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CLASSICAL AND HIGHER SYMMETRIES OF CONTROL SYSTEMS

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Abstract: We study classical and higher infinitesimal symmetries of control systems. Defining equations for classical external symmetries are obtained in the general and affine cases. For computing higher symmetries we suggest a simple procedure involving algebraic operations and differentiation but not integration. Relations between classical symmetries and first integrals of control systems are established. An example is considered to illustrate our methods.

Keywords: Differential geometric methods, symmetries, decomposition of control systems, flatness.

1. INTRODUCTION

Consider a nonlinear control system

$$\dot{x} = f(t, x, u), \qquad (1)$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the state, $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ is the control input, f: $E = \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth (C^∞) function, $\dot{x} \equiv dx/dt$. Let $\operatorname{rank}(\partial f/\partial u) = m$. Following (Aranda–Bricaire *et al*, 1995) and (Fliess *et al*, 1995) we regard system (1) as an underdetermined system of ordinary differential equations. This viewpoint allows to consider system (1) in the framework of geometrical theory of differential equations (see Krasil'shchik *et al*, 1999).

In this framework, system (1) is naturally related to two manifolds ($\mathcal{E}$ and $\mathcal{E}^\infty$ below) with two distributions. The first manifold ($\mathcal{E}$) is finite-dimensional, the second one ($\mathcal{E}^\infty$) is infinite-dimensional. Both distributions are called the Cartan ones. A map from one of these manifolds to itself is called a symmetry of system (1) if the map preserves the corresponding Cartan distribution. In the first case ($\mathcal{E}$), a symmetry is called classical. In the case $\mathcal{E}^\infty$, it is called higher. Any classical symmetry generates a higher symmetry. Any symmetry of system (1) takes each solution of (1) to a solution again.

An infinitesimal version of the above construction leads to the concepts of infinitesimal classical and higher symmetries. If two control systems are static feedback equivalent, then the Lie algebras of their classical infinitesimal symmetries are isomorphic. Similarly, an equivalence by endogenous dynamic feedback generates an isomorphism of the corresponding Lie algebras of higher infinitesimal symmetries.

This paper is devoted to methods for calculation of classical and higher infinitesimal symmetries of control systems. To find classical infinitesimal symmetries, one needs to solve some system of partial differential equations. We give this system in the general (see Theorems 1 and 2) and affine (see Theorems 5 and 2) cases. Our method for calculation of higher infinitesimal symmetries is based on the infinitesimal Brunovsky form introduced in (Aranda–Bricaire *et al*, 1995). We assign a higher symmetry to an arbitrary collection of m functions and a classical symmetry of some system of ordinary differential equations (see Theorem 7 for details).

The results exposed in this paper were applied by the authors to the decomposition problem for control systems (Kanatnikov *et al*, 1994) and the flatness problem (Chetverikov, 2001).

The paper is organized as follows. Classical symmetries are studied in Sections 2—5. The method for calculation of higher infinitesimal symmetries is presented in Sections 6—8. In Sections 2 and 6 we give the two geometric interpretations of control systems ($\mathcal{E}$ and $\mathcal{E}^\infty$ respectively). The conditions for classical infinitesimal symmetries are obtained in Section 3 in the general case and in Section 5 in the affine case. A relationship between classical symmetries and first integrals of control systems is discussed in Section 4. In Section 7 a construction introduced in (Aranda–Bricaire *et al*, 1995) is generalized to the nonautonomous case. This generalization is used in Section 8, where higher infinitesimal symmetries of control systems are described. Finally, Section 9 contains an example of calculation of classical and higher infinitesimal symmetries.

2. THE FIRST GEOMETRIC INTERPRETATION

System (1) determines the trivial bundle $\pi\colon E \to \mathbb{R}^1$, $\pi(t,x,u) = t$. Consider the 1-jet space $J^1\pi$ of this bundle (see Krasil'shchik *et al*, 1999). Let (t,x,u,p,q) be local coordinates on $J^1\pi$ with $p = (p_1,\dots,p_n)$ corresponding to $\dot{x}(t)$, $q = (q_1,\dots,q_m)$ corresponding to $\dot{u}(t)$. System (1) can be written as $p = f(t,x,u)$. Therefore it can be interpreted as the submanifold

$$\mathcal{E} = \big\{(t,x,u,p,q) \in J^1\pi |\ p - f(t,x,u) = 0\big\}$$

of codimension n in $J^1\pi$. Each section $(x(t),u(t))$ of the bundle π has a prolongation onto $J^1\pi$ as a curve l_{xu} of the form $t \mapsto (t, x(t), u(t), \dot{x}(t), \dot{u}(t))$. A section $(x(t),u(t))$ is a solution of (1) if and only if $l_{xu} \subset \mathcal{E}$.

