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ASYMPTOTIC PROPERTIES AND STABILITY OF ZEROS OF SAMPLED MULTIVARIABLE SYSTEMS

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Abstract: Unstable zeros limit the achievable control performance. When a continuous-time system is discretized using the zero-order hold, there is no simple relation which shows how the zeros of the continuous-time system are transformed by sampling. This paper analyzes the asymptotic behavior of the limiting zeros for multivariable systems and derives a new condition for the zeros to be stable for sufficiently small sampling periods. Furthermore, the result is applied to a collocated matrix second-order system. *Copyright © 2002 IFAC*

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1. INTRODUCTION

It is well known that unstable zeros limit the achievable control performance, particular if zero cancellation techniques are used (Åström *et al.*, 1984). When a continuous-time plant is discretized using zero-order hold, poles p_i are transformed as $p_i \to \exp(p_i T)$, where T is the sampling period. However, the transformations of zeros are much more complicated and the stability of zeros is not preserved in the discretization process in some cases (Åström *et al.*, 1984). Since it is generally impossible to derive a closed-form expression between the continuous-time zeros and the discrete-time ones, the efforts were devoted to the analysis of the limiting zeros in the earlier research studies (Åström *et al.*, 1984; Hagiwara *et al.*, 1993; Weller, 1999). Here, the limiting zeros mean the zeros of a discete-time system in the limiting case when the sampling period tends to zero.

For single-input, single-output systems, at least one of the limiting zeros lies strictly outside the unit circle if the relative degree of a continuous-time transfer function is greater than or equal to three (Åström *et al.*, 1984). This fact indicates that even though all the zeros of such a continuous-time system are stable, the corresponding discrete-time system has an unstable zero in the limiting case as T tends to zero. Second, when the relative degree of a transfer function is one or two, all the limiting zeros are located just on the unit circle, i.e., in the marginal case of the stability. Thus, the asymptotic behavior of the limiting zeros is an interesting issue because the limiting zeros are stable for sufficiently small T if they approach the unit circle from inside as T tends to zero. Åström *et al.* (1984) and Hagiwara *et al.* (1993) analyzed the asymptotic behavior of the limiting zeros and derived stability conditions of the limiting zeros for sufficiently small T.

For multivariable systems, Hayakawa *et al.* (1983) and Weller (1999) studied the limiting zeros. The properties of the zeros for multivariable systems are characterized by the degrees of the infinite elementary divisors. It was shown in Hayakawa *et al.* (1983) that when a continuous-time system has a degree of the infinite elementary divisors greater than four, at least one of the limiting zeros of the corresponding discrete-time system is unstable. This implies that the stability of zeros is not preserved in the above cases. Meanwhile, when all the degrees of the infinite elementary divisors are two or three, then the limiting zeros lie just on the unit circle. Therefore, attention should be directed to the asymptotic behavior of the limiting zeros from the view point of the stability of the zeros for sufficiently small T.

This paper investigates how the limiting zeros reach the unit circle in the cases of all the degrees of the infinite elementary divisors two or three when T goes to zero, and derives stability conditions of the limiting zeros for sufficiently small T. Furthermore, the result is applied to collocated matrix second-order systems.

The asymptotic behavior of the limiting zeros for multivariable systems was presented in the previous report (Ishitobi, 2000), but only a limited set of particular cases was treated and the result could not be applied to matrix second-order systems.

