}(i) \triangleq \hat{c}(i|0)$, $R_x = C^T R_y C$, $R_y = R_y^T > 0$, $R_u = R_u^T > 0$. Then, the following upper bound for (9) will be derived

$$\bar{J} \triangleq x^T(0) R_x x(0) + \sum_{i=0}^{N-1} J_i, \tag{16}$$

such that

$$\max_{\hat{p}(i) \in S_{\hat{p}(i)}} \hat{x}^T(i+1|0) R_x \hat{x}(i+1|0) + \hat{c}^T(i) R_u \hat{c}(i) \leq J_i, \tag{17}$$

$i = 0, 1, \ldots, N-2$ and

$$\max_{\hat{p}(N) \in S_{\hat{p}(N)}} \hat{x}^T(N|0) P \hat{x}(N|0) + \hat{c}^T(N-1) R_u \hat{c}(N-1) \leq J_{N-1}, \tag{18}$$

so that the optimal synthesis problem stated in the previous section can be rephrased as the following guaranteed cost problem

Problem 1. Find a sequence of inputs $u(\cdot|0)$ such that the upper bound $\bar{J}$ is minimized.

In order to shorten the notational burden we will define the following matrices, $\Theta_G^k \triangleq \left[\, \Phi_K^{k-1} G \;\; \Phi_K^{k-2} G \;\; \dots \;\; G \,\right]$, $\Theta_{B_p}^k \triangleq \left[\, \Phi_K^{k-1} B_p \;\; \Phi_K^{k-2} B_p \;\; \dots \;\; B_p \,\right]$, $k = 1, \dots, N$,

$$\chi_k \triangleq \left[\, x(0)^T \;\; \hat{c}(0)^T \;\; \hat{c}(1)^T \;\; \dots \;\; \hat{c}(k)^T \;\; \hat{p}(0)^T \;\; \hat{p}(1)^T \;\; \cdots \;\; \hat{p}(k)^T \,\right]^T, \quad (19)$$

$$Y^k \triangleq \left[\, \Phi_K^k \;\; \Theta_G^k \;\; \Theta_{B_p}^k \,\right], \quad (20)$$

$$\begin{aligned}
H_{k-1}^k &\triangleq \left[\, C_q\Phi_K^{k-1} \;\; C_q\Theta_G^{k-1} \;\; D_q \;\; C_q\Theta_{B_p}^{k-1} \;\; 0_{n_q\times n_p} \,\right] \\
H_{k-2}^k &\triangleq \left[\, C_q\Phi_K^{k-2} \;\; C_q\Theta_G^{k-2} \;\; D_q \;\; 0_{n_q\times n_u} \;\; C_q\Theta_{B_p}^{k-2} \;\; 0_{n_q\times 2n_p} \,\right] \\
&\;\;\vdots \\
H_0^k &\triangleq \left[\, C_q \;\; D_q \;\; 0_{n_q\times((k-1)n_u+kn_p)} \,\right] \\
E_u^k &\triangleq \operatorname{diag}\left(0_{(n_x+(k-1)n_u)\times(n_x+(k-1)n_u)}, R_u, 0_{kn_p\times kn_p}\right) \quad (21) \\
E_{p,k-1}^k &\triangleq \operatorname{diag}\left(0_{(n_x+kn_u)\times(n_x+kn_u)}, 0_{(k-1)n_p\times(k-1)n_p}, I_{n_p}\right) \\
E_{p,k-2}^k &\triangleq \operatorname{diag}\left(0_{(n_x+kn_u)\times(n_x+kn_u)}, 0_{(k-2)n_p\times(k-2)n_p}, I_{n_p}, 0_{n_p\times n_p}\right) \\
&\;\;\vdots \\
E_{p,0}^k &\triangleq \operatorname{diag}\left(0_{(n_x+kn_u)\times(n_x+kn_u)}, I_{n_p}, 0_{(k-1)n_p\times(k-1)n_p}\right)
\end{aligned}$$