The *Cartan distribution* on $J^1\pi$ is determined by the 1-forms $\omega_i = dx_i - p_i dt$, $\tau_j = du_j - q_j dt$, $i = 1,2,\dots,n$, $j = 1,2,\dots,m$. The curves l_{xu} are integral curves of the Cartan distribution. Let $\pi_1\colon J^1\pi \to \mathbb{R}^1$ be the natural projection, i. e., $\pi_1(t,x,u,p,q) = t$. It is known (see Krasil'shchik *et al*, 1999) that if a one–dimensional integral submanifold of the Cartan distribution is locally maximal, then it locally coincides with a curve l_{xu} (except for singular points of the map $\pi_1\colon l_{xu} \to \mathbb{R}^1$). These manifolds are called *R-manifolds.* We shall call R-manifolds contained in $\mathcal{E}$ *generalized solutions* of system (1). The restriction of the Cartan distribution on $J^1\pi$ to $\mathcal{E}$ is called the *Cartan distribution* on $\mathcal{E}$. Generalized solutions are locally maximal integral curves of this distribution.

A diffeomorphism from $J^1\pi$ to itself is called a *Lie transformation* if it preserves the Cartan distribution. Lie transformations of $J^1\pi$ send any R-manifold to an R-manifold again. If a Lie transformation translates the submanifold $\mathcal{E}$ into itself (and consequently any generalized solution to a generalized solution again), then it is called a (*classical external*) *symmetry* of (1). It can be proved (see Krasil'shchik *et al*, 1999) that in the case $n + m > 1$ each Lie transformation is lifted from $J^0\pi = E$.

One stated above is transferred on one–parameter groups of Lie transformations. These groups are connected with their infinitesimal generators — vector fields named Lie fields. In our case, when Lie transformations are lifted from the manifold E, Lie fields are also obtained as lifting of vector fields on E. Namely if a vector field X on E has the form

$$X = \xi(t,x,u)\frac{\partial}{\partial t} + \sum_{i=1}^{n} \eta_i(t,x,u)\frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \vartheta_j(t,x,u)\frac{\partial}{\partial u_j}, \quad (2)$$

then its lifting $X^{(1)}$ on $J^1\pi$ is the vector field

$$X^{(1)} = X + \sum_{i=1}^{n} \zeta_i(t,x,u,p,q)\frac{\partial}{\partial p_i} + \sum_{j=1}^{m} \varepsilon_j(t,x,u,p,q)\frac{\partial}{\partial q_j}, \quad (3)$$

where $\zeta = (\zeta_1,\dots,\zeta_n)$ and $\varepsilon = (\varepsilon_1,\dots,\varepsilon_m)$ are obtained by formulas

$$\zeta = D\eta - pD\xi, \quad \varepsilon = D\vartheta - qD\xi, \quad (4)$$

with $\eta = (\eta_1,\dots,\eta_n)$, $\vartheta = (\vartheta_1,\dots,\vartheta_m)$, and

$$D = \frac{\partial}{\partial t} + \sum_{i=1}^{n} p_i\frac{\partial}{\partial x_i} + \sum_{j=1}^{m} q_j\frac{\partial}{\partial u_j}$$

being the *total derivative with respect to* t on $J^1\pi$ (see Krasil'shchik *et al*, 1999).

If Lie field (3) is tangent to the submanifold $\mathcal{E}$, then Lie transformations of its one–parameter group translates $\mathcal{E}$ into itself. In this case the vector field is called an (*infinitesimal classical external*) *symmetry* of system (1). The condition necessary and sufficient to field (3) being tangent to $\mathcal{E}$ is the relation

$$X^{(1)}(p_i - f_i)|_{\mathcal{E}} = 0, \quad i = 1,2,\dots,n, \quad (5)$$

where $(f_1,\dots,f_n) = f$.

