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H∞ CONTROLLER DESIGN FOR SINGULARLY PERTURBED SYSTEMS BY DELTA OPERATOR APPROACH

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Abstract: This paper presents a standard H∞ controller design for singularly perturbed systems with frequency domain by unified approach using the delta operators. Decomposition of the singularly perturbed systems into the fast and slow subsystems is shown. And the delta operator approach is implemented to improve a finite word-length (FWL) characteristics. The delta operator systems have the better FWL characteristics over the shift operator systems. The result is shown in the example.

Keywords: H∞ control, singularly perturbed systems, delta operator, finite word-length characteristics

1. INTRODUCTION

Singularly perturbed systems are decomposed into the fast and slow subsystems. In the continuous time and discrete time domains (Chang, 1974; Kokotovic, 1975; Chow and Kokotovic, 1976; Kokotovic *et al.*, 1986) and Naidu (1988) made valuable contributions in the systems decomposition by matrix block diagonalization, respectively. Luse (1985 and 1986) studied systems decomposition in the frequency domain. It is noted that A. Tikhonov, A.B. Vasileva, L. Fridman, and V.F. Butuzov studied the singularly perturbed systems mathematically. Their papers are too many to list in the reference.

It is shown that the unified approach using the delta operators has improved finite word-length (FWL) characteristics by reducing round off errors where compared with the discrete systems. Middleton and Goodwin made fundamental studies on the unified approach using the delta operators (Middleton and Goodwin, 1986 and 1990). Shim and Sawan (2001 and 2002) studied LQR design and State Feedback Control design for the singularly perturbed delta operator systems.

The shift (q) operators are used to write the discrete systems. But, these systems are inconvenient to use since they are chopped and lengthy compared to the continuous systems. Their resolution of the stability circle is coarse especially where the system poles gather near $1+0j$ point at small sampling interval. They have lager round-off errors.

The delta operator approach alleviates the problems of the discrete systems especially at higher sampling period. It unified both the continuous and the discrete systems; thus, the discrete system is handled and solved like the continuous one.

Zames (1981) first studied H∞ control in the frequency domain. Doyle *et al.* (1989) showed the state space formulations of the H∞ method. Zhou et al. (1995) introduces comprehensive descriptions of the robust control with the optimal sense. Stoorvogel (1992) worked for H∞ control for the discrete systems. Guillard et al. (1996) describe sufficient conditions for the existence and the construction of a feedback law that imposes a prescribed level of disturbance attenuation with internal stability. Loescharataramdee studied a standard H∞ controller design for two-time-scale continuous systems (1997). Collins and Song (1999) developed a method to directly design discrete-time H∞ controllers, represented using the delta operator.

In this paper, the authors extend Loescharataramdee (1997) by implementing a unified approach using the delta operators.

2. DELTA OPERATOR APPROACH

According to Middleton and Goodwin (1990), the delta operator is defined as follow:

$$\delta = \frac{(q-1)}{\Delta}. \qquad (2.1)$$

The shift and the delta operators have the following relations as

$$qx(k) \cong x(k+1),$$
$$\delta x(k) = \frac{qx(k) - x(k)}{\Delta}. \qquad (2.2)$$

where Δ is the sampling interval. Now, consider a linear, time-invariant continuous system

$$\frac{dx}{dt} = Ax(t) + Bu(t). \qquad (2.3)$$

where x is state vector with n by 1 and u is control vector with r by 1 dimensions. A has n by n and B has n by r matrices. The corresponding sampled-data system with zero-order hold and sampling interval Δ is then given by

$$x(k+1) = A_q x(k) + B_q u(k),$$
$$y(k) = C_q x(k), \qquad (2.4)$$

where $A_q = e^{A\Delta},\ B_q = \int_0^{\Delta} e^{A(\Delta-\tau)} B d\tau.$

Eq. (2.4) is rewritten using the relation between q and δ operators as,

$$\delta x(k) = A_\delta x(k) + B_\delta u(k), \qquad (2.5)$$

where $A_\delta = \frac{(A_q - I)}{\Delta},\ B_\delta = \frac{B_q}{\Delta},\ C_\delta = C_q.$

Eqs. (2.3) - (2.5) are written as a comprehensive form as,

$$\rho x(\tau) = A_\rho x(\tau) + B_\rho u(\tau),\ y(\tau) = C_\rho x(\tau). \qquad (2.6)$$

where $A_\rho = \begin{Bmatrix} A \\ A_q \\ A_\delta \end{Bmatrix}, \rho = \begin{Bmatrix} d/dt \\ q \\ \delta \end{Bmatrix},\ time = \begin{Bmatrix} t \\ k \\ \tau \end{Bmatrix}.$

The upper and lower rows denote continuous-time systems and discrete-time systems, respectively. When $\Delta \to 0$, then $A_\delta \to A,\ B_\delta \to B$. This means that, when the sampling time goes to zero, the discrete-like delta expression becomes that of continuous system.

