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STABILITY OF TIME-VARYING DISCRETE-TIME CASCADES

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Abstract. Stability results for analysis of time-varying discrete-time cascaded systems are presented. Most of these results parallel their continuous-time counterparts although there are some important differences in the proof techniques that are used. More importantly, some fundamental differences between stability properties of discrete-time and continuous-time cascades are pointed out and illustrated via examples.
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1. INTRODUCTION

Stability of continuous-time cascaded systems has attracted a lot of attention over the past ten years. Motivation for this research originated in geometric nonlinear control where it was shown that many systems can be transformed into a cascade via a local change of coordinates (see, for example, (Isidori 1995, Lemma 1.6.1)). Further motivation comes from many practical applications where one can use the cascade structure to check stability of the closed-loop system and in certain situations even design a control law with the aim of achieving a cascade structure that is easier to analyze. For instance, this approach has been successfully used in stabilization of a ship (Loría*et al.* 2000) and non-holonomic systems (Panteley *et al.* 1998).

While continuous-time results are important in their own right, the prevalence of computer controlled systems strongly motivates the investigation of stability properties of discrete-time cascaded systems. Indeed, results on stability of discrete-time cascaded systems are needed if the controller design is based on the discrete-time plant model.

First results on stability of time-invariant discrete-time cascades that we are aware of were presented in (Jiang and Wang 2001) where the input-to-state stability (ISS) property was used to provide sufficient conditions for global asymptotic stability of the cascade. It turns out, however, that the ISS property is often too restrictive and weaker conditions are needed. A range of such weaker conditions were found in (Arcak *et al.* To appear., Panteley and Loría 2001) for continuous-time cascades.

It is the main purpose of this paper to present results on stability of time-varying discrete-time cascaded systems that generalize the time-invariant discrete-time results in (Jiang and Wang 2001) and parallel similar continuous-time results in (Arcak *et al.* To appear., Panteley and Loría 2001). Our time-varying results are very important for tracking problems (see (Panteley and Loría 2001)) which often arise in practice. We emphasize that our results are not just a simple translation from continuous-time to discrete-time and there are notable differences in the proofs. Moreover, there are several aspects in which discrete-time results are very different from their continuous-time counterparts. For instance, we show for discrete-time cascades that if the bottom subsystem is dead-beat stable and the nominal upper subsystem is uniformly globally asymptotically stable (UGAS), then the cascade is UGAS irrespective of the growth of the interconnection term, which is in clear contrast with the continuous-time results. More precisely consider the following examples.

Example 1. Consider the system:

$$\begin{aligned} x_{k+1} &= ax_k + x_k^p y_k \\ y_{k+1} &= 0 \ , \end{aligned} \tag{1}$$

where $p \geq 0$ is arbitrary and $a \in (0,1)$. We claim that the discrete-time system is GAS for any $p \geq 0$. We prove this by constructing a Lyapunov function for the system. Let $\epsilon > 0$ be such that $a + \epsilon - 1 < 0$ and define the Lyapunov function

$$V(x,y) := |ax + x^p y| + \epsilon\,|x| + |y| \ . \tag{2}$$

This function is obviously positive definite and radially unbounded for any value of $p \geq 0$. Finally, the first difference of the Lyapunov function is:

$$\begin{aligned} \Delta V &= |a(ax + x^p y)| + \epsilon\,|ax + x^p y| - |ax + x^p y| \\ &\quad - \epsilon\,|x| - |y| \\ &= (a + \epsilon - 1)\,|ax + x^p y| - \epsilon\,|x| - |y| \ , \end{aligned} \tag{3}$$

which is negative definite for any $p \geq 0$ since $a + \epsilon - 1 < 0$ and $\epsilon > 0$. □

Hence, the discrete-time cascade is GAS for any value of $p \geq 0$, as opposed to the continuous-time case where not even forward completeness can be guaranteed.