3. DEFINING EQUATIONS FOR CLASSICAL SYMMETRIES

Using (2)–(4) relation (5) in coordinate terms reduces to

$$\xi\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\eta+\frac{\partial f}{\partial u}\vartheta-\frac{\partial \eta}{\partial t}-\frac{\partial \eta}{\partial x}f-\frac{\partial \eta}{\partial u}q + f\left(\frac{\partial \xi}{\partial t}+\frac{\partial \xi}{\partial x}f+\frac{\partial \xi}{\partial u}q\right)=0, \quad (6)$$

the latter being valid for all (t,x,u,q). System (6) is linear with respect to q and therefore decomposes into two subsystems

$$\xi\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\eta+\frac{\partial f}{\partial u}\vartheta-\frac{\partial \eta}{\partial t}-\frac{\partial \eta}{\partial x}f + f\frac{\partial \xi}{\partial t}+f\frac{\partial \xi}{\partial x}f=0, \quad (7)$$

$$\frac{\partial \eta}{\partial u}-f\frac{\partial \xi}{\partial u}=0. \quad (8)$$

We shall call equations (7)–(8) the *defining equations for classical infinitesimal symmetries* of system (1).

Any symmetry (2)–(3) of a control system is uniquely determined by its components $\xi,\eta_1,\dots,\eta_n$ (see (4) and (10)). Let

$$H=\xi\frac{\partial}{\partial t}+\sum_{i=1}^{n}\eta_i\frac{\partial}{\partial x_i},\quad F=\frac{\partial}{\partial t}+\sum_{i=1}^{n}f_i\frac{\partial}{\partial x_i}.$$

Denote by $\mathcal{F}_u$ the distribution generated by fields $F_j=[\partial/\partial u_j,F], j=1,\dots,m$. Note that $\dim\mathcal{F}_u=\operatorname{rank}(\partial f/\partial u)=m$.

Theorem 1. (Kanatnikov *et al*, 1994) System (1) possesses a symmetry X of the form

$$X=H+\sum_{j=1}^{m}\vartheta_j(t,x,u)\frac{\partial}{\partial u_j} \quad (9)$$

if and only if the vector field $[F,H]-F(\xi)F$ lies in the distribution $\mathcal{F}_u$ and system (8) holds. In this case, the components $\vartheta_1,\dots,\vartheta_m$ of X are uniquely determined by the condition

$$[F,H]-F(\xi)F=\sum_{j=1}^{m}\vartheta_jF_j. \quad (10)$$

Theorem 2. (Kanatnikov *et al*, 1994) If a vector field X (2) is a symmetry of system (1) and $\operatorname{rank}(\partial f/\partial u)\geq 2$ everywhere on E, then the components $\xi,\eta_1,\dots,\eta_n$ of X are independent of u and system (8) is trivial.

4. FIRST INTEGRALS OF CONTROL SYSTEMS AND SYMMETRIES

A first integral of system (1) is a function $\alpha(t,x,u)$ which is constant along any solution $(x(t),u(t))$ of the system. In other words, a first integral is a function α with its time–derivative $\dot\alpha(t,x,u)|_{(1)}$ according to system (1) being equal to 0. Hence

$$\frac{\partial\alpha}{\partial u_j}=0,\quad j=1,\dots,m,$$

and

$$F(\alpha)=\frac{\partial\alpha}{\partial t}+\sum_{i=1}^{n}f_i\frac{\partial\alpha}{\partial x_i}=0.$$

First integrals of system (1) form a ring under the standard addition and multiplication.

Theorem 3. (Kanatnikov *et al*, 1994) If a vector field X is a symmetry of system (1) then for each first integral α the field αX is also a symmetry of system (1). The set of all symmetries of system (1) is a module over the ring of first integrals.

Theorem 4. (Kanatnikov *et al*, 1994) The family of all symmetries X of system (1) of the form (2) with ξ being a first integral, is involutive.