2. PRELIMINARIES

Consider a time-invariant, controllable, observable, m-input m-output n-th order linear system

$$S_C : \begin{cases} \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C\boldsymbol{x}(t) \end{cases} \tag{1}$$

with a state vector $\boldsymbol{x}(t) \in \boldsymbol{R}^n$, an input vector $\boldsymbol{u}(t) \in \boldsymbol{R}^m$, an output vector $\boldsymbol{y}(t) \in \boldsymbol{R}^m$, and the discretized system

$$S_D : \begin{cases} \boldsymbol{x}((k+1)T) = \Phi\boldsymbol{x}(kT) + \Psi\boldsymbol{u}(kT) \\ \boldsymbol{y}(kT) = C\boldsymbol{x}(kT) \end{cases} \tag{2}$$

with $\boldsymbol{x}(kT) \in \boldsymbol{R}^n$, $\boldsymbol{u}(kT) \in \boldsymbol{R}^m$, $\boldsymbol{y}(kT) \in \boldsymbol{R}^m$, where

$$\Phi = e^{AT}, \quad \Psi = \int_0^T e^{At} B dt \tag{3}$$

It is assumed that S_C is invertible. Then, the definitions of system zeros, invariant zeros and transmission zeros for S_C coincide. Thus, these zeros are simply called the zeros of S_C throughout the paper. The zeros of S_D have the same properties for a sufficiently small T (Hayakawa *et al.*, 1983).

We are here interested in the relations of the zeros of S_D to those of S_C.

The zeros of S_C are given by the roots of $|\Gamma(s)| = 0$ where $\Gamma(s)$ denotes the system matrix or the pencil of S_C defined by

$$\Gamma(s) = \begin{bmatrix} A - sI_n & B \\ C & O_m \end{bmatrix} \tag{4}$$

The zeros of S_D are similarly calculated using the system matrix $\Gamma_T(z)$ of S_D where

$$\Gamma_T(z) = \begin{bmatrix} \Phi - zI_n & \Psi \\ C & O_m \end{bmatrix} \tag{5}$$

The properties of the zeros are characterized by the degrees of the infinite elementary divisors (Gantmacher, 1959; Rosenbrock, 1970). We denote here by $\mu_1, \cdots, \mu_m$ $(2 \le \mu_1 \le \cdots \le \mu_m)$ the degrees of the infinite elementary divisors of $\Gamma(s)$. Then, S_C has $(n - \sum_{i=1}^m (\mu_i - 1))$ zeros (Suda and Mutsuyoshi, 1978). Hayakawa *et al.* (1983) showed that S_D has $(n-m)$ limiting zeros if the difference between the largest and smallest degrees of the infinite elementary divisors of $\Gamma(s)$ is less than two, for instance, $\mu_1 = \cdots = \mu_\ell = \mu$ and $\mu_{\ell+1} = \cdots = \mu_m = \mu + 1$, and that $(n - (\mu-1)\ell)$ limiting zeros among them are located at the point $z = 1$ and the remaining $((\mu-1)\ell - m)$ limiting zeros coincide with the roots of the equation

$$B^{\ell}_{\mu-1}(z) B^{m-\ell}_{\mu}(z) = 0 \tag{6}$$

where

$$\begin{aligned} B_\tau(z) &= b_1^\tau z^{\tau-1} + b_2^\tau z^{\tau-2} + \cdots + b_\tau^\tau \\ b_k^\tau &= \sum_{\ell=1}^{k} (-1)^{k-\ell} \ell^\tau \begin{pmatrix} \tau+1 \\ k-\ell \end{pmatrix}, \\ k &= 1, 2, \cdots, \tau \end{aligned}$$

The former $(n - (\mu-1)\ell)$ limiting zeros are called intrinsic zeros and the latter $((\mu-1)\ell - m)$ limiting zeros are discretization zeros. The polynomials $B_\tau(z)$ are listed for a few values of τ as follows. $B_1(z) = 1$, $B_2(z) = z + 1$, $B_3(z) = z^2 + 4z + 1$ and $B_4(z) = z^3 + 11z^2 + 11z + 1$. A similar result was obtained for decouplable systems by Weller (1999).