Example 2. Consider the cascade

$$\dot{x} = -x + x^2 z \tag{4a}$$
$$\dot{z} = f(z) \tag{4b}$$

where $z, x \in \mathbb{R}$ and where $f(\cdot)$ is locally Lipschitz and has the property that the system (4b) is GAS. Note that the nominal system $\dot{x} = -x$ is GAS. We show that this cascade is not GAS no mater how we choose $f(\cdot)$. Indeed, let f be arbitrary but fixed and such that the system (4b) is GAS and let $z(t, z_\circ)$ denote a solution of this system. Let $z_\circ > 0$ be arbitrary. Then, $0 \leq z(t, z_\circ) \leq z_\circ, \forall t \geq 0$. Moreover, the solutions of the subsystem (4a) can be explicitly written as (see (Sepulchre *et al.* 1997, pg. 128)):

$$x(t, x_\circ, z_\circ) = \frac{e^{-t}}{\frac{1}{x_\circ} - \int_0^t e^{-s} z(s, z_\circ) ds} \ . \tag{5}$$

Hence, for each initial condition $x_\circ$ that satisfies

$$x_\circ > \left(\int_0^\infty e^{-s} z(s, z_\circ) ds \right)^{-1}$$

there exists some time $t_e > 0$ such that the denominator in (5) becomes zero and hence the solution $x(t, x_\circ, z_\circ)$ escapes to infinity as $t \to t_e$. □

The rest of the paper is organized as follows. In Section 2 we present preliminary results and definitions we need in the sequel. Section 3 contains the main results. Due to space constraints, some proofs are omitted. Conclusions and suggestions for further research are given in the last section.

2. PRELIMINARIES

Throughout this paper we denote by $\mathbb{Z}$ the set of integer numbers and by $\mathbb{R}$ the set of reals. $|\cdot|$ stands for the 1-norm of vectors, i.e. $|x| := \sum_i |x_i|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is continuous, strictly increasing and zero at zero; $\alpha \in \mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for all $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$, for all $s > 0$, $\beta(s, \cdot)$ is decreasing and $\lim_{t \to \infty} \beta(s,t) = 0$.

In this paper we consider systems of the form:

$$x_{k+1} = F(k, x_k) \,. \tag{6}$$

The solution of the system (6) at time k, starting at initial time $k_\circ$ and emanating from the initial condition $x_\circ = x(k_\circ)$, is denoted as $x(k, k_\circ, x_\circ)$ or x_k if $k_\circ, x_\circ$ are clear from the context. We will say that the system (6) is forward complete if all solutions of (6) exist for all $k \geq k_\circ \geq 0$.

We say that the forward complete system (6) is:
(i) *uniformly* globally asymptotically stable (UGAS), if there exists a function $\beta \in \mathcal{KL}$ and such that

$$|x_k| \leq \beta(|x_\circ|\,, k - k_\circ) \tag{7}$$

for all $k \geq k_\circ \geq 0$ and $x_\circ \in \mathbb{R}^n$.
(ii) uniformly globally dead-beat stable (UGDS), if there exist π and $k^* \in \mathcal{K}_\infty$ such that

$$|x_k| \leq \pi(|x_\circ|) \qquad \forall\, k \in [k_\circ,\, k_\circ + k^*(|x_\circ|)] \tag{8}$$
$$x_k \equiv 0 \qquad \forall\, k \geq k_\circ + k^*(|x_\circ|) \tag{9}$$

for all $x_\circ \in \mathbb{R}^n$ and $k_\circ \geq 0$.
(iii) uniformly globally bounded (UGB), if there exist $\kappa \in \mathcal{K}_\infty$ and $c > 0$ such that

$$|x_k| \leq \kappa(|x_\circ|) + c \ , \tag{10}$$

for all $k \geq k_\circ \geq 0$ and all $x_\circ \in \mathbb{R}^n$.

We are interested in studying sufficient conditions for UGAS of forward complete time-varying discrete-time systems of the form

$$x_{k+1} = F_1(k, x_k) + G(k, x_k, z_k) \tag{11a}$$
$$z_{k+1} = F_2(k, z_k) \tag{11b}$$

where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$. A standing assumption will be that the subsystem (11b) and the "nominal" system

$$x_{k+1} = F_1(k, x_k) \tag{12}$$

are both UGAS.

We also assume that there exist a nondecreasing function γ_1, class $\mathcal{K}$ functions γ_2 and ϕ_i, $i = 1, 2$ such that for all $k \geq 0$, $x \in \mathbb{R}^{n_x}$ and $z \in \mathbb{R}^{n_z}$ we have

$$|G(k,x,z)| \leq \gamma_1(|x|)\gamma_2(|z|) \tag{13}$$
$$|F_1(k,x)| \leq \phi_1(|x|) \tag{14}$$
$$|F_2(k,z)| \leq \phi_2(|x|) \,. \tag{15}$$

Note that these assumptions guarantee forward completeness of (11) as opposed to more restrictive assumptions needed in the continuous-time systems context (see e.g. (Sepulchre *et al.* 1997)). In the sequel, we will denote the state of (11) by the column vector $\xi := \text{col}[x, z]$ of dimension $n := n_x + n_z$.