5. AFFINE SYSTEMS

Affine control system

$$\dot x=a(t,x)+\sum_{j=1}^{m}b_j(t,x)u_j, \quad (11)$$

where a, b_1, …, b_m: $\mathbb{R}^1\times\mathbb{R}^n\to\mathbb{R}^n$ are smooth functions, corresponds uniquely to vector fields

$$A=\frac{\partial}{\partial t}+\sum_{i=1}^{n}a_i\frac{\partial}{\partial x_i},$$
$$B_j=\sum_{i=1}^{n}b_{ij}\frac{\partial}{\partial x_i},\quad j=1,\dots,m,$$

with $(a_1,\dots,a_n)=a(t,x)$, $(b_{1j},\dots,b_{nj})=b_j(t,x)$. The field F has the form

$$F=A+\sum_{j=1}^{m}u_jB_j.$$

Denote by $\mathcal{B}$ the distribution generated by fields $B_1,\dots,B_m$.

Theorem 5. (Kanatnikov *et al*, 1994) Let the components $\xi,\eta_1,\dots,\eta_n$ of a vector field H be independent of u. A vector field X of the form (9) is a symmetry of system (11) if and only if the vector fields

$$[A,H]-A(\xi)A,\quad [B_k,H]-B_k(\xi)A \quad (12)$$

for $k=1,\dots,m$ lie in the distribution $\mathcal{B}$. The components $\vartheta_1,\dots,\vartheta_m$ of X are uniquely determined by the conditions

$$\vartheta_k = \vartheta_k^{(0)}(t,x) + \sum_{j=1}^{m} \vartheta_{kj}^{(1)}(t,x)u_j - F(\xi)u_k,$$

$$[A,H] - A(\xi)A = \sum_{j=1}^{m} \vartheta_j^{(0)} B_j, \qquad (13)$$

$$[B_k,H] - B_k(\xi)A = \sum_{j=1}^{m} \vartheta_{jk}^{(1)} B_j, \quad k=1,\ldots,m.$$

6. THE SECOND GEOMETRIC INTERPRETATION

For system (1), the *diffiety* (or *infinitely prolonged system*) is an infinite-dimensional manifold $\mathcal{E}^\infty$ with coordinates

$$(t, x, u^{(0)}, u^{(1)}, \ldots, u^{(l)}, \ldots), \qquad (14)$$

where $u^{(l)}$ denotes the vector variable corresponding to the lth order derivative of u with respect to t. The *Cartan distribution* on $\mathcal{E}^\infty$ is one-dimensional and is determined by the vector field

$$D = \frac{\partial}{\partial t} + f(t,x,u^{(0)})\frac{\partial}{\partial x} + u^{(1)}\frac{\partial}{\partial u^{(0)}} + \ldots + u^{(s+1)}\frac{\partial}{\partial u^{(s)}} + \ldots, \qquad (15)$$

which is called the *total derivative with respect to* t on $\mathcal{E}^\infty$. The Lie derivative along D is simply the time–derivative according to system (1). We denote by $D\omega$ the Lie derivative of the form ω along D.

A smooth function on $\mathcal{E}^\infty$ is a function smoothly depending on a finite (but arbitrary) number of coordinates (14). By $\mathcal{F}(\mathcal{E})$ denote the $\mathbb{R}$–algebra of smooth functions on $\mathcal{E}^\infty$. Differential forms on $\mathcal{E}^\infty$ are finite sums, whereas vector fields may be given by infinite sums with coefficients in $\mathcal{F}(\mathcal{E})$ (see, for example, (15)).

A vector field of the form $hD, h \in \mathcal{F}(\mathcal{E})$, is called *horizontal*. A vector field on $\mathcal{E}^\infty$ without a term $\partial/\partial t$ is called *vertical*. A vertical field X on $\mathcal{E}^\infty$ is called a *higher* (*infinitesimal*) *symmetry* of system (1) if $[X, D] = 0$.