Now, the stability regions for various operators are introduced. For continuous system, operator is d/dt, transform variable is s and stability region Re{s}< 0. For discrete systems, operator is q, transform variable is z and stability region $|z|$ <1. For unified system, operator is δ, transform variable is γ and stability region is

$$\frac{\Delta}{2}|\gamma|^2 + \mathrm{Re}\{\gamma\} < 0. \qquad (2.7)$$

As Δ closes to zero, Eq. (2.7) is identified to that of the continuous system.

Remark 2.1:
When truncated power series is used to evaluate the matrix exponential as,

$$e^{A\Delta} \cong \sum_{k=0}^{N} \frac{(A\Delta)^k}{k!}. \qquad (2.8)$$

selection of the sampling time Δ as in $\|\Delta A\|_2$ should not be close to 1 because of numerical difficulty for computing this finite power series.

3. SINGULARLY PERTURBED SYSTEMS

3.1 Two-Frequency-Scale Systems

One can describe a system by a transfer function matrix H(s,ε) where ε is a small parameter. If there exists an integer m and function f analytic at ε = 0 such that

$$g(\varepsilon) = f(\varepsilon)/\varepsilon^m,\ g \in \Re_\varepsilon. \qquad (3.1)$$

H(s,ε) is required to be rational in s over the field $\Re_\varepsilon$. Also notations are described in (Luse and Khalil ,1985).

Definition 3.1:
A matrix H(s,ε) rational in s over the field $\Re_\varepsilon$ is two-frequency-scale (TFS) if,

(i) H(s,ε) is proper in s, (ii) H(s,0) is defined and proper, (iii) $H(\frac{p}{\varepsilon},\varepsilon)\Big|_{\varepsilon=0}$ is defined and proper, (iv) the following relations hold.

$$s_p(\varepsilon) = \sum_{j=0}^{\infty} b_j \varepsilon^{j/q},$$
$$s_p(\varepsilon) = \frac{1}{\varepsilon}\sum_{j=0}^{\infty} b_j \varepsilon^{j/q},\ \ b_0 \neq 0. \qquad (3.2)$$

where $a_n(\varepsilon)s^n + \ldots + a_1(\varepsilon)s + a_0(\varepsilon) = 0.$

It is noted that (3.2) has $\mathrm{Re}(b_0) < 0$ for stability of H(s,ε).

Corollary 3.1:
If H(s,ε) is two frequency scale, then $H_S(\infty) = H_F(0)$ and

$$\left[H(\frac{p}{\varepsilon},\varepsilon)\Big|_{p=\infty} \right]_{\varepsilon=0} = H_F(p)\Big|_{p=\infty}. \qquad (3.3)$$

S and F denote the slow and the fast, respectively. Time scale expression of the systems are transformed in the frequency domain as a unified transfer matrices as

$$H_s(\gamma_s) = C_{\delta s}(\gamma_s I - A_{\delta s})^{-1} B_{\delta s} + D_{\delta s},$$
$$H_f(\gamma_p) = C_{\delta 2}(\gamma_p I - A_{\delta 22})^{-1} B_{\delta 2} + D_{\delta f}. \quad (3.4)$$

Note that $\gamma_p = \varepsilon \gamma_s$ as high frequency variable.

Proof:
See (Luse and Khalil, 1985).

Lemma 3.1:
H(s,ε) is a stable two-frequency-scale transfer function matrix if and only if its subsystem matrices $H_s(\gamma_s)$ and $H_f(\gamma_p)$ are stable, all lost poles are stable too. While performing system approximation, some poles are lost due to order reduction.