3. MAIN RESULTS

The first main result (Theorem 3) is essentially the discrete-time counterpart of (Panteley and Loría 2001, Lemma 2). This theorem states necessary and sufficient conditions for stability of the system (11). Due to space reasons we are able to provide only a sketch of the proof. For clarity of exposition this is done in the Appendix.

Theorem 3. (UGAS + UGAS + UGB ⇔ UGAS). The system (11) is UGAS if and only if (11b) and (12) are UGAS and the solutions of (11) are UGB. □

While this result is fundamental since it states necessary and sufficient conditions for UGAS of (11), these conditions are often hard to check and in particular the third (UGB) condition. Therefore, we provide below several sufficient conditions for UGB.

In particular, we investigate how the notion of integral Input-to-State Stability (iISS) can be used to state sufficient conditions for stability of (11). We will present a discrete-time counterpart of continuous-time results in (Arcak *et al.* To appear.). We note that iISS is strictly more general than ISS that was used in (Jiang and Wang 2001) for investigation of stability of discrete-time cascades. iISS of subsystem (11a) (where z is regarded as an input) allows us to quantify the convergence rate of the subsystem (11b).

We also discuss the particular case where the subsystem (11b) is assumed to be dead-beat stabile. We will show that in general, we do not need any restrictions on the growth of the interconnection term, to conclude UGB and consequently (via Theorem 3) UGAS. Thus, we will make formal the statement illustrated in Examples 1.

Finally, sufficient conditions for UGB via growth restrictions on the interconnection term are presented. These are similar to the continuous-time results in (Panteley and Loría 2001).

3.1 *Conditions involving integral ISS*

We first explore sufficient conditions for UGAS of cascades in terms of the property *Integral Input-to-State Stability* (iISS) (see (Sontag 1998)). The iISS framework allows us to study general cascades of the form

$$x_{k+1} = f_1(k, x_k, z_k) \tag{16a}$$
$$z_{k+1} = f_2(k, z_k)\,. \tag{16b}$$

For the purposes of this paper we use the following definitions, which are similar to those given in (Angeli 1999) for autonomous discrete-time systems.

Definition 4. (iISS). The system (16a) is integral Input-to-State Stable (iISS) –with gain μ and input z_k– if there exists α, $\mu \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$ such that for all initial conditions, all inputs z_k and all $k \geq k_\circ \geq 0$

$$\alpha(|x_k|) \leq \beta(|x_\circ|\,, k - k_\circ) + \sum_{k=k_\circ}^{k-1} \mu(|z_k|)\,. \qquad \square$$

Definition 5. (iISNS). The system (16a) is Integral Input-to-State Neutrally Stable (iISNS) –with gain μ and input z_k– if there exists α, γ and $\mu \in \mathcal{K}_\infty$ such that for all initial conditions, all inputs z_k and all $k \geq k_\circ \geq 0$

$$\alpha(|x_k|) \leq \gamma(|x_\circ|) + \sum_{k=k_\circ}^{k-1} \mu(|z_k|)\,. \qquad \square$$

Note that iISS implies iISNS with $\gamma(s) := \beta(s, 0)$. Necessary and sufficient condition for iISS and iISNS of discrete-time time-invariant systems can be found in (Angeli 1999).

The following Proposition captures a similar result to that contained in (Arcak *et al.* To appear., Theorem 1). This result shows that in order to have UGAS of the system (16), there is a tradeoff between the rate of convergence of trajectories of the system (16b) and the magnitude/shape of the iISNS gain of the system (16a).

Proposition 6. If there exist μ, $\sigma \in \mathcal{K}_\infty$, $\kappa \in \mathcal{K}$ and λ, $c > 0$ such that:

(i) the system (16b) satisfies (7) with $\beta(r, t) = \sigma(\kappa(r)\mathrm{e}^{-\lambda t})$

(ii) the system $x_{k+1} = f_1(k, x_k, 0)$ is UGAS and (16a) is iISNS with gain μ and input z_k such that

$$\int_0^1 \frac{\mu \circ \sigma(s)}{s} ds \leq c < \infty\,. \tag{17}$$

Then, the cascade (16) is UGAS. □

We remark that the assumption in item (i) of Proposition 6 is not restrictive since it was shown in (Sontag 1998) that given any $\beta \in \mathcal{K}L$, there exist $\sigma \in \mathcal{K}_\infty$, $\kappa \in \mathcal{K}$ such that $\beta(r, t) \leq \sigma(\kappa(r)e^{-t}), \forall r, t$. However, the second item is restrictive and it shows a tradeoff between the iISNS gain μ of (16a) and the convergence rate of (16b).

