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ROBUST RECEDING HORIZON CONTROL FOR NONLINEAR DISCRETE-TIME SYSTEMS

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Abstract: This paper describes an approach based on Receding Horizon (RH) control for the solution of the state-feedback H_∞ control problem for discrete-time nonlinear systems. The control law is obtained through the solution of a finite-horizon dynamic game and guarantees robust stability in the face of a class of bounded disturbances and/or parameter uncertainties.

Keywords: Robust Control, Model Predictive Control, Receding Horizon control, Discrete-time systems

1. INTRODUCTION

This paper is motivated by the problem of designing closed-loop controllers ensuring robust stability in the face of bounded disturbances and/or parameter uncertainties for discrete-time nonlinear systems. A classical way to address this problem is to resort to H_∞ control, see (Wei and Byrnes, 1995), (Lin and Byrnes, 1996), (Lin and Xie, 1998), (Basar and Olsder, 1995) and (James and Baras, 1995), where a solution is presented for the state, the full-information, and the output-feedback case. The derivation of the H_∞ control law, however, calls for the solution of a Hamilton-Jacobi-Isaacs equation (Lin and Byrnes, 1996); this is a difficult computational task which hampers the application to real systems. In order to overcome this problem, at least partially, the Receding Horizon (RH) paradigm appears to be a promising approach. In an H_∞ setting, RH schemes were first introduced in (Tadmor, 1992) and (Lall and Glover, 1994) for linear unconstrained systems and were recently studied in (Chen *et al.*, 1998) and (Scokaert and Mayne, 1998) for constrained linear systems, while in (Chen *et al.*, 1997) and (Magni *et al.*, 2001*b*) H_∞-RH control algorithms for nonlinear continuous time systems have been proposed. The basic ingredients of these nonlinear RH controllers are an auxiliary controller $\hat{\kappa}(x)$ (typically obtained by linearization techniques) and a computable invariant set $\Omega(\hat{\kappa})$, inside which the auxiliary controller $\hat{\kappa}(x)$ solves the H_∞ problem. The design of the RH controller is motivated by the desire to ensure the solution of the H_∞ problem in a set Ω^{RH} larger than $\Omega(\hat{\kappa})$.

In this respect, the approach proposed in (Chen *et al.*, 1997) does not guarantee that $\Omega^{RH} \subseteq \Omega(\hat{\kappa})$ because only open-loop sequences are considered in the optimization problem, while this is ensured by the methods described in (Scokaert

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and Mayne, 1998), (Magni *et al.*, 2001b) where closed-loop strategies are used. As a matter of fact, guaranteeing that $\Omega^{RH} \supseteq \Omega(\hat{\kappa})$ is still not completely satisfactory. In fact, $\Omega(\hat{\kappa})$ is just an easy-to-compute invariant set associated with the auxiliary controller $\hat{\kappa}$, and could be considerably smaller than the largest region of attraction $\Omega^M(\hat{\kappa})$, where the auxiliary controller solves the H_∞ control problem.

In the present paper we propose a new solution to the H_∞-RH control problem so as to obtain $\Omega^{RH} \supseteq \Omega^M(\hat{\kappa})$ without a substantial increase of complexity. This is achieved by introducing flexibility in the problem formulation, i.e. by using different control and prediction horizons. The use of two horizons has been already discussed in the context of nonlinear RH control in (Magni *et al.*, 2001a), where it has been shown that it leads to significant improvements in terms of performance, domain of attraction and computational burden.

The organization of the paper is as follows. In Section 2, the problem is formulated, while in Section 3 the RH control solution is introduced with the main result. Section 4 contains some conclusions.

