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BOUNDARY CONTROL OF A CLASS OF UNSTABLE PARABOLIC PDES VIA BACKSTEPPING [2]

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Abstract: In this paper a family of stabilizing boundary feedback control laws for a class of linear parabolic PDEs motivated by engineering applications is presented. The design procedure presented here can handle systems with an arbitrary finite number of open-loop unstable eigenvalues and is not restricted to a particular type of boundary actuation. The stabilization is achieved through the design of coordinate transformations that have the form of recursive relationships. The fundamental difficulty of such transformations is that the recursion has an infinite number of iterations. The problem of feedback gains growing unbounded as grid becomes infinitely fine is resolved by a proper choice of the target system to which the original system is transformed. We show how to design coordinate transformations such that they are sufficiently regular (not continuous but L_∞). We then establish closed–loop stability, regularity of control, and regularity of solutions of the PDE. The result is accompanied by a simulation study for a linearization of a tubular chemical reactor around an unstable steady state.

Keywords: Boundary Control, Linear Parabolic PDEs, Stabilization, Backstepping, Coordinate Transformations

1. INTRODUCTION

Motivated by the model for the chemical tubular reactor, the model of unstable burning in solid rocket propellants, and other PDE systems that appear in various engineering applications, we present an algorithm for global stabilization of a broader class of linear parabolic PDEs. The result presented here is a generalization of the ideas of Balogh and Krstić (2001). The goal is to obtain an L_∞ coordinate transformation and a boundary control law that renders the closed–loop system asymptotically stable, and additionally establish regularity of control and regularity of solutions for the closed–loop system.

The key issue with arbitrarily unstable linear parabolic PDE systems is the target system to which one is transforming the original system by coordinate transformation. For example, if one takes the standard backstepping route leading to a tri–diagonal form, the resulting transformations, if thought of as integral transformations, end up with "kernels" that are not even finite. A proper selection of the target system will result in a bounded kernel and the solutions corresponding to the controlled problem are going to be at least continuous.

The class of parabolic PDEs considered in this paper is

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + Bu_x(x,t) + \lambda(x)\, u(x,t) + \int_0^x f(x,\xi)u(\xi,t)d\xi, \quad x \in (0,1), \quad t > 0, \qquad (1)$$

where $\varepsilon > 0$ and B are constants, $\lambda(x) \in L_\infty(0,1)$ and $f(x,y) \in L_\infty([0,1]\times[0,1])$, with initial condition $u(x,0) = u^0(x)$, for $x \in [0,1]$. The boundary condition at $x = 0$ is homogeneous Dirichlet,

$$u(0,t) = 0, \qquad t > 0, \qquad (2)$$

while the Dirichlet boundary condition at the other end

$$u(1,t) = \alpha(u(t)), \qquad t > 0 \qquad (3)$$

is used as the control input, where the linear operator α represents a control law to be designed to achieve stabilization. It is assumed that the initial distribution is compatible with (2), i.e. $u^0(0) = 0$.

Our interest in systems described by (1) is twofold. First, the physical motivation for considering equation (1) is that it represents the linearization of the class of reaction–diffusion–convection equations that model many physical phenomena. Examples are numerous and among others include the problem of compressor rotating stall (Hagen *et al.*, 1999), and the linearization of an adiabatic chemical tubular reactor (Hlaváček and Hofmann, 1970).

Second, from the perspective of control theory, systems described by (1) are interesting since their discretization appears in the most general strict-feedback form (Krstic *et al.*, 1995). Therefore, developing backstepping control algorithms for such a class of problems is of great importance as the first step in an attempt to fully extend the existing backstepping techniques from the finite dimensional setup to the infinite dimensional one.

We use a backstepping method for the finite difference semi–discretized approximation of (1) with Dirichlet

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boundary conditions to derive a boundary feedback control law that makes the infinite dimensional closed loop system stable with an arbitrary prescribed stability margin. We show that the integral kernel in the control law resides in the function space $L_\infty(0,1)$ and that solutions corresponding to the controlled problem are classical. Our method can be generalized for different combinations of the boundary condition at $x=0$ (Dirichlet or Neumann), and control applied at $x=1$ (Dirichlet or Neumann).