Proof. The proof follows closely that of (Arcak *et al.* To appear., Theorem 1). By assumption, there exist

α, γ, $\mu \in \mathcal{K}_\infty$, $\kappa \in \mathcal{K}$ and $\lambda > 0$ such that for all $x_\circ \in \mathbb{R}^{n_x}$, $z_\circ \in \mathbb{R}^{n_z}$ and $k \geq k_\circ \geq 0$

$$\alpha(|x_k|) \leq \gamma(|x_\circ|) + \sum_{k=k_\circ}^{\infty} \mu \circ \sigma(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(k-k_\circ)})\,. \tag{18}$$

The sum on the right hand side of the inequality above satisfies

$$\sum_{k=k_\circ}^{\infty} \mu \circ \sigma(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(k-k_\circ)}) = \mu \circ \sigma \circ \kappa(|z_\circ|) + \sum_{k=k_\circ+1}^{\infty} \mu \circ \sigma(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(k-k_\circ)})$$

and since $\mu \circ \sigma(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(k-k_\circ)})$ is monotonically decreasing in k, the last term on the right hand side of this inequality satisfies

$$\sum_{k=k_\circ+1}^{\infty} \mu \circ \sigma(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(k-k_\circ)}) \leq \int_{t_\circ:=k_\circ}^{\infty} \mu \circ \sigma(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(t-t_\circ)})\, dt\,.$$

Define as in (Arcak *et al.* To appear.), $s := \kappa(|z_\circ|)\mathrm{e}^{-\lambda(t-t_\circ)}$ then

$$\int_{t_\circ:=k_\circ}^{\infty} \mu \circ \sigma(\kappa(|z_\circ|)\mathrm{e}^{-\lambda(t-t_\circ)})\, dt = \int_0^{\kappa(|z_\circ|)} \frac{\mu \circ \sigma(s)}{\lambda\, s}\, ds\,.$$

From item (ii) of the Proposition we have that

$$\kappa_1(s) := \frac{1}{\lambda}\int_0^s \frac{\mu \circ \sigma(t)}{t}\, dt$$

exists for all $s \geq 0$ and it is class $\mathcal{K}$ because $\kappa_1(0) = 0$ and $\frac{\mu\circ\sigma(t)}{t} > 0$ for all $t > 0$. Putting all these bounds together and using item (i) of the Proposition we obtain that

$$\alpha(|x_k|) \leq \gamma(|x_\circ|) + [\mu \circ \sigma \circ \kappa(|z_\circ|) + \kappa_1 \circ \kappa(|z_\circ|)]$$

for all $k \geq k_\circ \geq 0$ and since the system (16b) is UGAS, the solutions of (16) are UGB. The result follows invoking Theorem 3. ■

We note that it was proved in (Angeli 1999) for autonomous systems with inputs

$$x_{k+1} = f(x_k, u_k)\ , \tag{19}$$

that the system (19) is iISS if and only if the zero input system $x_{k+1} = f(x_k, 0)$ is GAS. This result is not true for continuous-time systems. Actually, it was shown in (Angeli *et al.* 2000) for continuous-time systems with inputs

$$\dot{x} = f(x, u)\ , \tag{20}$$

that if the system $\dot{x} = f(x, 0)$ is GAS and moreover (20) is forward complete, this still does not imply that (20) is iISS. These results and Proposition 6 indicate that one can expect large differences between continuous-time and discrete-time cascade results. Indeed, following results of (Angeli 1999) and Proposition 6 for time-invariant cascades

$$x_{k+1} = f_1(x_k, z_k) \tag{21a}$$
$$z_{k+1} = f_2(z_k) \tag{21b}$$

we can state the following:

Corollary 7. If $x_{k+1} = f_1(x_k, 0)$ is GAS, then there exists a GAS subsystem (21b) so that the cascade (21) is GAS. □

Proof. Note that since $x_{k+1} = f_1(x_k, 0)$ is GAS, then results of (Angeli 1999) guarantee that (21a) is iISS with some gain μ. Then any GAS subsystem (21b) with $\beta_2(r,t) = \mu^{-1}(\kappa(r)e^{-\lambda t})$ satisfies all conditions of Proposition 6 hence the cascade (21) is GAS. ■

We will show in the next section that even a stronger statement is true for discrete-time time-varying cascades if the bottom system is dead-beat stable. However, statement of Corollary 7 is not true in continuous-time even for time-invariant systems, as illustrated by Example 2.