2. PROBLEM FORMULATION

Consider the nonlinear discrete-time dynamic system

$$\begin{cases} x(k+1) = f(x(k), u(k), w(k)), & k \geq 0 \\ z(k) = h(x(k), u(k), w(k)) \end{cases} \tag{1}$$

where $x \in I\!R^n$, $u \in I\!R^m$, $w \in l_2([0,T], I\!R^p)$, for every positive integer T, $z \in R^s$, f and h are known C^2 functions with $f(0,0,0) = 0$, $h(0,0,0) = 0$ and $x(0) = x_0$.

Assumption A1: Given a non-empty set $\bar{\Omega}$, containing the origin as an interior point, the system (1) is zero-state detectable in $\bar{\Omega}$, i.e., $\forall x_0 \in \bar{\Omega}$ and $\forall u(\cdot)$ such that $x(k) \in \bar{\Omega}$, $\forall k \geq t$, we have

$$z(k)|_{w=0} = 0, \quad \forall k \geq t \Longrightarrow \lim_{k\to\infty} x(k) = 0$$

■

In this paper we want to synthesize a state-feedback controller that ensures robust stability in the face of all disturbances w satisfying the following assumption:

Assumption A2: Given a positive constant γ_Δ, the disturbance w is such that

$$\|w(k)\|^2 \leq \gamma_\Delta^2 \|z(k)\|^2, \quad k \geq t. \tag{2}$$

The space of admissible disturbances will be denoted by $\mathcal{W}(\gamma_\Delta)$. ■

As is well known, equation (2) also represents a wide class of modeling errors, with respect to which robust stability is desired. To this end, we consider the H_∞ control problem defined below.

P1 H_∞ **control problem:** Design a state-feedback control law

$$u = \kappa(x) \tag{3}$$

guaranteeing that the closed-loop system (1)-(3) with input $w \in \mathcal{W}(\gamma_\Delta)$ and output z has a finite $L_2 - gain \leq \gamma$ in a finite positively invariant set Ω, that is, $\forall x_0 \in \Omega$,

i) $x(k) \in \Omega$, $\forall k \geq 0$;
ii) there exists a finite quantity $\beta(x_0)$, $\beta(0) = 0$, such that $\forall T \geq 0$,

$$\sum_{i=0}^{T} \|z(i)\|^2 \leq \gamma^2 \sum_{i=0}^{T} \|w(i)\|^2 + \beta(x_0)$$

for any nonzero $w \in \mathcal{W}(\gamma_\Delta)$. ■

Once such a control law is applied, it follows from the small-gain theorem that the closed-loop system (1)-(3) will be robustly stable in Ω for all uncertainties $w \in \mathcal{W}(\gamma_\Delta)$ provided that $\gamma_\Delta < 1/\gamma$, see (van der Schaft, 1996).

A partial result is derived in (Lin and Xie, 1998) where it is shown that, under some regularity assumptions, a nonlinear H_∞ control problem is locally solved by the linear H_∞ control law synthesized on the linearization of (1). The main limitation of this result is that the invariance of the region Ω is not established, but it is only shown that the condition (ii) holds for all disturbances $w \in l_2[0,\infty)$ such that the state trajectory of the system (1)-(3) does not leave Ω. Unfortunately, for a general nonlinear system it is not possible to guarantee the invariance of the finite set Ω without imposing some limitations on the disturbance w. Among the few results presented in literature, we recall (Lu, 1995) where the construction of invariant subsets of the state space for nonlinear systems with persistent bounded disturbances is investigated.

Hereafter, given a controller $u = \kappa(x)$ that solves $P1$, the symbol $\Omega(\kappa, \gamma, \gamma_\Delta)$ will denote the invariant set Ω mentioned in the statement of $P1$.

Following (Lin and Xie, 1998), it can be shown that, under Assumption $A2$, if the H_∞ control problem for the linearized system is solvable, then, there exists a finite region $\Omega(K^\infty, \gamma, \gamma_\Delta)$ where the linear H_∞ control law $u = K^\infty(x) = Kx$ is a solution for the nonlinear H_∞ control problem $P1$. In this respect the following assumption is introduced.