The prior work on stabilization of general parabolic equations includes, among others, the results of Lasiecka and Triggiani (1983) who developed a general framework for the structural assignment of eigenvalues in parabolic problems through the use of semigroup theory. The stabilization problem can be also approached using the abstract theory of boundary control systems developed by Fattorini (1968) that results in a dynamical feedback controller (see (Curtain and Zwart, 1995, Section 3.5)). The first result, to our knowledge, where backstepping was applied to a PDE is the control design for a rotating beam by Coron and d'Andréa Novel (1998). They designed a nonlinear feedback torque control law for a hyperbolic PDE model of rotating beam with no damping and no control on the free boundary. The scalar control input, applied in a distributed fashion, is used to achieve global asymptotic stabilization of the system. In addition, authors show regularity of control inputs. Backstepping was successfully applied to parabolic PDEs in (Liu and Krstic, 2000; Bošković and Krstić, 2000; Bošković and Krstić, 2001*b*; Bošković and Krstić, 2001*a*) in settings with only a finite number of steps.

Our work is also related to results of Burns *et al.* (1996). Although their result is quite different because of the different control objective (theirs is LQR optimal control, ours is stabilization), and the fact that their plant is open–loop stable but with the spatial domain of dimension higher than ours, the technical problem of proving some regularity of the gain kernel ties the two results together.

The backstepping control design for linear parabolic PDEs presented here has advantages of its own. First, compared to the pole placement type of designs it has the standard advantage of a Lyapunov based approach that the designer does not have to look for the solution of the uncontrolled system to find the controller that stabilizes it. The problem of finding modal data in the case of spatially dependent $\lambda(x)$ and $f(x,y)$ becomes nontrivial and finding closed form expressions for the system eigenvalues and eigenvectors appears highly unlikely in the general case. In that case finding eigenvalues and eigenvectors numerically becomes inevitable, which might be computationally very expensive if a large number of grid points is necessary for simulating the system. To obtain a backstepping controller that stabilizes the system, on the other hand, the designer has to obtain a kernel given by a simple recursive expression that is computationally inexpensive. Second, from applications point of view, numerical results both for the nonlinear (Bošković and Krstić, 2000; Bošković and Krstić, 2001*b*; Bošković and Krstić, 2001*a*) and linear (linearization of the chemical tubular reactor presented here) parabolic PDEs suggest that reduced order backstepping control laws that use only a few state measurements can successfully stabilize the system for a variety of different simulation settings.

2. MOTIVATION

The semi–discretized version of system (1) with (2) and (3) using central differencing in space is the finite dimensional system:

$$u_0 = 0, \tag{4}$$

$$\dot{u}_i = \varepsilon\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} + B\frac{u_{i+1}-u_i}{h} + \lambda_i u_i, \quad + h\sum_{k=1}^{i-1} f_{i,k}u_i \qquad i=1,\ldots,n, \tag{5}$$

$$u_{n+1} = \alpha_n(u_1,u_2,\ldots,u_n), \tag{6}$$

where $n\in N$, $h=\frac{1}{n+1}$, $u_i=u(ih,t)$, $\lambda_i=\lambda(ih)$ and $f_{i,k}=f(ih,kh)$ for $k=1,\ldots,i-1$ and $i=0,\ldots,n+1$. With u_{n+1} as control, this system is in the strict–feedback form and hence it is readily stabilizable by standard backstepping. However the naive version of backstepping would result in a control law with gains that grow unbounded as $n\to\infty$.

Our approach is to transform the system, but keep its parabolic character, i.e., keep the second spatial derivative in the transformed coordinates. Towards this end, we start with a finite–dimensional backstepping–style coordinate transformation

$$w_0 = u_0 = 0, \tag{7}$$

$$w_i = u_i - \alpha_{i-1}(u_1,\ldots,u_{i-1}), i=1,\ldots,n, \tag{8}$$

$$w_{n+1} = 0, \tag{9}$$

for the discretized system (4)–(6), and seek the functions α_i such that the transformed system has the form

$$w_0 = 0, \tag{10}$$

$$\dot{w}_i = \varepsilon\frac{w_{i+1}-2w_i+w_{i-1}}{h^2} + B\frac{w_{i+1}-w_i}{h} - cw_i \qquad i=1,\ldots,n, \tag{11}$$

$$w_{n+1} = 0. \tag{12}$$

The finite–dimensional system (10)-(12) is the semi–discretized version of the infinite–dimensional system

$$w_t(x,t) = \varepsilon w_{xx}(x,t) + Bw_x(x,t) - cw(x,t), \tag{13}$$

for $x\in(0,1)$, $\quad t>0$, with boundary conditions

$$w(0,t)=0, \qquad w(1,t)=0, \tag{14}$$

which is exponentially stable for $c > -\varepsilon\pi^2 - \frac{B^2}{4\varepsilon}$.