3.2 *Dead-beat stability conditions*

We show next that if the bottom subsystem is dead-beat stable, then one can allow for *any growth* of $G(k, \cdot, z)$. Thereby formalizing the observation made in Example 1.

Proposition 8. If (12) is UGAS and (11b) is UGDS then the cascade (11) is UGAS. □

Proof. The proof is based on the fact that forward completeness of (11a) and the inequalities (8), (13), (14) imply that the solutions of (11a) are uniformly bounded over any bounded interval. In particular, there exists a continuous function $\Phi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is zero at zero and such that

$$|x_k| \leq \Phi(k - k_\circ, |\xi_\circ|) \qquad \forall\, k \geq k_\circ\,. \tag{22}$$

Moreover, we can assume without loss of generality that $\Phi(\cdot,\cdot)$ is strictly increasing in both arguments. Let now the dead-beat assumption on (11b) generate $k^* \in \mathcal{K}_\infty$ such that (8) and (9) hold. Using all these facts and (22) we obtain that

$$|x_k| \leq \Phi(k^*(|z_\circ|), |\xi_\circ|) \leq \Phi_1(|\xi_\circ|)\ , \tag{23}$$

$\forall\, k \in [k_\circ, k_\circ + k^*(|z_\circ|)]$, where $\Phi_1(s) := \Phi(k^*(s), s)$ is clearly continuous strictly increasing. Let $\beta_1 \in \mathcal{KL}$ come from UGAS of (12). Then we can write that

$$|x_k| \leq \beta_1(|x_{k_\circ+k^*}|\,, k - k^* - k_\circ)\ ,$$

for all $k \geq k^* + k_\circ$. Moreover, since for any $\xi_\circ$ we have $|x_{k_\circ+k^*}| \leq \Phi_1(|\xi_\circ|)$ and noting that $\beta_1(s,0) \geq s$ for all $s \geq 0$ it follows that

$$|x_k| \leq \beta_1(\Phi_1(|\xi_\circ|), 0) \qquad \forall k \geq k_\circ\,. \tag{24}$$

The result follows combining this bound with (8) and (9) to conclude that the system (11) is UGB and then invoking Theorem 3 to conclude UGAS of (11). ■

Note that in the proof of Proposition 8 we have actually shown that the following is true:

Corollary 9. If (11b) is UGDS, then for any system (11a) such that (12) is UGAS we have that the cascade (11) is UGAS. □

Corollary 9 shows, in particular, that if (11b) is UGDS, then we can allow arbitrary growth in the interconnection term. Example 2 illustrated that this was not true for continuous-time systems and illustrates that arbitrary growth in the interconnection term is indeed allowed for discrete-time systems for which (11b) is UGDS.

3.3 *Conditions on growth rates*

The following corollary is the discrete-time counterpart of (Panteley and Loría 2001, Theorem 4). It presents sufficient conditions on the growth of the interconnection term that would guarantee UGB and hence UGAS of the discrete-time cascade.

Proposition 10. Assume that (11b) and (12) are UGAS, there exist a positive semidefinite function $W(\cdot)$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, $\eta > 0$, $\lambda > 0$ and a continuous Lyapunov function $V(\cdot,\cdot)$ which satisfies for all k, x

$$\alpha_1(|x|) \leq V(k,x) \leq \alpha_2(|x|) \tag{25}$$

$$V(k+1, F_1) - V(k,x) \leq -W(x) \tag{26}$$

and for all $z \in \mathbb{R}^{n_z}$ and $k \geq 0$,

$$\{|x| \geq \eta\} \implies \tag{27}$$
$$V(k+1, F_1+G) - V(k+1, F_1) \;\leq\; \lambda W(x)\sigma(|z|)\;.$$