Assumption A3: The constant γ is such that problem P1 for the linearized system is solvable. ■

Note that, the computation of the largest invariant set $\Omega^M(K^\infty, \gamma, \gamma_\Delta)$ is in general, an impossible task, so that only a smaller invariant set $\Omega(K^\infty, \gamma, \gamma_\Delta)$ can be supposed to be known.

Starting from an available auxiliary control law $u = \hat{\kappa}(x)$ (for example $u = K^\infty(x) = Kx$) with an associated invariant set $\Omega(\hat{\kappa}, \gamma, \gamma_\Delta)$, we want to obtain a control law $u = \kappa^{RH}(x)$ and an associated invariant set $\Omega(\kappa^{RH}, \gamma, \gamma_\Delta)$ such that:

(a) $\Omega(\kappa^{RH}, \gamma, \gamma_\Delta) \supseteq \Omega(\hat{\kappa}, \gamma, \gamma_\Delta)$;
(b) $\Omega(\kappa^{RH}, \gamma, \gamma_\Delta)$ tends to $\bar{\Omega}^M$ by a proper choice of the design parameters, where $\bar{\Omega}^M$ denotes the largest domain of attraction achievable by a control law that solves $P1$.

3. RECEDING HORIZON CONTROL LAW

3.1 *Problem statement*

The derivation of the *RH* control law is based on the solution of a finite-horizon zero-sum differential game, where u is the input of the minimizing player (the controller) and w is the input of the maximizing player ("the nature"). More precisely, the controller chooses the input $u(k)$ as a function of the current state $x(k)$ so as to ensure that the effect of the disturbance $w(\cdot)$ on the output $z(\cdot)$ is sufficiently small for any choice of $w(\cdot)$ made by "nature".

In the following, according to the *RH* method, we will focus on a finite time interval $[t, t+N_p-1]$. At a given time t, the controller will have to choose a vector of feedback control strategies $\kappa_{t,t+N_c-1} := [\kappa_0(x(t)), \ldots, \kappa_{N_c-1}(x(t+N_c-1))]$ where $\kappa_i(\cdot) : I\!R^n \to I\!R^m$ will be called *policy* and N_c is the *control horizon*. At the end of the control horizon an auxiliary state-feedback control law $u = \hat{\kappa}(x)$ is used. The sequence of disturbances chosen by "the nature" will be denoted by $w_{t,t+N_p-1} := [w(t), \ldots, w(t+N_p-1)]$, where $N_p > N_c$ is the *prediction horizon*.

Differently from the standard *RH* approach, in this case it is not convenient to consider open-loop control strategies since open-loop control would not account for changes in the state due to unpredictable inputs played by "the nature" (see also (Scokaert and Mayne, 1998)). Hence, at each time t, the minimizing player optimizes his sequence $\kappa_{t,t+N_c-1}$ of policies, i.e. the minimization is carried out in an infinite-dimensional space. Conversely, in open-loop it would be sufficient to minimize with respect to the sequence $[u(t), u(t+1), \ldots, u(t+N_c-1)]$ of future control actions, a sequence which belongs to a finite-dimensional space.

Consider an auxiliary control law $u = \hat{\kappa}(x)$ that solves the problem $P1$, with a domain of attraction $\Omega(\hat{\kappa}, \gamma, \gamma_\Delta)$ whose boundary is assumed to be a level line of a positive (storage) function $V_\kappa(x)$ such that

$$\begin{gathered} V_{\hat{\kappa}}(f(x, \hat{\kappa}(x), w)) - V_{\hat{\kappa}}(x) \\ < -\frac{1}{2}\left(\|z\|^2 - \gamma^2 \|w\|^2\right), \\ \forall x \in \Omega(\hat{\kappa}, \gamma, \gamma_\Delta), \quad \forall w \in \mathcal{W}(\gamma_\Delta) \end{gathered} \tag{4}$$

and $V_{\hat{\kappa}}(0) = 0$. A practical way to compute $\hat{\kappa}$ and $V_{\hat{\kappa}}$ is by means of a linear H_∞ controller (which exists in view of A3) for the linearized system and an associated quadratic storage function as described in (Lin and Xie, 1998).