The backstepping coordinate transformation is obtained by combining (4)–(6), (7)–(9) and (10)–(12) and solving the resulting system for the α_i's. Namely, subtracting (11) from (5), expressing the obtained equation in terms of $u_k - w_k$, $k=i-1,i,i+1$, and applying (8) we obtain the recursive form

$$\begin{aligned}\alpha_i = (\varepsilon+Bh)^{-1}\Bigg\{&\left(2\varepsilon+Bh+ch^2\right)\alpha_{i-1} - \varepsilon\alpha_{i-2} \\ &-(\lambda_i+c)h^2u_i - h^3\sum_{k=1}^{i-1}f_{i,k}u_k \\ &+\frac{\partial\alpha_{i-1}}{\partial u_1}\left((\varepsilon+Bh)u_2 - \left(2\varepsilon+Bh-\lambda_1h^2\right)u_1\right) \\ &+\sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial u_j}\Bigg((\varepsilon+Bh)u_{j+1} - \left(2\varepsilon+Bh-\lambda_jh^2\right)u_j \\ &+\varepsilon u_{j-1} + h^3\sum_{k=1}^{j-1}f_{j,k}u_k\Bigg)\Bigg\}\end{aligned} \tag{15}$$

Writing the α_i's in the linear form

$$\alpha_i = \sum_{j=1}^{i} k_{i,j}u_j \tag{16}$$

and performing simple calculations we obtain the general recursive relationship for the kernel as

$$k_{i,j} = \frac{h^2}{\varepsilon + Bh}(c+\lambda_j)\,k_{i-1,j} + k_{i-1,j-1}$$
$$+\frac{\varepsilon}{\varepsilon + Bh}(k_{i-1,j+1} - k_{i-2,j}) - \frac{h^2}{\varepsilon + Bh}(c+\lambda_i)\,\delta_{i,j}$$
$$-\frac{h^3}{\varepsilon + Bh} f_{i,j} + \frac{h^3}{\varepsilon + Bh}\sum_{l=j+1}^{i-1} f_{l,j} k_{i-1,l}$$
$$j = 1, \ldots, i \qquad (17)$$

for $i = 1, \ldots, n$ with convention

$$k_{i,j} = 0, \quad \text{for} \quad j > i \quad \text{and} \quad i,j \le 0 \qquad (18)$$

Initial conditions are obtained by using (18). For the simple case when $\lambda(x) \equiv \lambda$ = constant and $f(x,y) \equiv f$ = constant, equations (17) can be solved explicitly to obtain

$$k_{i,i-j} = -\binom{i}{j+1} L_n^{j+1} - (i-j)\sum_{l=1}^{[j/2]} \frac{1}{l}\binom{j-l}{l-1} M_n^l$$
$$\times \binom{i-l}{j-2l} L_n^{j-2l+1} - (i-j)\sum_{l=1}^{[(j+1)/2]} \frac{1}{l} P_n^l \sum_{m=0}^{[(j+1)/2]-l} M_n^m$$
$$\times \binom{l+m-1}{l-1} \sum_{k=0}^{j+1-2l-2m} \binom{j-l-2m-k}{l-1}$$
$$\times \binom{k+l+m-1}{k}\binom{i-m}{k+l-1} L_n^k \qquad (19)$$

for $i = 1, \ldots, n,\ j = 1, \ldots, i$, where $L_n = \frac{h^2}{\varepsilon + Bh}(c+\lambda)$, $M_n = \frac{\varepsilon}{\varepsilon + Bh}$ and $P_n = \frac{h^3}{\varepsilon + Bh} f$. The linearity of the control law in (16) suggests a stabilizing boundary feedback control of the form