Then, the cascade (11) is UGAS. □

Proof. Let W, V, σ, λ and η be generated the Proposition. Let $\beta_2 \in \mathcal{KL}$ come from UGAS of (11b). Let $k^*(\cdot)$ be a continuous nondecreasing function such that such that $\sigma(\beta_2(s, k^*(s))) \leq 1/\lambda, \forall s \geq 0$. Following verbatim the proof of Proposition 8 we obtain that (23) holds for all $k \in [k_\circ, k_\circ + k^*(|z_\circ|)]$ —we remark that in the derivation of $\Phi_1(\cdot)$ in the proof of Proposition 8 we have used (8) but for the purposes of the present proof one can construct $\Phi_1(\cdot)$ using $\pi(s) := \beta_2(s,0)$. Consider arbitrary initial state $\xi_\circ = (x_\circ^T\ z_\circ^T)^T$ and the corresponding trajectory ξ_k. Then, it follows from (27) that for all $k \geq k_\circ + k^*(|z_\circ|)$ and all x_k such that $|x_k| \geq \eta$, we have that

$$v_{k+1} - v_k \leq -W(x_k)(1 - \lambda\sigma(|z_k|)) \;\leq\; 0\,, \tag{28}$$

where $v_k := V(k, x_k)$. The first inequality follows directly from (27) while the second one follows from the fact that $\sigma(|z_k|) \leq 1/\lambda$ for all $k \geq k_\circ + k^*(|z_\circ|)$. Then, from (25) we have that

$$\begin{aligned} \{k \geq k_\circ + k^*(|z_\circ|),\, |x_k| \geq \eta\} \;&\Rightarrow\; v_{k+1} \leq v_k \\ &\Rightarrow\; |x_{k+1}| \leq \alpha_1^{-1} \circ \alpha_2(|x_k|)\,. \end{aligned} \tag{29}$$

Notice also that our choice of that k^* and λ and the fact that (11b) is UGAS imply that[1] $|z_k| \leq \sigma^{-1}(1/\lambda)$ for all $k \geq k_\circ + k^*(|z_\circ|)$. It follows using (13), (14) and (15) that,

$$\{k \geq k_\circ + k^*(|z_\circ|),\, |x_k| \leq \eta\} \;\Rightarrow \tag{30}$$
$$|x_{k+1}| \leq \phi_1(\eta) + \gamma_1(\eta)\gamma_2 \circ \sigma^{-1}(1/\lambda) \;:=\; c(\eta,\lambda)\,.$$

Thus, from (23), (29) and (30) and using induction we obtain that

$$|x_k| \leq \max\{\, c(\eta,\lambda),\ \alpha_1^{-1} \circ \alpha_2(|x_{k_\circ+k^*}|)\,\}\;, \tag{31}$$

for all $k \geq k_\circ + k^*(|z_\circ|)$.

Since $\alpha_1^{-1} \circ \alpha_2(s) \geq s, \forall s \geq 0$ and $|x_{k_\circ+k^*}| \leq \Phi_1(|\xi_\circ|)$ for any $\xi_\circ$, we finally obtain that

$$|x_k| \leq \max\{\, c(\eta,\lambda),\ \alpha_1^{-1} \circ \alpha_2(\Phi_1(|\xi_\circ|))\,\}\;,$$

for all $k \geq k_\circ$, that is, the cascade is UGB. The result follows invoking Theorem 3. ■

4. CONCLUSION

Conditions for stability of discrete-time time-varying cascade systems were presented. Necessary and sufficient conditions are in general hard to check and we presented several sufficient conditions that are easier to check. Several differences between continuous-time and discrete-time cascade results were commented on and illustrated by examples. In particular, it was shown that the growth of the interconnection term in x is not so restrictive for stability of discrete-time cascades as it is for continuous-time cascades.

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[1] Notice that we can assume that λ is sufficiently large so that $1/\lambda$ be in the domain of $\sigma^{-1}(\cdot)$.

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Appendix A. SKETCH OF PROOF OF THEOREM 1

The conditions of the theorem imply the following statements.