Finite-Horizon Optimal Dynamic Game ($FHODG$): Minimize with respect to $\kappa_{t,t+N_c-1}$, and maximize with respect to $w_{t,t+N_p-1}$, the cost function

$$\begin{aligned} &J(\bar{x}, \kappa_{t,t+N_c-1}, w_{t,t+N_p-1}, N_c, N_p) \\ &= \frac{1}{2}\sum_{i=t}^{t+N_p-1}\left\{\|z(i)\|^2 - \gamma^2\|w(i)\|^2\right\} + V_{\hat{\kappa}}(x(t+N_p)) \end{aligned}$$

subject to (1) with $x(t) = \bar{x}$, and

$$x(t+N_p) \in \Omega(\hat{\kappa}, \gamma, \gamma_\Delta) \subset R^n$$

$$u(i) = \begin{cases} \kappa_{i-t}(x(i)), & t \le i < t+N_c \\ \hat{\kappa}(x(i)), & t+N_c \le i < t+N_p \end{cases}$$

■

In the previous definition, γ is a constant, which can be interpreted as the disturbance attenuation level.

For a given initial condition $\bar{x} \in R^n$, we denote by

$$\begin{aligned} &\kappa^o_{t,t+N_c-1} \\ &= \arg \min_{\kappa_{t,t+N_c-1}} \max_{w_{t,t+N_p-1}} J(\bar{x}, \kappa_{t,t+N_c-1}, w_{t,t+N_p-1}, N_c, N_p) \end{aligned}$$

and

$$\begin{aligned} &w^o_{t,t+N_p-1} \\ &= \max_{w_{t,t+N_p-1}} J(\bar{x}, \kappa^o_{t,t+N_c-1}, w_{t,t+N_p-1}, N_c, N_p) \end{aligned}$$

the saddle point solution, if exists, of the zero-sum $FHODG$.

According to the *RH* method, we obtain the value of the feedback control law as a function of $\bar{x}$ by solving the $FHODG$ and setting

$$\kappa^{RH}(\bar{x}) = \kappa^o_0(\bar{x}) \tag{5}$$

where $\kappa^o_0(\bar{x})$ is the first column of $\kappa^o_{t,t+N_c-1} := [\kappa^o_0(x(t)), \ldots, \kappa^o_{N_c-1}(x(t+N_c-1))]$.

3.2 *Algorithm*

In summary the implementation of the proposed *RH* controller consists of the following steps.

Off-line computations: Computation of $\hat{\kappa}(x)$, $\Omega(\hat{\kappa},\gamma,\gamma_\Delta)$, and $V_{\hat{\kappa}}(x)$ according to (Lin and Xie, 1998).

On-line computations:

(1) At each time instant t compute $\kappa^o_{t,t+N_c-1}$ by solving the Finite-Horizon Optimal Dynamic Game;
(2) Apply the control action $u(t) = \kappa^{RH}(x(t))$ where κ^{RH} is defined in (5).

3.3 *Properties*

In order to establish the closed-loop stability properties of the *RH* controller, we first introduce the following definitions.

Definition 1. : Let $\mathcal{K}(\bar{x}, N_c, N_p)$ be the set of all policies $\kappa_{t,t+N_c-1}$ such that starting from $\bar{x}$, it results $x(t+N_p) \in \Omega(\hat{\kappa},\gamma,\gamma_\Delta)$ for every admissible disturbance sequences $w_{t,t+N_p-1}$.

Definition 2. (Vincent and Grantham, 1997) Let $\Omega^{RH}(N_c, N_p)$ be the set of initial states $\bar{x}$ such that $\mathcal{K}(\bar{x}, N_c, N_p)$ is nonempty.