$$\alpha(u) = \int_0^1 k(x)\,u(x)\,dx, \qquad (20)$$

where $k(x)$ is obtained as a limit of $\{(n+1)k_{n,j}\}_{j=1}^n$ as $n \to \infty$. From the complicated expression (19) it is not clear if such limit exists. A quick numerical simulation (see Figure 1) shows that the coefficients $\{(n+1)k_{n,j}\}_{j=1}^n$ remain bounded but it also shows their oscillation, and increasing n only increases the oscillation. A similar type of behavior was encountered in the related work of Balogh and Krstić (2001). Clearly, there is no hope for pointwise convergence to a continuous kernel $k(x)$. However, as we will see in the next sections, there is weak* convergence in L_∞ as we go from the finite dimensional case to the infinite dimensional one. As a result, we obtain a solution to our stabilization problem (1) with boundary conditions (2) and (3).

3. MAIN RESULT

The precise formulation of the main result is summarized in the following theorem.

Theorem 1. For any $\lambda(x) \in L_\infty(0,1)$, $f(x,y) \in L_\infty([0,1] \times [0,1])$ and ε, $c > 0$ there exists a function $k \in L_\infty(0,1)$ such that for any $u_0 \in L_\infty(0,1)$ the unique classical solution $u(t,x) \in C^1((0,\infty); C^2(0,1))$ of system (1), (2), (3) is exponentially stable in the $L_2(0,1)$ and maximum norms with decay rate c. The precise statements of stability properties are the following: There exists positive constant M^{***} such that for all $t > 0$

$$\|u(t)\| \le M\|u_0\|\,e^{-ct} \qquad (21)$$

and

$$\max_{x\in[0,1]} |u(t,x)| \le M \sup_{x\in[0,1]} |u_0(x)|\,e^{-ct}. \qquad (22)$$

Remark 1. For a given integral kernel $k \in L_\infty(0,1)$ the existence and regularity results for the corresponding solution $u(x,t)$ follows from trivial modifications in the proof of *(Levine, 1988, Thm 4.1).*

4. PROOF OF MAIN RESULT

The proof of Theorem 1 requires four lemmas.

Lemma 1. The elements of the sequence $\{k_{i,j}\}$ defined in (17) satisfy

$$|k_{i,i-j}| \le \binom{i}{j+1} L_n^{j+1} + (i-j)\sum_{l=1}^{[j/2]} \frac{1}{l}\binom{j-l}{l-1} M_n^l$$
$$\times \binom{i-l}{j-2l} L_n^{j-2l+1} + (i-j)\sum_{l=1}^{[(j+1)/2]} \left\{ \sum_{m=0}^{[(j+1)/2]-l} M_n^m \right.$$
$$\times \binom{l+m-1}{l-1} \sum_{k=0}^{j+1-2l-2m} \binom{j-l-2m-k}{l-1}$$
$$\left. \times \binom{k+l+m-1}{k}\binom{i-m}{k+l-1} L_n^k \right\} \qquad (23)$$

where $\lambda = \max_{x\in[0,1]} |\lambda(x)|$ and $f = \sup_{(x,y)\in[0,1]\times[0,1]} |f(x,y)|$.

Remark 2. There is equality in (23) when $\lambda(x) =$ constant > 0 and $f(x,y) \equiv f =$constant> 0.

Proof 1. We first obtain estimates for the initial values of k's, and then go from $j = i$ backwards to obtain

$$|k_{i,i}| \le iL_n, \quad |k_{i,i-1}| \le \frac{i(i-1)}{2} L_n^2 + (i-1)P_n \qquad (24)$$

Finally we obtain inequality (23) of Lemma 1 using the general identity (17) and mathematical induction.

We now introduce notations $q = \frac{j}{n} \in [0,1]$ so that we can write

$$|k_{n,n-j}| = |k_{n,n-qn}| \le \binom{n}{qn+1} L_n^{qn+1}$$
$$+(n-qn)\sum_{l=1}^{[qn/2]} \frac{1}{l}\binom{qn-l}{l-1}\binom{n-l}{qn-2l}$$
$$\times L_n^{qn-2l+1} M_n^l + (n-qn)$$
$$\times \sum_{l=1}^{[(qn+1)/2]} \frac{1}{l} P_n^l \sum_{m=0}^{[(qn+1)/2]-l} \binom{l+m-1}{l-1} M_n^m$$
$$\times \sum_{k=0}^{qn+1-2l-2m} \binom{qn-l-2m-k}{l-1}\binom{k+l+m-1}{k}$$
$$\times \binom{n-m}{k+l-1} L_n^k \qquad (25)$$

$^{***}M$ grows with c, λ and $1/\varepsilon$.