A motivation of the last definition is the following. A vector field on $\mathcal{E}^\infty$ is called *integrable* if it possess a one–parameter group of diffeomorphisms (a flow). Since $\mathcal{E}^\infty$ is an infinite-dimensional manifold, vector fields on $\mathcal{E}^\infty$ are not usually integrable. Nevertheless let us first consider an integrable field Y. Suppose all diffeomorphisms of its flow take each solution of (1) to a solution again. Since solutions of (1) are integral curves of the field D, we have

$$[Y, D] = aD \qquad (16)$$

for some function a on $\mathcal{E}^\infty$. Consider now an arbitrary (may be nonintegrable) vector field Y on $\mathcal{E}^\infty$ satisfying (16). It is uniquely represented as the sum of a vertical field X and a horizontal field hD for some function h on $\mathcal{E}^\infty$, i. e., $Y = X + hD$. Condition (16) means that $[X, D] = 0$ and h is an arbitrary function on $\mathcal{E}^\infty$. Thus the set of all fields satisfying (16) is split in equivalence classes and each class contains a higher symmetry.

7. INFINITESIMAL BRUNOVSKÝ FORM FOR NONAUTONOMOUS SYSTEMS

Here we remind some concepts from (Aranda–Bricaire *et al*, 1995) and simultaneously generalize them to the nonautonomous case.

Let $\mathcal{C}^1\Lambda(\mathcal{E})$ be the $\mathcal{F}(\mathcal{E})$–module of differential 1–forms on $\mathcal{E}^\infty$ belonging to the codistribution corresponding to the Cartan distribution, i. e.,

$$\omega \in \mathcal{C}^1\Lambda(\mathcal{E}) \quad \Leftrightarrow \quad \omega(D) = 0.$$

Define the operator $d_\mathcal{C} : \mathcal{F}(\mathcal{E}) \longrightarrow \mathcal{C}^1\Lambda(\mathcal{E})$ by the rule $f \mapsto df - D(f)dt$. The operator $d_\mathcal{C}$ possesses many properties of the differential d. In particular,

$$d_\mathcal{C} f(x, u, \ldots) = \sum_i \frac{\partial f}{\partial x_i} d_\mathcal{C} x_i + \sum_j \frac{\partial f}{\partial u_j} d_\mathcal{C} u_j + \ldots.$$

However $d_\mathcal{C} f = 0$ iff f is a function of t.

Obviously, in coordinate system (14) the module $\mathcal{C}^1\Lambda(\mathcal{E})$ is generated by forms

$$d_\mathcal{C}x_1, \ldots, d_\mathcal{C}x_n, d_\mathcal{C}u_1^{(0)}, \ldots, d_\mathcal{C}u_m^{(0)}, \ldots, d_\mathcal{C}u_1^{(l)}, \ldots, d_\mathcal{C}u_m^{(l)}, \ldots.$$

Denote by $\mathcal{H}_0$ the $\mathcal{F}(\mathcal{E})$–submodule of $\mathcal{C}^1\Lambda(\mathcal{E})$ generated by forms $d_\mathcal{C}x_1, \ldots, d_\mathcal{C}x_n$. By definition, put

$$\mathcal{H}_{k+1} = \{\omega \in \mathcal{H}_k | D\omega \in \mathcal{H}_k\}, \quad k \geq 0.$$

The *dimension* of some submodule $\mathcal{H} \subset \mathcal{C}^1\Lambda(\mathcal{E})$ at a point $\theta \in \mathcal{E}^\infty$ is the dimension of the space of covectors $\{\omega_\theta | \omega \in \mathcal{H}\}$. A point $\theta \in \mathcal{E}^\infty$ is called *Brunovský–regular* (or shortly *B–regular*) if in a neighborhood of θ one has $\operatorname{rank}(\partial f/\partial u) = m$ and for any $k > 0$ the dimensions of $\mathcal{H}_k$ and $\mathcal{H}_k + D(\mathcal{H}_k)$ are constant.

Note that the dimension of $\mathcal{H}_k$ at any point is finite and $\mathcal{H}_{k+1} \subset \mathcal{H}_k$. It follows that in a neighborhood of a B–regular point there exists an integer k^* such that $\mathcal{H}_{k+1} = \mathcal{H}_k = \mathcal{H}_{k^*}$ for $k \geq k^*$. By ρ denote the dimension of $\mathcal{H}_{k^*}$ in a neighborhood of a B–regular point under consideration.