3.1 *Conditions for J_0*

Given $x(0)$, the following inequality

$$\max_{p(0)\in S_{p(0)}} \hat{x}(1|0)^T R_x \hat{x}(1|0) + \hat{c}(0)^T R_u \hat{c}(0) \le J_0 \quad (22)$$

must be satisfied, where $\hat{x}(1|0)$ denotes the one-step prediction. The inequality (22) is convex and its extremum is reached on the boundary of $S_{p(0)}$, say it $\partial S_{p(0)}$, where

$$\|\hat{p}(0)\|_2 = \left\|C_q x(0) + D_q \hat{c}(0)\right\|_2$$

Note that the previous equality can be rewritten as

$$\chi_0^T \left(H_0^{0,T} H_0^0 - E_{p,0}^0\right) \chi_0 \triangleq \chi_0^T D_{p(0)}^0 \chi_0 = 0\,. \quad (23)$$

By observing that $\hat{x}(1|0) = \Phi_K x(0) + G\hat{c}(0) + B_p \hat{p}(0)$, the argument of (22) becomes

$$\chi_0^T \left(Y^{0,T} R_x Y^0 - E_u^0\right) \chi_0 \triangleq \chi_0^T D_{J_0}^0 \chi_0\,,$$

and the related inequality to be satisfied is

$$\chi_0^T D_{J_0}^0 \chi_0 \le J_0\,. \quad (24)$$

As a consequence, the conditioned inequality (22) can be equivalently stated as

$$\begin{cases} J_0 - \chi_0^T D_{J_0}^0 \chi_0 \;\;\ge\;\; 0 \\ \\ \forall \chi_0 \in \mathbb{R}^{n_x+n_u+n_p} \;\; \text{s.t.} \\ \chi_0^T D_{p(0)}^0 \chi_0 \;\;=\;\; 0\,. \end{cases} \quad (25)$$

By applying the S-procedure (see (Yakubovich, 1992)), (25) is satisfied if there exists a no-zero scalar $\tau_{p(0)}^0 \in \mathbb{R}$ such that

$$J_0 - \chi_0^T \left(D_{J_0}^0 + \tau_{p(0)}^0 D_{p(0)}^0\right) \chi_0 \ge 0\,. \quad (26)$$

$\forall \chi_0 \in \mathbb{R}^{n_x+n_u+n_p}$. The scalar is chosen by solving off-line the following GEVP problem

$$\begin{cases} \min\limits_{\tau_{p(0)}^0 \neq 0} \quad \bar{\lambda}\left(D_{J_0}^0 + \tau_{p(0)}^0 D_{p(0)}^0\right) \\ \text{subject to} \\ \qquad \left(D_{J_0}^0 + \tau_{p(0)}^0 D_{p(0)}^0\right) > 0\,. \end{cases} \quad (27)$$

Once such a coefficient is determined, the following equivalent LMI condition

$$\begin{bmatrix} J_0 & \chi_0^T \\ \chi_0 & \left(D_{J_0}^0 + \tau_{p(0)}^0 D_{p(0)}^0\right)^{-1} \end{bmatrix} \ge 0\,. \quad (28)$$

results for (26) from the use of the Schur's complements.

3.2 *Conditions for J_1*

LMI (28) allows one to define the set Z_1^1, an ellipsoidal set that provides an outer approximation of the set of all admissible one-step ahead state predictions $\hat{x}(1|0)$, as follows

$$Z_1^1 \triangleq \left\{\hat{x}(1|0) \;\middle|\; \hat{x}(1|0)^T R_x \hat{x}(1|0) + \hat{c}(0)^T R_u \hat{c}(0) \le J_0 \right\}. \quad (29)$$

The above set can be equivalently formulated in terms of χ_0 as

$$\tilde{Z}_1^1 \triangleq \left\{\chi_0 \;\middle|\; \chi_0^T \left(D_{J^0}^0 + \tau_{p(0)}^0 D_{p(0)}^0\right) \chi_0 \le J_0 \right\} \quad (30)$$