Robustness and sensitivity results for linear feedback systems typically involve properties of stable rational matrices along the imaginary axis. The following theorem shows that under certain stability conditions, the values of $H_s(\gamma_s)$ and $H_f(\gamma_p)$ along the imaginary axis determine a uniform O(ε) approximation of H(s,ε) along the imaginary axis. If such a rational matrix represents a signal gain, then $H_s(j\omega)$ and $H_f(j\varepsilon\omega)$ are approximate signal gains for low and high frequency sinusoidal inputs. The reciprocal of singular value graphs used for robustness evaluation can be approximated from $H_s(\gamma_s)$ and $H_f(\gamma_p)$.

Theorem 3.1:
Let H(s,ε) be a two-frequency-scale rational matrix. Suppose that $H_S(s)$ and $H_F(p)$ have no pure imaginary poles and that $H(s,\varepsilon)$ has no pure imaginary lost poles. Then

$$\sup_{s\in D} \left\| H(s,\varepsilon) - H_S(s) - H_F(\varepsilon s) + W \right\| = O(\varepsilon). \qquad (3.5)$$

where $W = H_S(\infty) = H_F(0)$ holds and ||.|| is some matrix norm, and D is the imaginary axis.

Proof:
See (Luse and Khalil, 1985).

Theorem 3.1 gives an approximation as

$$H(s,\varepsilon) \cong H_S(s) + H_F(p) - H_S(\infty). \qquad (3.6)$$

3.2 Two-Time-Scale Systems

Singularly perturbed systems with noise input are given as,

$$\delta x_1(t) = A_{\delta 11}x_1(t) + A_{\delta 12}x_2(t) + G_{\delta 1}w_1(t) + B_{\delta u1}u(t),$$
$$\varepsilon\delta x_2(t) = A_{\delta 21}x_1(t) + A_{\delta 22}x_2(t) + G_{\delta 2}w_1(t) + B_{\delta u2}u(t),$$
$$y(t) = C_{y\delta 1}x_1(t) + C_{y\delta 2}x_2(t) + w_2(t),$$
$$z(t) = \begin{bmatrix} H_{\delta 1}x_1(t) + H_{\delta 2}x_2(t) \\ u(t) \end{bmatrix}. \qquad (3.7)$$

where x_1 and x_2 are the slow and fast state vectors. ε is called a singular perturbation parameter. ω is disturbance input, z is performance variable, u is control input, and y is measurement used for feedback. For the standard H∞ control diagram, y and u are used input and output in the controller design.

By taking matrix block diagonalization, the slow subsystem (3.7) and the fast subsystem (3.8) are obtained as below.

$$\delta x_s(t) = A_{\delta s}x_s(t) + B_{\delta ws}w_s(t) + B_{\delta us}u_s(t),$$
$$z_s(t) = C_{\delta zs}x_s(t) + D_{\delta 11s}w_s(t) + D_{\delta 12s}u_s(t),$$
$$y_s(t) = C_{\delta ys}x_s(t) + D_{\delta 21s}w_s(t) + D_{\delta 22s}u_s(t). \qquad (3.8)$$

where $A_{\delta s} = A_{\delta 11} - A_{\delta 12}A_{\delta 22}^{-1}A_{\delta 21},\ B_{\delta ws} = \begin{bmatrix} G_{\delta s} & 0 \end{bmatrix}$

$B_{\delta us} = B_{\delta u1} - A_{\delta 12}A_{\delta 22}^{-1}B_{\delta u2},\ G_{\delta s} = G_{\delta 1} - A_{\delta 12}A_{\delta 22}^{-1}G_{\delta 2}$

$C_{\delta ys} = C_{\delta y1} - C_{\delta y2}A_{\delta 22}^{-1}A_{\delta 21},\ C_{\delta zs} = \begin{bmatrix} H_{\delta s} \\ 0 \end{bmatrix}$

$H_{\delta s} = H_{\delta 1} - H_{\delta 2}A_{\delta 22}^{-1}A_{\delta 21},\ D_{\delta 12s} = \begin{bmatrix} -H_{\delta 2}A_{\delta 22}^{-1}B_{\delta 2} \\ I \end{bmatrix}$

$D_{\delta 11s} = \begin{bmatrix} -H_{\delta 2}A_{\delta 22}^{-1}G_{\delta 2} & 0 \\ 0 & 0 \end{bmatrix},\ D_{22s} = -C_2 A_{22}^{-1}B_2$

$D_{\delta 21s} = \begin{bmatrix} -C_{\delta 2}A_{\delta 22}^{-1}G_{\delta 2} & I \end{bmatrix},\ z_f(\tau) = \begin{bmatrix} H_{\delta 2}x_f(t) \\ u_f(t) \end{bmatrix}$

$$\delta x_f(\tau) = A_{\delta 22} x_f(\tau) + G_{\delta 2} w_{1f}(\tau) + B_{\delta u2} u_f(\tau),$$
$$y_f(\tau) = C_{\delta y2} x_f(\tau) + w_{2f}(\tau). \tag{3.9}$$

where $\tau = t/\varepsilon$.