Remark 1. In the autonomous case we can consider only functions, differential forms, and vector fields that are independent of t. In this case, $d_\mathcal{C} f = df$, $\mathcal{E}^\infty$ is a manifold with coordinates $(x, u^{(0)}, \ldots, u^{(l)}, \ldots)$ (without t), $\mathcal{C}^1\Lambda(\mathcal{E})$ is identified with $\Lambda^1(\mathcal{E}^\infty)$. Also, all concepts and facts from this section are transformed to concepts and facts from (Aranda–Bricaire *et al*, 1995).

Theorem 6. In a neighborhood of a B–regular point for system (1) there exist ρ functions $\chi_1, \dots, \chi_\rho$ of $t, x_1, \dots, x_n$ and m forms $\omega_1, \dots, \omega_m$ from $\mathcal{H}_0$ such that
(1) $\{d_{\mathcal{C}}\chi_1, \dots, d_{\mathcal{C}}\chi_\rho\}$ is a basis of the module $\mathcal{H}_{k^*}$;
(2) the functions $\chi_1, \dots, \chi_\rho$ and their total derivatives with respect to t satisfy a system of the form

$$\{\dot{\chi}_i = \gamma_i(t, \chi_1, \dots, \chi_\rho), \quad i = 1, \dots, \rho; \quad (17)$$

(3) $\{d_{\mathcal{C}}\chi_1, \dots, d_{\mathcal{C}}\chi_\rho\} \cup \{D^j(\omega_k) | k = 1, \dots, m, j \geq 0\}$ is a basis of the module $\mathcal{C}^1\Lambda(\mathcal{E})$.

The proof is similar to that of the corresponding theorem from (Aranda–Bricaire *et al*, 1995). Moreover, the infinitesimal Brunovský form given in the same work for the autonomous case can be generalized to the nonautonomous case.

8. HIGHER SYMMETRIES OF CONTROL SYSTEMS

By Theorem 6, it follows that in a neighborhood of a B–regular point there exist functions $\{g_{l,i}, h_{l,k,j}\}$ on $\mathcal{E}^\infty$ such that

$$d_{\mathcal{C}}x_l = \sum_{i=1}^{\rho} g_{l,i} d_{\mathcal{C}}\chi_i + \sum_{k=1}^{m}\sum_{j=0}^{r_k} h_{l,k,j} D^j \omega_k \quad (18)$$

for any $l = 1, \dots, n+m$ and some $r_1, \dots, r_m$, where $x_l = u_{l-n}$ for $l = n+1, \dots, n+m$.

Theorem 7. (Chetverikov, 1999) In a neighborhood of a B–regular point any higher symmetry of system (1) has the form

$$\square_\varphi = \sum_{i=1}^{n} \varphi_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{m}\sum_{j=0}^{\infty} D^j \varphi_{i+n} \frac{\partial}{\partial u_i^{(j)}}, \quad (19)$$

where

$$\varphi_l = \sum_{i=1}^{\rho} g_{l,i} a_i + \sum_{k=1}^{m}\sum_{j=0}^{r_k} h_{l,k,j} D^j \psi_k, \quad (20)$$

for $l = 1, \dots, n+m$, $\psi_1, \dots, \psi_m$ are arbitrary functions on $\mathcal{E}^\infty$, $a_1, \dots, a_\rho$ are arbitrary functions of $t, \chi_1, \dots, \chi_\rho$ such that

$$\frac{\partial a_i}{\partial t} + \sum_{\alpha=1}^{\rho} \gamma_\alpha \frac{\partial a_i}{\partial \chi_\alpha} = \sum_{\alpha=1}^{\rho} a_\alpha \frac{\partial \gamma_i}{\partial \chi_\alpha}, \quad (21)$$

for $i = 1, \dots, \rho$.

The vector function $\varphi = (\varphi_1, \dots, \varphi_{n+m})$ is called the *generating function* of symmetry (19).

Remark 2. The vector field

$$\sum_{\alpha=1}^{\rho} a_\alpha \frac{\partial}{\partial \chi_\alpha} \quad (22)$$

is a symmetry of the system of ordinary differential equations (17). Condition (21) means that the commutator of the fields (22) and

$$\frac{\partial}{\partial t} + \sum_{\alpha=1}^{\rho} \gamma_\alpha \frac{\partial}{\partial \chi_\alpha}$$

vanishes.