Then, the condition which must be imposed on J_1 is

$$\max_{\substack{\hat{p}(1)\in\partial S_{p(1)} \\ \hat{p}(0)\in\partial S_{p(0)} \\ \hat{x}(1|0)\in\partial Z_1^1}} \left\{\hat{x}(2|0)^T R_x \hat{x}(2|0) + \hat{c}(1)^T R_u \hat{c}(1)\right\} \le J_1\,, \quad (31)$$

where $\hat{x}(2|0) = \left[\, \Phi_K^2 \;\; \Phi_K G \;\; G \;\; \Phi_K B_p \;\; B_p \,\right] \chi_1$ represents two-steps ahead state predictions. The argument of (31) can be rewritten as

$$\chi_1^T \left(Y^{2,T} R_x Y^2 + E_{u,1}^2\right) \chi_1 = \chi_1^T D_{J_1}^1 \chi_1\,,$$

and the conditions $\hat{p}(1) \in \partial S_{p(1)}$, and $\hat{p}(0) \in \partial S_{p(0)}$ become

$$\chi_1^T \left(H_1^{2,T} H_1^2 - E_{p,1}^2\right) \chi_1 \triangleq \chi_1^T D_{p(1)}^1 \chi_1,$$

$$\chi_1^T \left(H_0^{2,T} H_0^2 - E_{p,0}^2\right) \chi_1 \triangleq \chi_1^T D_{p(0)}^1 \chi_1\,.$$

The condition on $\hat{x}(1|0) \in \partial Z_1^1$ can be expressed as $\chi_0^T \left(D_{J^0}^0 + \tau_{p(0)}^0 D_{p(0)}^0\right) \chi_0 = J_0$. This equation can be rewritten in terms of χ_1 by suitably adding zero blocks inside $\left(D_{J^0}^0 + \tau_{p(0)}^0 D_{p(0)}^0\right)$ becoming $\chi_1^T D_{J(0)}^1 \chi_1 = J_0$. Then, condition (31) is true if

$$\begin{cases} \chi_1^T D_{J_1}^1 \chi_1 \le J_1 \;\; \forall \chi_1 \;\; \text{s.t.} \\ \\ \qquad \chi_1^T D_{J_0}^1 \chi_1 = J_0 \\ \qquad \chi_1^T D_{p(1)}^1 \chi_1 = 0 \\ \qquad \chi_1^T D_{p(0)}^1 \chi_1 = 0\,. \end{cases}$$

This conditioned inequality can be reduced to an unconditioned one via the S-procedure by finding no-zero scalars $\tau_{J_0}^1, \tau_{p(1)}^1, \tau_{p(0)}^1 \in \mathbb{R}$ such that

$$J_1-\chi_1^T\left(D^1_{J_1}+\tau^1_{p(1)}D^1_{p(1)}+\tau^1_{p(0)}D^1_{p(0)}\right)\chi_1+\tau^1_{J_0}\left(J_0-\chi_1^TD^1_{J_0}\chi_1\right)\geq 0\,. \quad (32)$$

The scalars $\tau^1_{J_0},\tau^1_{p(1)},\tau^1_{p(0)}$ can be chosen by solving off-line the following convex optimization problem

$$\begin{cases} \min\limits_{\tau^1_{J_0},\tau^1_{p(1)},\tau^1_{p(0)}\neq 0} & \bar{\lambda}\left(D^1_{J_1}+\tau^1_{p(1)}D^1_{p(1)}+\tau^1_{p(0)}D^1_{p(0)}+\tau^1_{J_0}D^1_{J_0}\right) \\ \text{subject to} & \\ & \left(D^1_{J_1}+\tau^1_{p(1)}D^1_{p(1)}+\tau^1_{p(0)}D^1_{p(0)}+\tau^1_{J_0}D^1_{J_0}\right)>0\,. \end{cases} \quad (33)$$

Once such scalars have been determined, the inequality (32) can be rewritten as an LMI constraint using the Schur's complements as follows

$$\begin{bmatrix} J_1+\tau^1_{J_0}J_0 & \chi_1^T \\ \chi_1 & \left(D^1_{J_1}+\tau^1_{p(1)}D^1_{p(1)}+\tau^1_{p(0)}D^1_{p(0)}+\tau^1_{J_0}D^1_{J_0}\right)^{-1}\end{bmatrix}\geq 0\,. \quad (34)$$

3.3 *Conditions for J_k*

For the k-th term we have to satisfy

$$\max_{\substack{\hat{p}(i)\in\partial S_{p(i)}\\ \hat{x}(j|0)\in\partial Z^k_{j-1}}}\left\{\hat{x}(k|0)^TR_x\hat{x}(k|0)+\hat{c}(k-1)^TR_u\hat{c}(k-1)\right\}\leq J_k\,, \quad (35)$$