Definition 3. Given the initial state $\bar{x}$ and a policy vector $\kappa_{t,t+N_c-1} \in \mathcal{K}(\bar{x}, N_c, N_p)$, the set of admissible disturbances $w_{t,t+N_p-1}$ is denoted by $\mathcal{W}(\bar{x}, \kappa_{t,t+N_c-1}, N_p)$. ■

In the following, the optimal value of the *FHODG* will be denoted by $V(\bar{x}, N_c, N_p)$, i.e. $V(\bar{x}, N_c, N_p) := J(\bar{x}, \kappa^o_{t,t+N_c-1}, w^o_{t,t+N_p-1}, N_c, N_p)$. Now, the main result can be stated.

Theorem 1. Consider two positive constants γ and γ_Δ with $\gamma_\Delta\gamma < 1$, and the closed-loop system

$$\Sigma^{RH} : \begin{cases} x(k+1) = f(x(k), \kappa^{RH}(x(k)), w(k)) \\ z(k) = h(x(k), \kappa^{RH}(x(k)), w(k)) \end{cases}$$

Then, under Assumptions A1-A2-A3,

i) $\Omega^{RH}(N_c, N_p)$ is a positively invariant set for Σ^{RH} and $\Omega^{RH}(N_c, N_p) \supseteq \Omega(\hat{\kappa},\gamma,\gamma_\Delta)$, $\forall N_c, N_p > 0$;
ii) Σ^{RH} is internally stable and has L_2-gain less than or equal to γ, in $\Omega^{RH}(N_c, N_p)$;
iii) $\Omega^{RH}(N_c{+}1, N_p) \supseteq \Omega^{RH}(N_c, N_p)$ and $lim_{N_c\to\infty}$ $\Omega^{RH}(N_c, N_p) = \bar{\Omega}^M$, where $\bar{\Omega}^M$ denotes the largest domain of attraction achievable by a control law solving $P1$.
iv) $\forall N_c > 0, lim_{N_p\to\infty}\Omega^{RH}(N_c, N_p) \supseteq \Omega^M(\hat{\kappa},\gamma,\gamma_\Delta)$.

Proof:

i) If $\bar{x} \in \Omega^{RH}(N_c, N_p)$, there exists a policy $\kappa_{t,t+N_c-1} \in \mathcal{K}(\bar{x}, N_c, N_p)$. Then, $\tilde{\kappa}_{t+1,t+N_c} = [\kappa_{t+1,t+N_c-1}, \hat{\kappa}(x(t+N_c))] \in \mathcal{K}(f(\bar{x}, \kappa^{RH}(\bar{x}), w(t)), N_c, N_p)$, $\forall w$ satisfying Assumption A2, so that $f(\bar{x}, \kappa^{RH}(\bar{x}), w(t)) \in \Omega^{RH}(N_c, N_p)$ that is $\Omega^{RH}(N_c, N_p)$ is a positively invariant set for Σ^{RH}. Moreover $\forall \overline{x} \in \Omega(\hat{\kappa},\gamma,\gamma_\Delta)$ there exists the policy $\kappa_{t,t+N_c-1} = [\hat{\kappa}(x(t)), \ldots, \hat{\kappa}(x(t+N_c-1))]$ such that starting from $\bar{x}$, it results $x(t+N_p) \in \Omega(\hat{\kappa},\gamma,\gamma_\Delta)$ for every disturbance $w_{t,t+N_p-1} \in \mathcal{W}(\bar{x}, \kappa_{t,t+N_c-1}, N_p)$ so that $\Omega^{RH}(N_c, N_p) \supseteq \Omega(\hat{\kappa},\gamma,\gamma_\Delta)$, $\forall N_c, N_p > 0$ and the origin is an interior point of $\Omega^{RH}(N_c, N_p)$ $\forall N_c, N_p > 0$.
ii) Given $\bar{w}_{t,t+N_p-1} = 0$, for every $\kappa_{t,t+N_c-1}$