From these results, in most cases when S_C has a degree of the infinite elementary divisors more than or equal to four, i.e. $\mu_i \geq 4$, at least one of the zeros of S_D is located strictly outside the unit circle for sufficiently small T. Therefore, it is obvious that the discussion of the stability conditions for zeros of discretized systems for sufficiently small T should be limited to a class of S_C with all the degrees of the infinite elementary divisors less than or equal to three, i.e. $\mu_i \leq 3$.

3. MAIN RESULTS

In this section, at first, the asymptotic behavior of the limiting zeros of a multivariable system is studied when the degrees of the infinite elementary divisors are two or three, i.e., $\mu_1 = \cdots = \mu_{m-k} = 2$ and $\mu_{m-k+1} = \cdots = \mu_m = 3$ for $0 \leq k \leq m$. Second, according to the result obtained, we derive a stability condition of the limiting zeros for sufficiently small sampling periods. The first result is described in the following theorem.

Theorem 1: Let S_C be a continuous-time system (1) with the assumption of invertibility and r_i $(i = 1, \cdots, n-m-k)$ be the zeros of S_C.

Case (a); $k = 0$, $\mu_1 = \cdots = \mu_m = 2$:

- All the zeros z_i $(i = 1, \cdots, n-m)$ of S_D are the intrinsic zeros and obey

$$z_i = 1 + r_i T + \frac{(r_i T)^2}{2} + O(T^3) \qquad (7)$$

Case (b); $\mu_1 = \cdots = \mu_{m-k} = 2,\ \mu_{m-k+1} = \cdots = \mu_m = 3$ $(1 \leq k \leq m-1)$:

- S_D has $(n-m-k)$ intrinsic zeros z_i $(i = 1, \cdots, n-m-k)$ and k discretization zeros $z_{n-m-k+i}$ $(i = 1, \cdots, k)$, and the intrinsic zeros z_i $(i = 1, \cdots, n-m-k)$ can be expressed as

$$z_i = 1 + r_i T + O(T^2) \qquad (8)$$

  and the remaining discretization zeros $z_{n-m-k+i}$ $(i = 1, \cdots, k)$ have the form

$$z_i = -1 - \lambda_i\{\Theta_2 {\Theta_1}^{-1}\} T + O(T^2) \qquad (9)$$

  where $\Theta_2 = H_{1B} C A^2 B G_{1R}$ and $\Theta_1 = H_{1B} CAB G_{1R}$. Here, $\lambda_i\{\cdot\}$ denotes an eigenvalue of a matrix, and H_{1B} and G_{1R} are submatrices of non-singular $m \times m$ matrices

$$H_1 = \begin{bmatrix} H_{1T} \\ H_{1B} \end{bmatrix},\ G_1 = \begin{bmatrix} G_{1L} & G_{1R} \end{bmatrix} \quad (10)$$

  satisfying

$$H_1 CB G_1 = \begin{bmatrix} I_{m-k} & \mathrm{O} \\ \mathrm{O} & \mathrm{O}_k \end{bmatrix} \qquad (11)$$

  and the dimensions of H_{1T}, H_{1B}, G_{L1} and G_{1R} are $(m-k) \times m$, $k \times m$, $m \times (m-k)$ and $m \times k$, respectively.

Case (c); $k = m$, $\mu_1 = \cdots = \mu_m = 3$:

- Among the zeros of S_D, there are $(n-m)$ intrinsic zeros and m discretization zeros, and the intrinsic zeros z_i $(i = 1, \cdots, n-2m)$ can be represented as

$$z_i = 1 + r_i T + O(T^2) \qquad (12)$$

  and, for the remaining discretization zeros z_{n-2m+i} $(i = 1, \cdots, m)$ there is

$$z_i = -1 - \lambda_i\{\Theta_{02} {\Theta_{01}}^{-1}\} T + O(T^2) \qquad (13)$$

where $\Theta_{02} = CA^2B$ and $\Theta_{01} = CAB$.

Theorem 1 will be proved in the appendix.