$i=0,\ldots,k-1,\ j=1,\ldots,k$. The k-steps ahead state prediction is given by

$$\hat{x}(k|0)=\left[\,\Phi^k_K\ \Phi^{k-1}_KG\ \ldots\ G\ \Phi^{k-1}_KB_p\ \ldots\ B_p\,\right]\chi_k\,.$$

The constraint $\hat{p}(k-1)\in\partial S_{p(k-1)}$ can be expressed as

$$\chi_k^TD^k_{p(k-1)}\chi_k=0,$$

where $D^k_{p(k-1)}\triangleq H^{k,T}_{k-1}H^k_{k-1}-E^k_{p,k-1}$. By following the previous reasoning scheme, the condition $\hat{p}(k-2)\in\partial S_{p(k-2)}$ becomes

$$\chi_k^TD^k_{p(k-2)}\chi_k=0,$$

where $D^k_{p(k-2)}\triangleq H^{k,T}_{k-2}H^k_{k-2}-E^k_{p,k-2}$, and so on until $p(0)\in\partial S_{p(0)}$, which is equivalently stated as

$$\chi_k^TD^k_{p(0)}\chi_k=0,$$

with $D^k_{p(0)}$ obtained as before. In order to better understand the constraints related to the state predictions at the steps $k-2,k-3,\ldots,1$, we will start with the constraint $\hat{x}(1|0)\in\partial Z^k_1$. This constraint, by considering the expression for ∂Z^1_1 from (29), translates into the following equality constraint

$$\chi_0^T\left(D^0_{J^0}+\tau^0_{p(0)}D^0_{p(0)}\right)\chi_0=0.$$

By adding a proper number of zero blocks to

$$D^0_{J^0}+\tau^0_{p(0)}D^0_{p(0)},$$

we obtain that the condition $\hat{x}(1|0)\in\partial Z^k_1$ can be expressed as

$$\chi_k^TD^k_{J_0}\chi_k=J_0\triangleq J^k_0.$$

Further, we have to satisfy $\hat{x}(2|0)\in\partial Z^k_2$ that, by considering (32), can be expressed in terms of χ_2 as

$$\chi_2^T\left(D^1_{J_1}+\tau^1_{p(1)}D^1_{p(1)}+\tau^1_{p(0)}D^1_{p(0)}+\tau^1_{J_0}D^1_{J_0}\right)\chi_2=J_1+\tau^1_{J_0}J_0\triangleq J^k_1\,.$$

By adding a proper number of zero blocks to the left hand matrix, the previous equality can be written in terms of χ_k as

$$\chi_k^TD^k_{J_1}\chi_k=J^k_1.$$

Iteratively, we can obtain the condition for $\hat{x}(2|0)\in\partial Z^k_{k-1}$ by considering that, in terms of χ_{k-1} we have

$$\chi_{k-1}^T\left(D^{k-1}_{J_{k-1}}+\sum_{i=0}^{k-2}\tau^{k-1}_{p(i)}D^{k-1}_{p(i)}+\sum_{i=0}^{k-2}\tau^{k-1}_{J_i}D^{k-1}_{J_i}\right)\chi_{k-1}$$
$$=J_{k-1}+\sum_{i=0}^{k-2}\left(\prod_{j=0}^{i}\tau^{k-2-j}_{J_{k-2-j}}\right)J_{k-2-i}\triangleq J^k_i\,,$$

which, finally, can be expressed in terms of χ_k,

$$\chi_k^TD^k_{J_{k-1}}\chi_k=J^k_i,$$

where $D^k_{J_{k-1}}$ can be obtained from $D^{k-1}_{J_{k-1}}+\sum_{i=0}^{k-2}\tau^{k-1}_{p(i)}D^{k-1}_{p(i)}+\sum_{i=0}^{k-2}\tau^{k-1}_{J_i}D^{k-1}_{J_i}$ by adding proper zero blocks. Finally, the argument of (35) is equal to

$$\chi_k^TD^k_{J_k}\chi_k\,,$$

where

$$D^k_{J_k}\triangleq Y^{k,T}R_xY^k+E^k_{u,k-1}$$

By grouping the inequality which represents (35) together with the constraints written so far, we have to satisfy

$$\begin{cases}\chi_k^TD^k_{J_k}\chi_k\leq J_k\ \forall\chi_k\ \text{s.t.}\\ \chi_k^TD^k_{J_0}\chi_k=J^k_0\\ \chi_k^TD^k_{J_1}\chi_k=J^k_1\\ \cdots\\ \chi_k^TD^k_{J_{k-1}}\chi_k=J^k_{k-1}\\ \chi_k^TD^k_{p(k-1)}\chi_k=0\\ \cdots\\ \chi_k^TD^k_{p(0)}\chi_k=0\,,\end{cases}$$