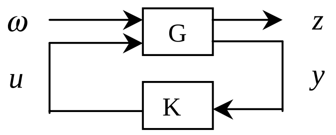

Fig. 4.1 Standard H∞ Control Diagram

4. H∞ DESIGN

4.1 The Fast Subsystem

Theorem 4.1:
From Eq. (3.3), suppose (i) $(A_{\delta 22}, H_{\delta 2})$ is detectable. (ii) There exist $X_f \geq 0$, $Y_f \geq 0$ which satisfy the following Algebraic Riccati Equation (ARE),

$$\begin{aligned} &A_{\delta 22}^T X_f + X_f A_{\delta 22} + \gamma^{-2} X_f G_{\delta 2} G_{\delta 2}^T X_f \\ &- X_f B_{\delta u2} B_{\delta u2}^T X_f + H_{\delta 2}^T H_{\delta 2} = 0, \\ &A_{\delta 22} Y_f + Y_f A_{\delta 22}^T + \gamma^{-2} Y_f H_{\delta 2}^T H_{\delta 2} Y_f \\ &- Y_f C_{\delta y2}^T C_{\delta y2} Y_f + G_{\delta 2} G_{\delta 2}^T = 0. \end{aligned} \tag{4.1}$$

(iii) The spectral radius is $\rho(X_f Y_f) < \gamma^2$, (iv) $A_{\delta 22} + \gamma^{-2} G_{\delta 2}^T G_{\delta 2} X_f - B_{\delta u2} B_{\delta u2}^T X_f$ is Hurwitz, then a dynamic controller that stabilizes the system (3.3) and guarantees the disturbance attenuation level, $\left\| T_{zf\omega f} \right\| \leq \gamma$ is given by

$$\begin{aligned} \delta \xi_f &= (A_{\delta 22} + \gamma^{-2} G_{\delta 2}^T G_{\delta 2} X_f - B_{\delta u2} B_{\delta u2}^T X_f \\ &\quad - L_f C_{\delta y2}) \xi_f + L_f y_f, \\ u_f &= -B_{\delta u2}^T X_f \xi_f, \\ L_f &= (I - \gamma^{-2} Y_f X_f)^{-1} Y_f C_{\delta y2}^T. \end{aligned} \tag{4.2}$$

Note: The proof in the discrete system is not done in this paper.

The controller's equation is given by

$$\begin{aligned} \delta \xi_f &= (A_{\delta 22} + B_{\delta u2} \kappa_f + G_{\delta 2} \kappa_{df} - L_f C_{\delta y2}) \xi_f \\ &\quad + L_f y_f. \end{aligned} \tag{4.3}$$

where $u_f = \kappa_f \xi_f$, and $\hat{\omega}_{1f} = \kappa_{df} \xi_f$ is the estimate of the fast disturbance ω_{1f}. It is needed to find the feedback gain L_f, the observer gain κ_{df} and disturbance estimate gain κ_f. Replacing the estimate with the estimate error, in order to de-couple the equation associated with feedback and observer design in the later stage, results in the following closed-loop fast sub-system:

$$\begin{aligned} \delta x_{fe} &= F_{\delta fe} x_{fe} + G_{\delta fe} w_f, \\ z_f &= H_{\delta fe} x_{fe}. \end{aligned} \tag{4.4}$$

where $x_{fe} = \begin{bmatrix} x_f \\ e_f \end{bmatrix}$, $\omega_f = \begin{bmatrix} \omega_{1f} \\ \omega_{2f} \end{bmatrix}$, $e_f = \xi_f - x_f$,

$$F_{\delta fe} = \begin{bmatrix} A_{\delta 22} + B_{\delta u2} \kappa_f & B_{\delta u2} \kappa_f \\ G_{\delta 2} \kappa_{df} & A_{\delta 22} + G_{\delta 2} \kappa_{df} - L_f C_{\delta y2} \end{bmatrix},$$

$$G_{\delta fe} = \begin{bmatrix} G_{\delta 2} & 0 \\ -G_{\delta 2} & L_f \end{bmatrix}, \quad H_{\delta fe} = \begin{bmatrix} H_{\delta 2} & 0 \\ \kappa_f & \kappa_f \end{bmatrix}.$$