Remark 1: Hayakawa *et al.* (1983) showed that the intrinsic zeros z_i can be approximated by $z_i = 1 + r_i T$. It is obvious that (7) provides a more accurate approximation for the intrinsic zeros in the case of all the dgrees of the infinite elementary divisors two, i.e., $\mu_1 = \cdots = \mu_m = 2$. This result means that if S_C has a pure imaginary zero $r_i = j\omega$, then for the corresponding intrinsic zero z_i of S_D, we obtain $|z_i|^2 = |1 + j\omega T - (\omega T)^2/2 + O(T^3)|^2 = 1 + O(T^3)$ and we cannot still determine whether the intrinsic zero z_i is located inside or outside the unit circle by the approximation of (7).

From Theorem 1, the following result is immediate. Case (a) was presented in also (Hayakawa *et al.*, 1983).

Theorem 2: Let S_C be a continuous-time system (1) with the assumption of invertibility.

Case (a); $k = 0$, $\mu_1 = \cdots = \mu_m = 2$:

- all the zeros of the discretized system are located strictly inside the unit circle for sufficiently small T if all the $(n-m)$ zeros of S_C are stable.

Case (b); $\mu_1 = \cdots = \mu_{m-k} = 2,\ \mu_{m-k+1} = \cdots = \mu_m = 3$ $(1 \leq k \leq m-1)$:

- all the zeros of the discretized system are located strictly inside the unit circle for sufficiently small T if all the $(n-m-k)$ zeros of

S_C are stable and

$$\Re\left[\lambda_i\left\{\Theta_2{\Theta_1}^{-1}\right\}\right] < 0, i = 1, \cdots, k \qquad (14)$$

Case (c); $k = m$, $\mu_1 = \cdots = \mu_m = 3$:

- all the zeros of the discretized system are located strictly inside the unit circle for sufficiently small T if all the $(n - 2m)$ zeros of S_C are stable and

$$\Re\left[\lambda_i\left\{\Theta_{02}{\Theta_{01}}^{-1}\right\}\right] < 0, i = 1, \cdots, m \qquad (15)$$

4. AN APPLICATION TO MATRIX SECOND-ORDER SYSTEMS

A linear model of a large space structure is known as an example of matrix second-order systems (Williams, 1989).

Consider an n'-mode, m-input, m-output second-order collocated system described by

$$M\ddot{\boldsymbol{q}}(t) + D\dot{\boldsymbol{q}}(t) + K\boldsymbol{q}(t) = V\boldsymbol{u}(t), \quad \boldsymbol{y}(t) = V^T\boldsymbol{q}(t) \qquad (16)$$

where $\boldsymbol{q} \in \boldsymbol{R}^{n'}$ is the vector of generalized coordinates, $\boldsymbol{u} \in \boldsymbol{R}^m$ that of applied actuator inputs, and $\boldsymbol{y} \in \boldsymbol{R}^m$ that of sensor outputs. Suppose that the mass, stiffness, and damping matrices of the system satisfy $M = M^T > O$, $D = D^T \geq O$ and $K = K^T \geq O$, respectively, while the control influence matrix V is of full column rank. We further assume that

$$\mathrm{rank}[D,\ V] = \mathrm{rank}[K,\ V] = n' \qquad (17)$$

It is possible to rewrite the system description (16) to the usual first-order state-space description by taking the state variable as $\boldsymbol{x}(t) = [\boldsymbol{q}^T(t),\ \dot{\boldsymbol{q}}^T(t)]^T$. Namely, it is obtained that

$$A = \begin{bmatrix} O & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad B = \begin{bmatrix} O \\ M^{-1}V \end{bmatrix}, \quad C = \begin{bmatrix} V^T & O \end{bmatrix} \qquad (18)$$

where $n = 2n'$ is the dimension of the state-space system. It is assumed that the system (18) is completely controllable and completely observable. A necessary and sufficient condition for the assumption can be given (Laub and Arnold, 1984) in the form of the second-order system (16).