By applying the S-procedure to this conditioned inequality we arrive, by determining no-zero scalars $\tau^k_{p(0)},\ldots,\tau^k_{p(k-1)},\ \tau^k_{J_0},\ldots,\tau^k_{J_{k-1}}$, to the following LMI constraint

$$\begin{bmatrix} J_k+\sum\limits_{i=0}^{k-1}\left(\prod\limits_{j=0}^{i}\tau^{k-1-j}_{J_{k-1-j}}\right)J_{k-1-i} & \chi_k^T \\ \chi_k & \left(D^k_{J_k}+\sum\limits_{i=0}^{k-1}\tau^k_{p(i)}D^k_{p(i)}+\sum\limits_{i=0}^{k-1}\tau^k_{J_i}D^k_{J_i}\right)^{-1}\end{bmatrix}\geq 0\,. \quad (36)$$

The procedure so explained is valid until $k=N-2$; in the next subsection we will express the conditions for J_{N-1} and for the belonging of the state trajectory at step N into the invariant terminal ellipsoid.

3.4 *Terminal state conditions*

The terminal condition on the state prediction is given by

$$\max_{\substack{\hat{p}(i)\in\partial S_{p(i)}\\ \hat{x}(j|0)\in\partial Z^N_j}}\left\{\hat{x}(N|0)^TP\hat{x}(N|0)+\hat{c}(N-1)^TR_u\hat{c}(N-1)\right\}\leq J_{N-1}\,, \quad (37)$$

$i = 0, \ldots, N-1,\ j = 1, \ldots, N-1$. It is straightforward to observe the similarity with condition (35), the difference being only on the weighting terminal state matrix that is P instead of R_x. By direct substitution, we have that the condition (37) is translated into the following terminal state LMI ($\hat{J}_N \triangleq J_{N-1} + \sum_{i=0}^{N-1} \left(\prod_{j=0}^{i} \tau_{J_{N-1-j}}^{N-1-j} \right) J_{N-1-i}$)

$$\begin{bmatrix} \hat{J}_N & \chi_N^T \\ \chi_N & \left(D_{J_N}^N + \sum_{i=0}^{N-1} \tau_{p(i)}^N D_{p(i)}^N + \sum_{i=0}^{N-1} \tau_{J_i}^N D_{J_i}^N \right)^{-1} \end{bmatrix} \geq 0 . \tag{38}$$

It remains to impose that

$$\hat{x}(N|0) \in C(P, \rho), \forall \hat{x}(N|0) \in \partial Z_N^N , \tag{39}$$

which translates into the following invariant ellipsoid LMI condition

$$\begin{bmatrix} \rho + \tau_F \hat{J}_N & \chi_N^T \\ \chi_N & \left(\tilde{D}_F^N + \tau_F \left(D_{J_N}^N + \sum_{i=0}^{N-1} \tau_{p(i)}^N D_{p(i)}^N + \sum_{i=0}^{N-1} \tau_{J_i}^N D_{J_i}^N \right) \right)^{-1} \end{bmatrix} \geq 0 , \tag{40}$$

where $\tau_F \neq 0$ results from the use of the S-procedure and can be obtained by solving off-line the following GEVP problem

$$\min_{\tau_F \neq 0} \overline{\lambda} \left(\tilde{D}_F^{N-1} + \tau_F D_F^* \right) \text{ subject to } \left(\tilde{D}_F^{N-1} + \theta_F D_F^* \right) > 0 \tag{41}$$

where

$$D_F^* \triangleq \left(\tilde{D}_F^N + \tau_F \left(D_{J_N}^N + \sum_{i=0}^{N-1} \tau_{p(i)}^N D_{p(i)}^N + \sum_{i=0}^{N-1} \tau_{J_i}^N D_{J_i}^N \right) \right).$$