9. EXAMPLE

Find classical and higher symmetries of the control system

$$\begin{cases} \dot{x}_1 = tx_1^2 \\ \dot{x}_2 = x_4 u_1 \\ \dot{x}_3 = u_1 \\ \dot{x}_4 = u_2. \end{cases} \quad (23)$$

In the case of classical symmetries, we use results of Section 5. We have $n = 4$, $m = 2$, and

$$A = \frac{\partial}{\partial t} + tx_1^2 \frac{\partial}{\partial x_1}.$$

The distribution $\mathcal{B}$ is generated by fields

$$B_1 = x_4 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \quad B_2 = \frac{\partial}{\partial x_4}.$$

The corresponding codistribution is generated by forms

$$\alpha_1 = dt, \quad \alpha_2 = dx_1, \quad \alpha_3 = dx_2 - x_4 dx_3.$$

By Theorem 2, it follows that the components $\xi, \eta_1, \dots, \eta_4$ of a desired symmetry X are independent of u_1 and u_2. By Theorem 5, the defining equations for symmetries can be expressed as

$$\alpha_i(Y) = 0 \quad (24)$$

for any field Y of the form (12) and $i = 1, 2, 3$. Since equalities (24) are trivial in the case $i = 1$, we obtain 6 differential equations for the components ξ, η_i. Introducing the functions

$$z = \eta_1 - tx_1^2\xi, \quad v = \eta_2 - x_4\eta_3, \quad (25)$$

these equations can be written as

$$A(z) = 2tx_1 z, \quad B_1(z) = 0, \quad B_2(z) = 0, \quad (26)$$
$$A(v) = 0, \quad B_1(v) = \eta_4, \quad B_2(v) = \eta_3. \quad (27)$$

Solving the system of equations (26), we get

$$z = a\Big(t^2 + \frac{2}{x_1}\Big)x_1^2,$$

where a is an arbitrary smooth function of one variable. From the first equation in (27) it follows that

$$v = b\Big(t^2 + \frac{2}{x_1},\ x_2,\ x_3,\ x_4\Big),$$

where b is an arbitrary function of four variable. Using the second and the third equations in (27), we get expressions for η_3 and η_4. Finally, using (25), we find η_1 and η_2.

Thus any infinitesimal classical symmetry of system (23) has the form

$$X = \xi\frac{\partial}{\partial t} + \sum_{i=1}^{4}\eta_i(t,x,u)\frac{\partial}{\partial x_i} + \sum_{j=1}^{2}\vartheta_j(t,x,u)\frac{\partial}{\partial u_j}$$

and is uniquely determined by functions a, b, and ξ. The component ξ is an arbitrary function of $t, x_1, \ldots, x_4$. Besides,

$$\eta_1 = tx_1^2\xi + a\Big(t^2 + \frac{2}{x_1}\Big)x_1^2,$$
$$\eta_2 = v + x_4\frac{\partial v}{\partial x_4},$$
$$\eta_3 = \frac{\partial v}{\partial x_4},$$
$$\eta_4 = x_4\frac{\partial v}{\partial x_2} + \frac{\partial v}{\partial x_3}.$$

From (10) it follows that

$$\vartheta_1 = F(\eta_3) - F(\xi)u_1, \quad \vartheta_2 = F(\eta_4) - F(\xi)u_2,$$

where $F = A + u_1B_1 + u_2B_2$.

To obtain higher symmetries of system (23), we use results of Section 7 and 8. In our case, any element of $\mathcal{H}_0$ has the form

$$\omega = \sum_{i=1}^{4} f_i d_{\mathcal{C}}x_i, \quad f_i \in \mathcal{F}(\mathcal{E}).$$

We get

$$D\omega = \sum_{i=1}^{4} Df_i d_{\mathcal{C}}x_i + f_1 2tx_1 d_{\mathcal{C}}x_1 + f_2(x_4 d_{\mathcal{C}}u_1 + u_1 d_{\mathcal{C}}x_4) + f_3 d_{\mathcal{C}}u_1 + f_4 d_{\mathcal{C}}u_2. \quad (28)$$

If $D\omega \in \mathcal{H}_0$, then the coefficients of $d_{\mathcal{C}}u_1$ and $d_{\mathcal{C}}u_2$ in (28) vanish. Whence

$$f_2x_4 + f_3 = 0, \quad f_4 = 0.$$

Therefore the module $\mathcal{H}_1$ is generated by $d_{\mathcal{C}}x_1$ and $d_{\mathcal{C}}x_2 - x_4d_{\mathcal{C}}x_3$.