The following result is obtained from Theorem 2.

Theorem 3. For a second-order linear system described by (16), if the matrix $V^TM^{-1}DMV$ is positive definite, then all the zeros of the corresponding discrete-time system are stable for sufficiently small sampling periods.

Proof of Theorem 3. First, from the fact that $CB = O$ and $|CAB| = |V^TM^{-1}V| \neq 0$ because of the positiveness of M, it is readily seen that the degrees of the infinite elementary divisors of S_C for (18) are all three. Second, all the zeros of the continuous-time system (16) with the collocated actuators and sensors are stable under the assumption (17) (Ikeda, 1990). In other words, all the zeros of S_C lie strictly in the open left-hand side of the complex plane. Third, simple calculation leads to

$$\Theta_{02}{\Theta_{01}}^{-1} = -\left(V^TM^{-1}DM^{-1}V\right) \times \left(V^TM^{-1}V\right)^{-1} \qquad (19)$$

and it is possible to show that the right-hand side of (19) is stable as follows. Then, all the conditions of Case (c) of Theorem 2 are satisfied.

In fact, the stability of (19) is shown below. Notice that λ_i implies an eigenvalue of (19), then the matrix $\lambda_iV^TM^{-1}V + V^TM^{-1}DM^{-1}V$ is singular; that is, there exists a nonzero vector $\boldsymbol{v}$ such that

$$\boldsymbol{v}^*\left(\lambda_iV^TM^{-1}V + V^TM^{-1}DM^{-1}V\right)\boldsymbol{v} = 0 \qquad (20)$$

where $\boldsymbol{v}^*$ denotes the complex conjugate and transpose vector of $\boldsymbol{v}$. Now, note that the influence matrix V is of full column rank and the matrix M is positive definite, then it follows

$$\alpha \equiv \boldsymbol{v}^*V^TM^{-1}V\boldsymbol{v} > 0 \qquad (21)$$

Further, the condition of Theorem 3 yields

$$\beta \equiv \boldsymbol{v}^*V^TM^{-1}DM^{-1}V\boldsymbol{v} > 0 \qquad (22)$$

Hence, we obtain $\lambda_i = -\beta/\alpha < 0$ which implies that $\Theta_{02}\Theta_{01}^{-1}$ is stable. Q.E.D.

5. CONCLUSIONS

This paper analyzes the asymptotic behavior of the limiting zeros for multivariable systems and gives a new stability condition of the zeros of the discrete-time systems for sufficiently small sampling periods.

The result can be applied to test the stability of zeros for collocated matrix second-order systems.

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APPENDIX: PROOF OF THEOREM 1

See (Kashiwamoto, 2000) and (Ishitobi and Ohba, 1999) for proofs of Case (a) and Case (b), respectively. Case (c) is proved below.

Under the assumption that the degrees of the infinite elementary divisors of S_C are all three, there exist (Suda and Mutsuyoshi, 1978) non-singular matrices P and Q which yield

$$\widehat{\Gamma}(s) \equiv \begin{bmatrix} \widehat{A} - sI_n & \widehat{B} \\ \widehat{C} & O_m \end{bmatrix} = P\Gamma(s)Q \quad (23)$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ O & P_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} P_{11}^{-1} & O \\ Q_{21} & Q_{22} \end{bmatrix}$$

the dimensions of P_{11}, P_{12}, P_{22}, Q_{21}, Q_{22} are $n \times n$, $n \times m$, $m \times m$, $m \times n$, $m \times m$, respectively, and

$$\widehat{A} = \left[\begin{array}{cc|c} O_m & I_m & \\ \hline O_m & O_m & O_{2m\times(n-2m)} \\ \hline \multicolumn{2}{c|}{O_{(n-2m)\times 2m}} & A_f \end{array}\right] \quad (24)$$

$$\widehat{B} = \left[\begin{array}{c} O_m \\ \hline I_m \\ \hline O_{(n-2m)\times m} \end{array}\right] \quad (25)$$

$$\widehat{C} = \left[\begin{array}{c|c} I_m & O_{m\times(n-m)} \end{array}\right] \quad (26)$$

The matrix manipulation by P and Q does not change the values of the zeros and the eigenvalues of A_f coincide with the zeros of S_C. The matrices P_{11} and P_{11}^{-1} are obtained by the following procedure.