3.5 *Algorithm NB-MPC*

An implementable MPC algorithm which summarizes all the previous conditions is as follows:

1. At time $t = 0$, given $x(0)$, find
$$[Y, Q] \triangleq \arg \min_{Y,Q,\rho,\Lambda} \rho \tag{42}$$
subject to (11), (12), (13) and (14). Compute τ_F as in (41) be used in (40) in step 2);
2. At each time $t \geq 0$ find $\hat{c}^*(t|t)$, $\hat{c}^*(t+1|t)$, $\ldots$, $\hat{c}^*(t+N-1|t)$, the minimizer of
$$\min_{J_i, \hat{c}(t+i|t), \hat{p}(i),\ i=0\ldots N-1} \overline{J} \tag{43}$$
subject to (36) for $k = 0, 1, \ldots, N-2$, (38), (40) and the input constraint $\left| \hat{u}(t+i|t)_j \right| \leq u_{j,\max}$, $i = 0, \ldots, N-1$, $j = 1, 2, \ldots, n_u$;
3. feed the plant by $\hat{u}^*(t|t) = Kx(t|t) + \hat{c}^*(t|t)$;
4. $t \leftarrow t+1$ and go to step 2.

Proposition 1. Let the NB-MPC scheme have solution at time $t = 0$. Then, it has solution at each future time instant t, satisfies the input constraints and yields an asymptotically (quadratically) stable closed-loop system.

Proof: The demonstration is omitted for space limitations. It is standard and can be obtained by following the same arguments used in (Casavola *et al.*, 2000).

4. A NUMERICAL EXPERIMENT

Consider the same two-carts/spring system of (Kothare *et al.*, 1996).

$$\begin{cases} \begin{bmatrix} x_1(\tau+1) \\ x_2(\tau+1) \\ x_3(\tau+1) \\ x_4(\tau+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.1\dfrac{K}{m_1} & 0.1\dfrac{K}{m_1} & 1 & 0 \\ 0.1\dfrac{K}{m_1} & -0.1\dfrac{K}{m_1} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \\ x_3(\tau) \\ x_4(\tau) \end{bmatrix} \\ \qquad + \begin{bmatrix} 0 \\ 0 \\ \dfrac{0.1}{m_1} \\ 0 \end{bmatrix} u(\tau), \\ y(\tau) = x_2(\tau). \end{cases} \tag{44}$$

Here, x_1 and x_2 are the positions of body 1 and 2, and x_3 and x_4 their respective velocities. m_1 and m_2 are the masses of the two bodies and K is the spring constant. For the actual system used in the simulation we consider $m_1 = m_2 = 1$, $K = 1$ with appropriate units. The spring constant is assumed to be uncertain in the range $K_{\min} := 0.25 \leq K \leq 1 =: K_{\max}$. The uncertainty on K is modelled as a norm bounded uncertainty on $\delta = (K - K_{\text{nom}})/K_{\text{dev}}$, $\delta^2 \leq 1$, and we have

$$\Phi = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.1K_{\text{nom}} & 0.1K_{\text{nom}} & 1 & 0 \\ 0.1K_{\text{nom}} & -0.1K_{\text{nom}} & 0 & 1 \end{bmatrix},$$

$$B_p = \begin{bmatrix} 0 \\ 0 \\ -0.1 \\ 0.1 \end{bmatrix},$$

$$C_q = \begin{bmatrix} K_{\text{dev}} & -K_{\text{dev}} & 0 & 0 \end{bmatrix}, D_q = 0$$

where $K_{\text{nom}} = \frac{1}{2}(K_{\max} + K_{\min})$ and $K_{\text{dev}} = \frac{1}{2}(K_{\max} - K_{\min})$. We shall assume that the state is available and the problem consists in unit-step output tracking of y. In all simulations we have used $R_u = 1$, $R_x = H' R_y H$, with $R_y = 1$ and saturation constraints $|u| \leq 0.1$. Fig. 1 shows the output and input for the proposed NB-MPC algorithm for $N = \{1, 2, 3, 4\}$ whereas Figs. 2-3 report comparisons between the NB-MPC and the polytopic MPC scheme A1 of (Casavola *et al.*, 2000) for $N = \{2, 4\}$ respectively. As there clearly results, a similar control performance has been obtained by using the two different descriptions for the uncertain system (structured uncertainty and polytopic one). Moreover, as expected, the use of increasingly larger control horizons improves the control performance at the expenses of increasingly larger computational