The ARE for the system (4.4) is given by

$$\begin{aligned} &F_{\delta fe}^T X_{fe} + X_{fe} F_{\delta fe} + \gamma^{-2} X_{fe} G_{\delta fe} G_{\delta fe}^T X_{fe} \\ &\quad + H_{\delta fe}^T H_{\delta fe} Y_f = 0. \end{aligned} \tag{4.5}$$

where $X_f = \begin{bmatrix} X_f & 0 \\ 0 & \tilde{X}_f \end{bmatrix}$. The (1,1) block of (4.5) is obtained as

$$\begin{aligned} &(A_{\delta 22} + B_{\delta u2} \kappa_f)^T X_f + X_f (A_{\delta 22} + B_{\delta u2} \kappa_f) \\ &+ \gamma^{-2} X_f G_{\delta 2} G_{\delta 2}^T X_f + \begin{bmatrix} H_{\delta 2}^T & \kappa_f^T \end{bmatrix} \begin{bmatrix} H_{\delta 2} \\ \kappa_f \end{bmatrix} = 0, \\ &\kappa_f = -B_{\delta u2}^T X_f. \end{aligned} \tag{4.6}$$

Block (1,2) and (2,1) of (4.5) is written as

$$\begin{aligned} &- X_f B_{\delta u2} B_{\delta u2}^T X_f + \kappa_{df}^T G_{\delta 2}^T \tilde{X}_f \\ &- \gamma^{-2} X_f G_{\delta 2} G_{\delta 2}^T \tilde{X}_f + \tilde{X}_f B_{\delta u2} B_{\delta u2}^T \tilde{X}_f = 0, \\ &\kappa_{df} = \gamma^{-2} G_{\delta 2}^T X_f. \end{aligned} \tag{4.7}$$

The (2,2) block of (4.5) is expressed as

$$\tilde{X}_f(A_{\delta 22}+\gamma^{-2}G_{\delta 2}G_{\delta 2}^T X_f - L_f C_{\delta y2}) + (A_{\delta 22}+\gamma^{-2}G_{\delta 2}G_{\delta 2}^T X_f - L_f C_{\delta y2})^T \tilde{X}_f + \tilde{X}_f(G_{\delta 2}G_{\delta 2}^T X_f + L_f L_f^T)\tilde{X}_f + X_f B_{\delta u2}B_{\delta u2}^T X_f = 0. \quad (4.8)$$

4.2 The Slow Subsystem

If (iv) of theorem 4.1 for the slow sub-system is satisfied, $\|T_{zsws}\| \le \gamma$ is guaranteed.

$$\begin{aligned} \delta\xi_s &= (A_{\delta s}+\gamma^{-2}G_{\delta s}^T G_{\delta s}X_s - B_{\delta us}B_{\delta us}^T X_s - L_s C_{\delta ys})\xi_s + L_s y_s, \\ u_s &= -B_{\delta us}^T X_s \xi_s, \\ L_s &= (I-\gamma^{-2}Y_s X_s)^{-1} Y_s C_{\delta ys}^T. \end{aligned} \quad (4.9)$$

The transfer matrix of the controller $K_s(s)$ is found from (4.9). y_s is the output and u_s is the input. From the two-frequency-scale property, the following identities hold as $K_s(\infty) = K_f(0) = K_\infty$. With all necessary parameters, all admissible slow controllers can be written using the lower Linear Fractional Transformation (LFT), F_l.