Define a matrix G_2 by $G_2 = (CAB)^{-1}$ on the basis of the fact that $|CAB| \neq 0$. Then, we can further define an $m \times (n-2m)$ matrix S_{2C} as

$$S_{2C} = CA(I_{n-2m} - ABG_2C) \quad (27)$$

Since there exist two matrices X_{2B} of dimension $(n-2m)\times n$ and Y_{2R} of dimension $n\times(n-2m)$ such that $\left[C^T,\ S_{2C}^T,\ X_{2B}^T\right]$ and $[ABG_2,\ BG_2,\ Y_{2R}]$ are non-singular, we can determine two matrices

$$S_{2B} = X_{2B}(I_n - ABG_2C - BG_2S_{2C}) \quad (28)$$

$$T_{2R} = (I_n - ABG_2C - BG_2S_{2C}) \times Y_{2R}(S_{2B}Y_{2R})^{-1} \quad (29)$$

Then the following relation holds

$$\begin{bmatrix} C \\ S_{2C} \\ S_{2B} \end{bmatrix} \begin{bmatrix} ABG_2 & BG_2 & T_{2R} \end{bmatrix} = I_n \quad (30)$$

Now, it is possible to obtain two matrices P and Q such as

$$P = \left[\begin{array}{c|cc} P_{11} & O & L_2 \\ \hline O & \multicolumn{2}{c}{I_m} \end{array}\right] \quad (31)$$

$$Q = \left[\begin{array}{c|c} P_{11}^{-1} & O \\ \hline O & \\ G_2 K_2 & G_2 \end{array}\right] \tag{32}$$

where

$$P_{11} = \begin{bmatrix} C \\ S_{2C} \\ S_{2B} \end{bmatrix}, L_2 = \begin{bmatrix} -CA^2BG_2 \\ -S_{2C}A^2BG_2 \\ -S_{2B}A^2BG_2 \end{bmatrix} \tag{33}$$

$$P_{11}^{-1} = \begin{bmatrix} ABG_2 & BG_2 & T_{2R} \end{bmatrix} \tag{34}$$

$$K_2 = \begin{bmatrix} O & O & -CA^2T_{2R} \end{bmatrix} \tag{35}$$

and, T_{2R}, K_2, S_{2B} and S_{2C} are of dimension $n \times (n-2m)$, $n \times n$, $(n-2m) \times n$ and $m \times n$, respectively. Here, we have $A_f = S_{2B}AT_{2R}$. For more details on this see Suda (1993).

In the next step, we consider a discrete-time system matrix (5).

Now, let $\widetilde{P}$ =block-diag(P_{11}, P_{22}) and $\widetilde{Q}$ =block-diag(P_{11}^{-1}, Q_{22}) where P_{11}, P_{22} and Q_{22} are the submatrices in P and Q, and let

$$\widetilde{\Gamma}_T(z) = \widetilde{P}\Gamma_T(z)\widetilde{Q} \tag{36}$$

Then it follows (Hayakawa *et al.*, 1983) from (23) that $\widetilde{\Gamma}_T(z)$ has the form

$$\widetilde{\Gamma}_T(z) = \begin{bmatrix} \widetilde{\Phi} - zI_n & \widetilde{\Psi} \\ \widehat{C} & O_m \end{bmatrix} \tag{37}$$

where

$$\widetilde{\Phi} = e^{\widetilde{A}T}, \ \widetilde{\Psi} = \int_0^T e^{\widetilde{A}t}\widehat{B}dt, \tag{38}$$