We now show the uniform boundedness of $\left\{(n+1)\,k_{i,j}\right\}_{i,j,n}$. Note that the binomial coefficients in inequality (23) are monotone increasing in i and hence it is enough to show the boundedness of terms $(n+1)\,k_{n,j}$.

Lemma 2. The sequence $\left\{(n+1)\,k_{n,j}\right\}_{j=1,\ldots,n\,n\geq 1}$ remains bounded uniformly in n and j as $n\to\infty$.

Proof 2. We can write, according to (25),

$$
\begin{aligned}
&(n+1)\left|k_{n,n-qn}\right| \leq (n+1)\binom{n}{qn+1}\left(\frac{E}{(n+1)^2}\right)^{qn+1}\\
&+(n+1)(n-qn)\sum_{l=1}^{[qn/2]}\frac{1}{l}\binom{qn-l}{l-1}\binom{n-l}{qn-2l}\\
&\times\left(\frac{E}{(n+1)^2}\right)^{qn-2l+1}M_n^l+(n+1)(n-qn)\\
&\times\sum_{l=1}^{[(qn-1)/2]}\frac{1}{l}P_n^l\sum_{m=0}^{[(qn+1)/2]-l}\binom{l+m-1}{l-1}M_n^m\\
&\times\sum_{k=0}^{qn+1-2l-2m}\binom{qn-l-2m-k}{l-1}\\
&\times\binom{k+l+m-1}{k}\binom{n-m}{k+l-1}\left(\frac{E}{(n+1)^2}\right)^k \qquad (26)
\end{aligned}
$$

where $E=2\frac{\lambda+c}{\varepsilon}$. The three terms in (26) can be estimated as

$$
(n+1)\binom{n}{qn+1}\left(\frac{E}{(n+1)^2}\right)^{qn+1}\leq Ee^{E/e}, \quad (27)
$$

$$
\begin{aligned}
&(n+1)(n-nq)\sum_{l=1}^{[qn/2]}\frac{1}{l}\binom{qn-l}{l-1}\binom{n-l}{qn-2l}\\
&\times\left(\frac{E}{(n+1)^2}\right)^{qn-2l+1}M_n^l\leq Ee^{R+E}, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
&(n+1)(n-qn)\sum_{l=1}^{[(qn+1)/2]}\frac{1}{l}P_n^l\\
&\times\sum_{m=0}^{[(qn+1)/2]-l}\binom{l+m-1}{l-1}M_n^m\\
&\times\sum_{k=0}^{qn+1-2l-2m}\binom{qn-l-2m-k}{l-1}\\
&\times\binom{k+l+m-1}{k}\binom{n-m}{k+l-1}\left(\frac{E}{(n+1)^2}\right)^k\\
&\leq H\left(1+\frac{R}{n}\right)^n\left(1+\frac{E}{n}\right)^n\left(1+\frac{H}{n}\right)^n\\
&\leq He^{(R+E+H)} \qquad (29)
\end{aligned}
$$

where $R=\frac{2|B|}{\varepsilon}$ and $H=\frac{2|f|}{\varepsilon}$. This proves the lemma.

As a result of the above boundedness, we obtain a sequence of piecewise constant functions

$$
k_n(x,y)=(n+1)\sum_{i=1}^{n}\sum_{j=1}^{i}k_{i,j}\chi_{I_{i,j}}(x,y) \qquad (30)
$$

for all $(x,y)\in[0,1]\times[0,1]$, $n\geq 1$, where

$$
I_{i,j}=\left[\frac{i}{n+1},\frac{i+1}{n+1}\right]\times\left[\frac{j}{n+1},\frac{j+1}{n+1}\right] \qquad (31)
$$

for all $j=1,\ldots,i,\quad i=1,\ldots,n,\quad n\geq 1$. The sequence (30) is bounded in $L_\infty([0,1]\times[0,1])$. The space $L_\infty([0,1]\times[0,1])$ is the dual space of $L_1([0,1]\times[0,1])$ hence, it has a corresponding weak*–topology. Since the space $L_1([0,1]\times[0,1])$ is separable, it follows now by Alaoglu's theorem, see, e.g. (Kato, 1966, pg. 140), that (30) converges in the weak*–topology to a function $\tilde{k}(x,y)\in L_\infty([0,1]\times[0,1])$. The uniform in $p\in\mathbb{N}$ weak convergence in each $L_p([0,1]\times[0,1])\supset L_\infty([0,1]\times[0,1])$, immediately follows.