In the same way, the condition

$$D\Big(f_1d_{\mathcal{C}}x_1 + f_2\left(d_{\mathcal{C}}x_2 - x_4d_{\mathcal{C}}x_3\right)\Big) \in \mathcal{H}_1,$$

means that $f_2 = 0$ and $d_{\mathcal{C}}x_1 \in \mathcal{H}_2$. We see that $k^* = 2$ and $d_{\mathcal{C}}x_1 \in \mathcal{H}_{k^*}$. Thus $\rho = 1, \chi_1 = x_1$, system (17) consists of the first equation of system (23), and $\omega_1 = d_{\mathcal{C}}x_2 - x_4d_{\mathcal{C}}x_3 \in \mathcal{H}_1$. The 1–form ω_2 should be chosen such that

$$\{d_{\mathcal{C}}\chi_1, \omega_1, D\omega_1, \omega_2\}$$

is a basis of the module $\mathcal{H}_0$. We put $\omega_2 = d_{\mathcal{C}}x_3$. In this case, the set of B–regular points is $\{u_1 \neq 0\}$ and we obtain

$$d_{\mathcal{C}}x_1 = d_{\mathcal{C}}\chi_1, \quad d_{\mathcal{C}}x_2 = \omega_1 + x_4\omega_2,$$
$$d_{\mathcal{C}}x_3 = \omega_2, \quad d_{\mathcal{C}}x_4 = \frac{1}{u_1}D\omega_1 + \frac{u_2}{u_1}\omega_2,$$
$$d_{\mathcal{C}}u_1 = D\omega_2, \quad d_{\mathcal{C}}u_2 = D\left(d_{\mathcal{C}}x_4\right).$$

Condition (21) has the form

$$\frac{\partial a}{\partial t} + tx_1^2\frac{\partial a}{\partial x_1} = 2tx_1a.$$

Solving the last equation and using Theorem 7, we get generating functions of all higher symmetries:

$$\varphi_1 = x_1^2a\Big(t^2 + \frac{2}{x_1}\Big), \quad \varphi_2 = \psi_1 + x_4\psi_2,$$
$$\varphi_3 = \psi_2, \quad \varphi_4 = \frac{1}{u_1}D\psi_1 + \frac{u_2}{u_1}\psi_2,$$
$$\varphi_5 = D\psi_2, \quad \varphi_6 = D\varphi_4,$$

where a is an arbitrary function of one variable, ψ_1, ψ_2 are arbitrary functions on $\mathcal{E}^\infty$.

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REFERENCES

Aranda–Bricaire, E., C. H. Moog, and J.-B. Pomet (1995). An infinitesimal Brunovsky form for nonlinear systems with applications to dynamic linearization, in *Geometry in Nonlinear Control and Differential Inclusions*, B. Jakuczyk, W. Respondek, and T. Rzezuchowski, eds., Banach Center Publications, **32**, Warsaw, Poland, , Institute of Mathematics, Polish Academy of Sciences, 19–33.

Chetverikov, V.N. (1999). Higher symmetries and infinitesimal Brunovský form of control systems. *Nauchniy Vestnik MGTU GA* (16), 77–83 (in Russian).

Chetverikov, V.N. (2001). New flatness conditions for control systems, in *Proc. of NOLCOS'01*, 168–173.

Fliess, M., J. Lévine, Ph. Martin, and P. Rouchon. (1995). Flatness and defect of non–linear systems: Introductory theory and examples. *Int. J. Contr.* **61** (6), 1327–1361.

Kanatnikov, A.N. and A.P. Krishchenko. (1994). Symmetries and Decomposition of Nonlinear Systems. *Differential Equations.* **30** (11), 1735–1745.

Krasil'shchik, I.S. and A.M. Vinogradov, Eds. (1999). *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*. Amer. Math. Soc.. Providence, RI.