$$\hat{K}_s(s) = F_l(M_s(s), Q_s(s)) = T_{ysus}(s) \quad (4.10)$$

Let

$$\begin{aligned} AM_{\delta s} &= A_{\delta s}+\gamma^{-2}G_{\delta s}G_{\delta s}^T X_s - B_{\delta us}B_{\delta us}^T X_s - L_s C_{\delta ys}, \\ BM_{\delta s} &= \begin{bmatrix} L_s & (I-\gamma^{-2}Y_s X_s)^{-1}B_{\delta us} \end{bmatrix} \\ CM_{\delta s} &= \begin{bmatrix} -B_{\delta us}^T X_s \\ -C_{\delta ys} \end{bmatrix}, \; DM_{\delta s} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \\ M_{\delta s}(s) &= \begin{bmatrix} M_{\delta 11}(s) & M_{\delta 12}(s) \\ M_{\delta 21}(s) & M_{\delta 22}(s) \end{bmatrix} \end{aligned}$$

$M_{\delta s}(s)$ is a transfer function matrix associated with $AM_{\delta s}, BM_{\delta s}, CM_{\delta s}, DM_{\delta s}$. Q(s) is a stable rational transfer function holding the relation $\|Q(s)\|_\infty \le \gamma$. Thus

$$\hat{K}_s = M_{\delta 11}(s) + M_{\delta 12}(s)Q(s)(I - M_{\delta 22}(s)Q(s))^{-1}M_{\delta 21}(s) \quad (4.11)$$

If $Q(s) = K_f(0) = K_\infty$ is chosen, $\hat{K}_s(\infty) = K_f(0) = K_\infty$ is verified. So, such a simple choice of Q(s) results in the slow H∞ controller satisfying the constraint at infinity.

4.3 The composite controller

Adding up the strictly proper part of the slow controller composes a stabilizing composite controller.

$$K(s,\varepsilon) = K_f(p) + \hat{K}_s(s) - K_\infty. \quad (4.12)$$

The closed-loop original system has the follwing inequality as

$$\|T_{z\omega}\|_\infty \le \gamma + O(\varepsilon). \quad (4.13)$$

5. EXAMPLE

Consider the two-time-scale-system given as

$$\begin{aligned} \delta x &= A_{\delta 11}x + A_{\delta 12}z + B_{\delta 1}u, \; x(0) = x_0, \\ \varepsilon\delta z &= A_{\delta 21}x + A_{\delta 22}z + B_{\delta 2}u, \; z(0) = z_0, \\ y &= C_{\delta 1}x + C_{\delta 2}z + D_\delta u. \end{aligned} \quad (5.1)$$

The parameters with ε = 0.1 are given as

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \end{bmatrix}, D = 1$$

According to Eq. (3.5), the H-infinity norms are obtained as

(I) For the continuous-time systems:

$$\|H(s)\|_{\infty Cont-Exact} = 7.8465, \; \|H(s)\|_{\infty Cont-Slow} = 7.8544$$

$$\|H(p)\|_{\infty Cont-Fast} = 1.9981, \; \|H(p=0)\|_{\infty Cont-Fast} = 2.0$$

Here, error of Eq.(3.5) is 0.006 that is within O(ε).

(II) For the delta operator systems:

$$\|H(s)\|_{\infty Delta-Exact} = 7.8400, \; \|H(s)\|_{\infty Delta-Slow} = 7.8482$$

$$\|H(p)\|_{\infty Delta-Fast} = 2.0860, \; \|H(p=0)\|_{\infty Delta-Fast} = 2.09726$$

Here, error of Eq.(3.5) is 0.003 that is within O(ε).

6. CONCLUSION

In this paper, the system decomposition in the frequency domain was succesfully done. It is shown that the delta operator systems have an improved finite word-length characteristics than those of the continuous systems. Continuous systems have less error than the discrete systems in the numerical computation. Therefore, the delta operator systems have few errors than the discrete systems.

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APPENDIX

Linear Quadratic Regulator Design in the delta operating systems, for example, is introduced as

$$0 = KA + A^T K + \frac{Q}{\Delta} + \Delta A^T KA - G^T\left(\frac{R}{\Delta} + \Delta B^T KB\right)G,$$

$$G(\varepsilon) = \left(\frac{R}{\Delta} + \Delta B^T K(\varepsilon)B\right)^{-1} B^T K(I + A\Delta),$$

$$u(\tau)_{opt} = -G(\varepsilon)\bar{x}(\tau).$$

$$J = \tfrac{1}{2} S_{\tau=0}^{\infty}\{[x^T(\tau)z^T(\tau)]^T Q[x^T(\tau)z^T(\tau)] + u(\tau)^T Ru(\tau)\}d\tau.$$

Note the equations in the delta form are too lengthy to come up with the 6 pages limit in this paper.

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