Lemma 3. The map $\tilde{k}:[0,1]\to L_\infty(0,1)$ is weakly continuous.

Proof 3. From the uniform boundedness in i of (23) we obtain that

$$
\begin{aligned}
&\sum_{j=1}^{[nx]}k_{[nx],j}u_j=\sum_{j=1}^{[nx]}\left((n+1)\,k_{[nx],j}\right)u_j\frac{1}{n+1}\xrightarrow{n\to\infty}\\
&\int_0^x\tilde{k}(x,\xi)\,u(\xi)\,d\xi\,\forall u\in L_1(0,1)\,,\forall\, x\in[0,1]\,. \qquad (32)
\end{aligned}
$$

Here $[nx]$ denotes the largest integer not larger than nx and the convergence is uniform in x. For an arbitrary $x\in[0,1]$ we now fix an $n>N(\varepsilon/2)$ and choose a $\delta>0$ such that $[nx]=[n(x+\delta)]$. We obtain

$$
\begin{aligned}
&\left|\int_0^1\tilde{k}(x,\xi)\,u(\xi)\,d\xi-\int_0^1\tilde{k}(x,\xi)\,u(\xi)\,d\xi\right|\\
&\leq\left|\int_0^x\tilde{k}(x,\xi)\,u(\xi)\,d\xi-\sum_{j=1}^{[nx]}k_{[nx],j}u_j\right|\\
&+\left|\sum_{j=1}^{[nx]}k_{[nx],j}u_j-\sum_{j=1}^{[n(x+\delta)]}k_{[n(x+\delta)],j}u_j\right|\\
&+\left|\sum_{j=1}^{[n(x+\delta)]}k_{[n(x+\delta)],j}u_j-\int_0^{x+\delta}\tilde{k}(x+\delta,\xi)\,u(\xi)\,d\xi\right|\\
&<\varepsilon/2+0+\varepsilon/2=\varepsilon \qquad (33)
\end{aligned}
$$

which proves the weak continuity.

The following lemma shows how norms change under the above transformation.

Lemma 4. (Balogh and Krstić (2001)). Suppose that two functions $w(x)\in L_\infty(0,1)$ and $u(x)\in L_\infty(0,1)$ satisfy the relationship

$$
w(x)=u(x)-\int_0^x\tilde{k}(x,\xi)\,u(\xi)\,d\xi \qquad \forall x\in[0,1]\,, \quad (34)
$$

where $\tilde{k}\in C([0,1];L_\infty(0,1))$. Then there exist positive constants m and M, whose sizes depend only on $\tilde{k}$, such that

$$
m\,\|w\|_\infty\leq\|u\|_\infty\leq M\,\|w\|_\infty \qquad (35)
$$

and

$$
m\,\|w\|\leq\|u\|\leq M\,\|w\|\,. \qquad (36)
$$

Proof 4. (Proof of Theorem 1). We now complete the proof of Theorem 1 by combining the results of Lemmas 1–4. In Lemma 1 we derived a coordinate transformation that transforms the finite dimensional system (4)–(6) into the finite dimensional system (10)–(12). As a result of the uniform boundedness of the transformation we obtained the coordinate transformation (34) that transforms the system (1), (2) into the asymptotically stable system (13)–(14). Due to the weak continuity proven in Lemma 3 the infinite dimensional coordinate transformation results in the specific boundary condition

$$u(1,t) = \alpha(u) = \int_0^1 k(\xi)\, u(\xi,t)\, d\xi, \tag{37}$$

where $k(\xi) = \tilde{k}(1,\xi)$, $\xi \in [0,1]$ with $k \in L_\infty(0,1)$. The well known (see, e.g. (Cannon, 1984)) stability properties of system (13)–(14) along with Lemma 4 proves the stability statements of Theorem 1.