Table 1. Comparison of numerical complexity per step

Flops per step	N=1	N=2	N=3	N=4
Algorithm NB-MPC	2167	4578	7828	11996
Algorithm A1	3557	10930	29842	76741

burdens for both types of uncertainty. However, as reported in Table 1 the NB-MPC algorithm shows a remarkable reduction of the computational complexity for all control horizons.

5. CONCLUSIONS

We have presented a new predictive controller which robustly asymptotically stabilizes an input constrained uncertain linear system with norm-bounded uncertainties. The receding horizon control strategy is based on the minimization, at each time instant, of a convex optimization problem costing an upper bound of a minmax quadratic index, under the constraint that all future states are robustly steered within N-steps into a feasible positively invariant set. The S-procedure plays a crucial role in determining the convex constraints of such an optimization problem. A significant reduction of the computational burden and no control performance loss with respect to the polytopic paradigm has been observed from the numerical experiments. This results especially true for large values of the control horizon N and when the number of vertices of the polytopic family is high.

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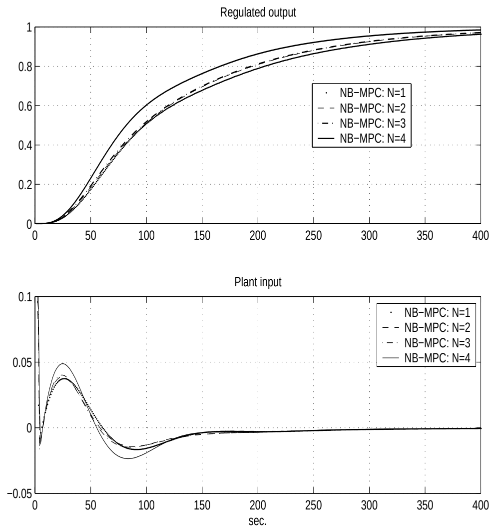

Fig. 1. Regulated plant output and input for $N = \{1, 2, 3, 4\}$

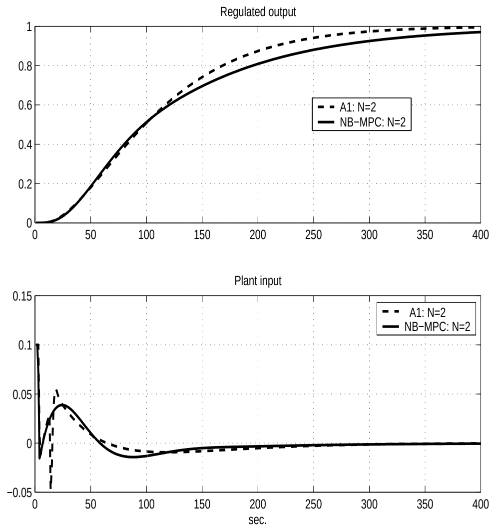

Fig. 2. Polytopic vs Norm-bounded: output and input for $N = 2$

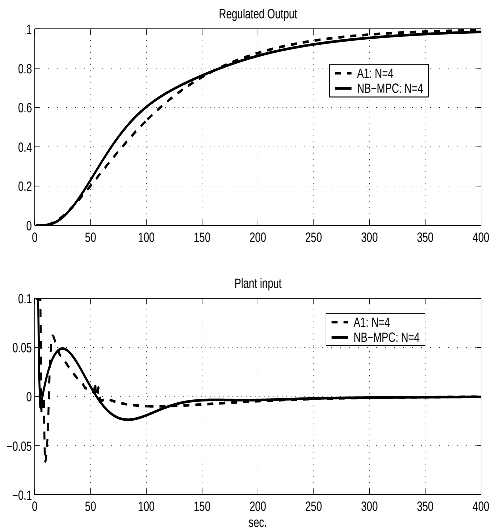

Fig. 3. Polytopic vs Norm-bounded: output and input for $N = 4$