$$J(\bar{x}, \kappa_{t,t+N_c-1}, \bar{w}_{t,t+N_p-1}, N_c, N_p)$$
$$= \sum_{i=t}^{t+N_p-1} \frac{1}{2}\left\{\|z(i)\|^2\right\} + V_{\hat{\kappa}}(x(t+N)) \geq 0$$

so that

$$V(x, N_c, N_p)(6)$$
$$\geq J(\bar{x}, \kappa_{t,t+N_c-1}, \bar{w}_{t,t+N_p-1}, N_c, N_p) \geq 0$$

$\forall \overline{x} \in \Omega^{RH}(N_c, N_p)$.

The monotonicity property $V(\bar{x}, N_c{+}1, N_p{+}1) \leq V(\bar{x}, N_c, N_p)$ is now proven. For ease of notation, define

$$S(z, w) = \frac{1}{2}\left\{\gamma^2\|w\|^2 - \|z\|^2\right\}$$

Suppose now that $\kappa^o_{t,t+N_c-1}$ is the solution of the *FHODG* with horizon N_c, and consider the following policy vector for the *FHODG* with horizon $N_c + 1$

$$\tilde{\kappa}_{t,t+N_c} = \begin{cases} \kappa^o_{t,t+N_c-1} & t \leq k \leq t+N_c-1 \\ \hat{\kappa}(x(t+N_c)) & k = t+N_c \end{cases}$$

then, letting $u(k) = \tilde{\kappa}(x(k))$ in (1), it results that

$$J(\bar{x}, \tilde{\kappa}_{t,t+N_c}, w_{t,t+N_p}, N_c+1, N_p+1)$$
$$= -\sum_{i=t}^{t+N_p} S(z(i), w(i)) + V_{\hat{\kappa}}(x(t+N_p+1))$$
$$= V_{\hat{\kappa}}(x(t+N_p+1)) - V_{\hat{\kappa}}(x(t+N_p))$$
$$-S(z(t+N_p), w(t+N_p))$$
$$-\sum_{i=t}^{t+N_p-1} S(z(i), w(i)) + V_{\hat{\kappa}}(x(t+N_p))$$

so that, in view of (4),

$$J(\bar{x}, \tilde{\kappa}_{t,t+N_c}, w_{t,t+N_c}, N_c+1, N_p+1)$$
$$\leq -\sum_{i=t}^{t+N_p-1} S(z(i), w(i)) + V_{\hat{\kappa}}(x(t+N_p))$$

which implies

$$V(\bar{x}, N_c+1, N_p+1)$$
$$\leq \max_{w_{t,t+N_p} \in \mathcal{W}(\bar{x}, \tilde{\kappa}_{t,t+N_c}, N_p+1)}$$
$$J(\bar{x}, \tilde{\kappa}_{t,t+Nc}, w_{t,t+N_p}, N_c+1, N_p+1)$$
$$\leq \max_{w_{t,t+N_p-1} \in \mathcal{W}(\bar{x}, \kappa^o_{t,t+N_c-1}, N_p)}$$
$$-\sum_{i=t}^{t+N_p-1} S(z(i), w(i)) + V_{\hat{\kappa}}(x(t+N_p)) \quad (7)$$
$$= V(\bar{x}, N_c, N_p)$$

which holds for all $x \in \Omega^{RH}(N_c, N_p)$. Note that in view of (6) and (7)

$$0 \leq V(0, N_c, N_p) \leq V_{\hat{\kappa}}(0) = 0$$

so that $V(0, N_c, N_p) = 0$. Moreover, in view of (7) and the definition of $V(x, N_c, N_p)$, it follows that $\forall \overline{x} \in \Omega^{RH}(N_c, N_p)$, and for a generic $w(t) \in \mathcal{W}(\bar{x}, \kappa^o_{t,t+N_c-1}, N_p)$:

$$V(\bar{x}, N_c, N_p)$$
$$= J(\bar{x}, \kappa^o_{t,t+N_c-1}, w^o_{t,t+N_p-1}, N_c, N_p)$$
$$\geq V(f(\bar{x}, \kappa^{RH}(\bar{x}), w(t)), N_c-1, N_p-1)$$
$$+\frac{1}{2}\left\{\left\|h(\bar{x}, \kappa^{RH}(\bar{x}), w(t))\right\|^2 - \gamma^2 \|w(t)\|^2\right\}$$
$$\geq V(f(\bar{x}, \kappa^{RH}(\bar{x}), w(t)), N_c, N_p)$$
$$+\frac{1}{2}\left\{\left\|h(\bar{x}, \kappa^{RH}(\bar{x}), w(t))\right\|^2 - \gamma^2 \|w(t)\|^2\right\} \quad (8)$$

Setting $w = 0$ in (8), by Assumption A1 and the continuity of f and h, it follows immediately from LaSalle's Invariance principle (LaSalle, 1986) that $x = 0$ is locally asymptotically stable when $w = 0$. Finally, with reference to Σ^{RH} with initial condition $x(t) = \bar{x}$, from (8) it follows that $\forall \overline{x} \in \Omega^{RH}(N_c, N_p)$

$$V(x(t+T+1), N_c, N_p) - V(\bar{x}, N_c, N_p)$$
$$\leq -\sum_{i=0}^{T} \frac{1}{2}\left\{\|z(t+i)\|^2 - \gamma^2 \|w(t+i)\|^2\right\}$$

and by (6)

$$0 \leq V(x(t+T+1), N_c, N_p)$$
$$\leq -\sum_{i=0}^{T} \frac{1}{2}\left\{\|z(t+i)\|^2 - \gamma^2 \|w(t+i)\|^2\right\}$$
$$+V(\bar{x}, N_c, N_p)$$

which implies that

$$\sum_{i=0}^{T} \frac{1}{2}\|z(t+i)\|^2 \leq \gamma^2 \sum_{i=0}^{T} \frac{1}{2}\|w(t+i)\|^2$$
$$+V(\bar{x}, N_c, N_p)$$

$\forall \overline{x} \in \Omega^{RH}(N_c, N_p), \forall T \geq 0, \forall w \in \mathcal{W}(\bar{x}, \kappa^o_{t,t+N_c}, N_p)$. Hence, we conclude that Σ^{RH} has L_2-gain less than or equal to γ, in $\Omega^{RH}(N_c, N_p)$.

iii) If $\kappa_{t,t+N_c-1} \in \mathcal{K}(\bar{x}, N_c, N_p)$, then $\hat{\kappa}_{t,t+N_c} = [\kappa_{t,t+N_c-1},\ \hat{\kappa}(x(t+N_c))] \in \mathcal{K}(\bar{x}, N_c+1, N_p)$ so that $\Omega^{RH}(N_c+1, N_p) \supseteq \Omega^{RH}(N_c, N_p)$, $\forall N_c > 0$. As $N_c \to \infty$ the problem becomes an infinite horizon H_∞ control problem implying that $\Omega^{RH}(N_c, N_p) \to \bar{\Omega}^M$.

iv) If $\bar{x} \in \Omega^M(\hat{\kappa}, \gamma, \gamma_\Delta)$ then there exists a finite $\bar{N}_p$ such that the control sequence $\kappa_{t,t+\bar{N}_p-1} = [\hat{\kappa}(x(t)), \ldots,\ \hat{\kappa}(x(t+\bar{N}_p-1))]$ satisfies $x(t+\bar{N}_p-1) \in \Omega(\hat{\kappa}, \gamma, \gamma_\Delta)$. This implies that $\forall N_c$, $\kappa_{t,t+N_c-1} = [\hat{\kappa}(x(t)), \ldots,\ \hat{\kappa}(x(t+N_c-1))] \in \mathcal{K}(\bar{x}, N_c, \bar{N}_p)$ or, equivalently, $\bar{x} \in \Omega^{RH}(N_c, \bar{N}_p)$.