5. SIMULATION STUDY

In this section we present the simulation results for a linearization of an adiabatic chemical tubular reactor. For the case when Peclet numbers for heat and mass transfer are equal (Lewis number of unity) the two equations for the temperature and concentration can be reduced to one equation (Hlaváček and Hofmann, 1970)

$$\theta_t = \frac{1}{Pe}\theta_{\xi\xi} - \theta_\xi + Da\,(b-\theta)\, e^{\frac{\theta}{1+\mu\theta}} \tag{38}$$

$$\theta_\xi(0,t) = Pe\,\theta(0,t) \tag{39}$$

$$\theta_\xi(1,t) = 0 \tag{40}$$

for $\xi \in (0,1)$, $t > 0$ where Pe stands for the Peclet number, Da for the Damköhler number, μ for the dimensionless activation energy, and b for the dimensionless adiabatic temperature rise. For a particular choice of system parameters ($Pe = 6$, $Da = 0.05$, $\mu = 0.05$, and $b = 10$) system (38)–(40) has three equilibria (Hlaváček and Hofmann, 1970). The middle profile is unstable while the outer two profiles are stable. The equilibrium profiles for this case are shown in Figure 2. Linearizing the system around the unstable equilibrium profile $\overline{\theta}(\xi)$ we obtain

$$\theta_t = \frac{1}{Pe}\theta_{\xi\xi} - \theta_\xi + Da\, G\left(\overline{\theta}(\xi)\right)\theta \tag{41}$$

$$\theta_\xi(0,t) = Pe\theta(0,t) \tag{42}$$

$$\theta_\xi(1,t) = 0 \tag{43}$$

where θ now stands for the deviation variable from the steady state $\overline{\theta}(\xi)$, and G is a spatially dependent coefficient defined as

$$G(\overline{\theta}) = \left[\frac{b-\overline{\theta}}{(1+\mu\overline{\theta})^2} - 1\right] e^{\frac{\overline{\theta}}{1+\mu\overline{\theta}}} \tag{44}$$

Although not obvious from the equations (41)–(43), it is physically justifiable to apply feedback boundary control at 0-end only. In real application control would be implemented through small variations of T_{in} and C_{in} (see (Varma and Aris, 1977) and (Hlaváček and Hofmann, 1970)). Since our control algorithm assumes actuation at 1-end we transform the original system (41)–(43) by introducing a variable change

$$u(x,t) = \theta(1-\xi)\,. \tag{45}$$

In the new set of variables the system (41)–(43) becomes

$$u_t(x,t) = \frac{1}{Pe}u_{xx}(x,t) + u_x(x,t) + Da\, g(x)\, u(x,t) \tag{46}$$

$$u_x(0,t) = 0 \tag{47}$$

$$u_x(1,t) = -Pe\,u(1,t) + \Delta u_x(1,t) \tag{48}$$

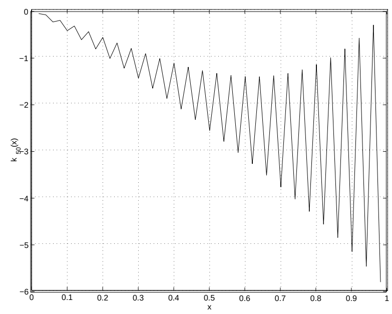

Fig. 1. Oscillation of the approximating kernel for $n = 50$, $\lambda \equiv 5$, $f \equiv 0$, $\varepsilon = 1$, $B = 1$, $c = 1$.

where $g(x)$ is defined as $g(x) = G(\overline{\theta}(1-\xi))$ and $\Delta u_x(1,t)$ stands for the control law to be designed. All simulations presented in this study were done using BTCS finite difference method for $n = 200$ and the time step equal to 0.001 s. Although we have tested the controller for several different combinations of initial distributions and target systems, we only present results for $c = 0.1$ and $u(x,0) = -\left(\frac{\omega}{Pe}\cos(\omega x) + \sin(\omega x)\right)$, $\omega = 1.48396$. This particular initial distribution has been constructed to exactly satisfy the imposed boundary conditions on both ends in the open loop case.