$$\widetilde{A} = P_{11}AP_{11}^{-1} \tag{39}$$

and the zeros of $\Gamma_T(z)$ are not changed by the matrix manipulation with $\widetilde{P}$ and $\widetilde{Q}$. Substituting (33) and (35) into (39) yields to

$$\widetilde{A} = \begin{bmatrix} \Theta_{02}G_2 & I_m & O \\ S_{2C}A^2BG_2 & O_m & S_{2C}AT_{2R} \\ S_{2B}A^2BG_2 & O & A_f \end{bmatrix} \tag{40}$$

Next, define

$$\bar{\Gamma}_T(z) = \widetilde{U}\widetilde{\Gamma}_T(z)\widetilde{V} \tag{41}$$

where $\widetilde{U}$=block-diag(V^{-1}, U), $\widetilde{V}$=block-diag(V, $T^{-1}I_n$), V=block-diag(TI_m, I_{n-m}) and $U = T^{-1}I_m$. Then, we get (Hayakawa *et al.*, 1983) that

$$\bar{\Gamma}_T(z) = \begin{bmatrix} \bar{\Phi} - zI_n & \bar{\Psi} \\ \widehat{C} & O_m \end{bmatrix} \tag{42}$$

where $\bar{\Phi} = e^{\bar{A}}$, $\bar{A} = V^{-1}\widetilde{A}VT$ and $\bar{\Psi} = \int_0^1 e^{\bar{A}t}\widehat{B}dt$.

For a sufficiently small T, the matrix $\bar{A}$ can be expressed as

$$\bar{A} \approx \begin{bmatrix} \Theta_{02}G_2 & I_m & O \\ O_m & O_m & A_{ST}T \\ O & O & A_fT \end{bmatrix} + O(T^2) \tag{43}$$

where $A_{ST} = S_{2C}AT_{2R}$.

From the fact that each element of $\bar{A}^3$ vanishes or consists of terms with the order greater than or equal to two with respect to T, this linear approximation gives

$$\bar{\Phi} = \begin{bmatrix} I_m + \Xi T & I_m + \frac{\Xi T}{2} & \frac{A_{ST}T}{2} \\ O_m & I_m & A_{ST}T \\ O & O & I_{n-2m} + A_fT \end{bmatrix} + O(T^2) \tag{44}$$

where $\Xi = CA^2BG_2$.

$$\bar{\Psi} = \begin{bmatrix} \frac{1}{2}I_m + \frac{1}{6}\Xi T \\ I_m \\ O \end{bmatrix} + O(T^2) \tag{45}$$

Taking account of $\widehat{C} = \begin{bmatrix} I_m & O_{m\times(n-m)} \end{bmatrix}$ with (44) and (45), we have

$$\begin{aligned} |\bar{\Gamma}_T(z)| &= (-1)^{nm} \begin{vmatrix} \bar{\Psi} & \bar{\Phi} - zI_n \\ O_m & \widehat{C} \end{vmatrix} \\ &= \beta|(1-z)I_{n-2m} + A_fT| \\ &\quad \times \left|(1+z)I_m + \frac{1}{3}\Xi T\right| + O(T^2) \end{aligned} \tag{46}$$

where $\beta = (-1)^{m(2n+m+1)}\,|I_m/2 + \Xi T/6|$. Hence, $(n-2m)$ zeros z_i for $i = 1, \cdots, n-2m$ of S_D can be expressed as $z_i = 1 + r_iT + O(T^2)$.

Furthermore, the remaining m zeros z_i for $i = n-2m+1, \cdots, n-m$ of S_D have the form of $z_i = -1 - T\lambda_i(CA^2BG_2)/3 + O(T^2)$ for $i = n-2m+1, \cdots, n-m$.

As a result, the proof is complete. Q.E.D.