3.4 *Comments*

- Point (i) is essential in order to ensure that it is worth replacing the auxiliary controller $\hat{\kappa}$ with the RH one κ^{RH}.
- Point (ii) states that the RH controller is indeed a solution to the H_∞ control problem for the nonlinear system.
- Point (iii) shows that, at the cost of an increase of the computational burden associated with optimization, the domain of attraction can be progressively enlarged towards the maximum achievable one. In other words, N_c is a tuning knob that regulates the complexity/performance trade off.
- Point (iv) shows that by suitably increasing the prediction horizon, the RH controller can be rendered better than the auxiliary one, not only in terms of the nominal set $\Omega(\hat{\kappa}, \gamma, \gamma_\Delta)$ (see (i)), but also in terms of the actual domain of attraction. Remarkably, this property holds irrespective of the value of the control horizon N_c. In particular, it is possible to let $N_c = 1$ in which case it is not even necessary to optimize over policies in an infinite dimensional space. In fact, $x(t) = \bar{x}$ is known and the first control move is just a vector belonging to an m-dimensional space.
- A major drawback of the approach proposed in the present paper is of computational type, since the implementation of the RH controller calls for optimization within the infinite dimensional space of control policies (at least for $N_c > 1$). A possible solution is to resort to a finite dimensional parametrization (Mayne, 2000) (e.g. polynomial control policies), at the cost of losing some flexibility. In this respect it is worth observing that the theoretical properties of the regulator remain unchanged provided that the auxiliary controller $\hat{\kappa}(x)$ belongs to the same finite dimensional class of control policies.
- In (Chen *et al.*, 1997) optimization is performed with respect to the sequence of future control moves. This is a (computationally

cheaper) open-loop strategy but, due to the action of disturbances, there is no guarantee that $\Omega^{RH} \supseteq \Omega(\hat{\kappa})$, so that the *RH* controller may not improve on the auxiliary one. Herein, we adopt a more effective closed-loop strategy, guaranteeing that $\Omega^{RH} \supseteq \Omega(\hat{\kappa})$.

- In (Magni *et al.*, 2001*b*) the property $\Omega^{RH} \supseteq \Omega(\hat{\kappa})$ is achieved by resorting to a closed-loop strategy (minimization with respect to polices). Conversely, there is no guarantee that $\Omega^{RH} \supseteq \Omega^{M}(\hat{\kappa})$, where $\Omega^{M}(\hat{\kappa})$ is the largest invariant set where the auxiliary controller $\hat{\kappa}$ solves the H_∞ problem, unless the control horizon (which is equal to the prediction one) is properly increased. Differently from (Magni *et al.*, 2001*b*), in this paper the use of a control horizon N_c shorter than the prediction horizon N_p allows us to ensure that (by a proper choice of N_p) $\Omega^{RH} \supseteq \Omega^{M}(\hat{\kappa})$ for any (short) control horizon, see point (*iv*) of Theorem 1.

4. CONCLUSIONS

In this paper it has been shown that the *RH* paradigm applied to the H_∞ control problem for discrete-time nonlinear systems can improve the domain of attraction provided by an available local solution, obtained for example through linearization. In particular, the *RH* control can enlarge the domain of attraction even for very short control horizons (e.g. $N_c = 1$) and at a reasonable computational cost. The key points of the algorithm are: (*i*) the adoption of a closed-loop strategy involving the optimization of the future control policies; (*ii*) the use of two distinct horizons: a prediction horizon over which system performance is evaluated, and a shorter control horizon over which control policies are optimized.

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