The open loop system ($\Delta u_x(1,t)$=0) is unstable as shown in Figure 3. We now obtain a coordinate transformation that transforms the discretization of (46)–(48) into discretization of the asymptotically stable system

$$w_t(x,t) = \frac{1}{Pe}w_{xx}(x,t) + w_x(x,t) - c\,w(x,t) \tag{49}$$

$$w_x(0,t) = 0 \tag{50}$$

$$w_x(1,t) = -Pe\,w(1,t) \tag{51}$$

The control is implemented as

$$\Delta u_x(1,t) = \frac{\alpha_n(u_1,\cdots,u_n) - \alpha_{n-1}(u_1,\cdots,u_{n-1})}{h} + Pe\,\alpha_n(u_1,\cdots,u_n), \tag{52}$$

The closed loop response of the system with controller designed for $n = 200$ and $c = 0.1$ and the corresponding control effort $\Delta u_x(1,t)$ are shown in Figure 4.

From applications point of view it is of interest to see whether the system (46)–(48) can be stabilized with a reduced version of the control law (52). The idea of using controllers designed using only a small number of steps of backstepping to stabilize the system for a certain range of the open–loop instability is based on the fact that in most real life systems only a finite number of open–loop eigenvalues is unstable. Indeed, simulation results show that we can successfully stabilize the unstable equilibrium using a kernel obtained with only two steps of backstepping (using only two state measurements $u(\frac{1}{3},t)$ and $u(\frac{2}{3},t)$) with the same $c = 0.1$. The closed loop response of the system with a reduced order controller and corresponding control effort $\Delta u_x(1,t)$ are shown in Figure 5.

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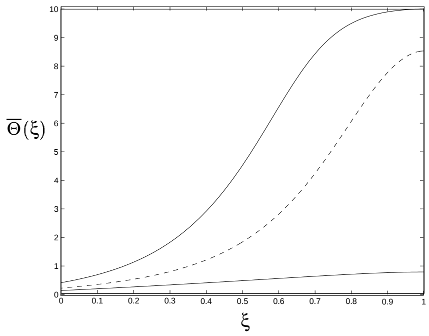

Fig. 2. Steady state profiles.

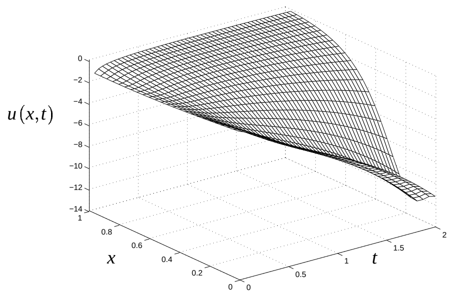

Fig. 3. Open loop response of the system.

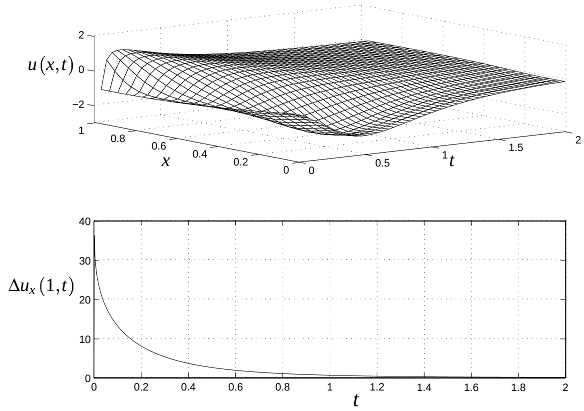

Fig. 4. Closed loop response of the system with controller that uses full state information. (First row: $u(x,t)$; Second row: The control effort $\Delta u_x(1,t)$.)

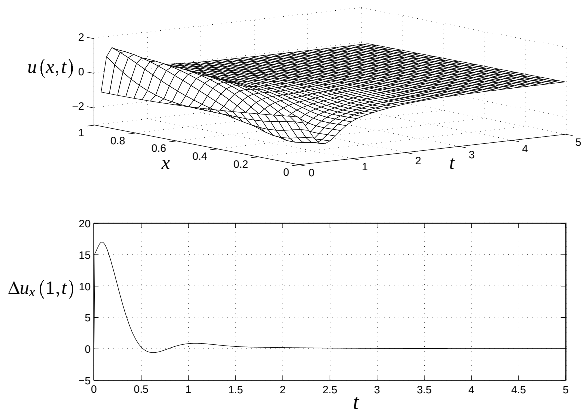

Fig. 5. Closed loop response of the system with controller designed using only two steps of backstepping. (First row: $u(x,t)$; Second row: The control effort $\Delta u_x(1,t)$.)

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