Fact 11. Since (12) and (11b) are UGAS, there exist two $\mathcal{KL}$ functions β_1 and β_2 such that for all $k \geq k_\circ \geq 0$, $x_\circ \in \mathbb{R}^{n_x}$ and $z_\circ \in \mathbb{R}^{n_z}$

$$|x_k| \leq \beta_1(|x_\circ|, k - k_\circ) \tag{A.1}$$

$$|z_k| \leq \beta_2(|z_\circ|, k - k_\circ)\,. \tag{A.2}$$

Fact 12. There exist α_1, $\alpha_2 \in \mathcal{K}_\infty$, α_3, $\alpha_4 \in \mathcal{K}$ and a continuous function $V : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{R}^{n_x}$ we have

$$\alpha_1(|x|) \leq V(k,x) \leq \alpha_2(|x|) \tag{A.3}$$

$$V(k+1, F_1(k,x)) - V(k,x) \leq -\alpha_3(|x|) \tag{A.4}$$

$$\left|\frac{\partial V}{\partial x}\right| \leq \alpha_4(|x|)\,. \tag{A.5}$$

Lemma 13. If (11b) is UGAS and (11) is UGB then, there exists $\beta_3 \in \mathcal{KL}$ such that the same $V(k,x)$ from Fact 12 satisfies the following inequality along trajectories [2] for all $\xi_\circ \in \mathbb{R}^n$, $k \geq k_\circ \geq 0$

$$|V(k+1, F_1 + G) - V(k+1, F_1)| \leq \beta_3(|\xi_\circ|, k - k_\circ)\,.$$

Let Fact 11 and Fact 12 and the Lemma generate the functions $\beta_i \in \mathcal{KL}$, $i = 1, 2, 3$ and let Fact 12 generate the function $V(\cdot,\cdot)$. Then, defining $v_k := V(k, x_k)$, using (A.3), (A.4) and invoking Lemma 13 we obtain that along trajectories

$$\begin{aligned} v_{k+1} - v_k &\leq -\alpha_3 \circ \alpha_2^{-1}(v_k) + \beta_3(|\xi_\circ|, k - k_\circ) \\ &= -\alpha(v_k) + \beta_3(|\xi_\circ|, k - k_\circ) \end{aligned} \tag{A.6}$$

for all $k \geq k_\circ \geq 0$ and where we defined $\alpha := \alpha_3 \circ \alpha_2^{-1}$. Define next $\zeta(t) := v_k + (t - k)(v_{k+1} - v_k)$ for all $t \in [k, k+1)$, $k \geq k_\circ \geq 0$. Notice that since $\zeta(t)$ is a linear interpolation from v_k to v_{k+1}, which are always non-negative, we have that

$$0 \leq \zeta(t) \leq \max\{v_k,\, v_{k+1}\} \tag{A.7}$$

for any $t \in [k, k+1)$, $k \geq k_\circ \geq 0$. Next it can be shown that the second inequality in (A.7) and the condition that { for all $t \in [k, k+1)$, $k \geq k_\circ \geq 0$

$$\zeta(t) \geq \alpha^{-1}(2\beta_3(|\xi_\circ|, k - k_\circ)) + \beta_3(|\xi_\circ|, k - k_\circ) \quad \} \tag{A.8}$$

imply by virtue of (A.6) that

$$v_k \geq \alpha^{-1}(2\beta_3(|\xi_\circ|, k - k_\circ)) \tag{A.9}$$

for all $t \in [k, k+1)$, $k \geq k_\circ \geq 0$ and, using (A.6) again we obtain that

$$v_{k+1} - v_k \leq -\frac{1}{2}\alpha(v_k)\,. \tag{A.10}$$

Consequently, defining the piecewise constant "input",

$$u(t) := \alpha^{-1}(2\beta_3(|\xi_\circ|, k - k_\circ) + \beta_3(|\xi_\circ|, k - k_\circ))\,, \tag{A.11}$$

for all $t \in [k, k+1)$, $k \geq k_\circ \geq 0$, and $t_\circ := k_\circ$, the following is true for all $t \in [t_\circ, T_{\max})$, $0 \leq T_{\max} \leq \infty$

$$\left\{ \zeta(t) \geq \left\|u_{[t_\circ,t)}\right\|_\infty \implies \dot{\zeta}(t) \leq -\frac{1}{2}\alpha(\zeta(t)) \right\} \tag{A.12}$$

for almost all $t \in [t_\circ, T_{\max})$. The implication (A.12) is known (see e.g. (Sontag and Wang 1995)) to imply the existence of $\beta_a \in \mathcal{KL}$ such that [3]

$$\zeta(t) \leq \beta_a(\zeta(t_\circ), t - t_\circ) + \max_{t_\circ \leq \tau \leq t} u(\tau) \quad \forall\, t \geq t_\circ \geq 0\,.$$

Thus, the proof can be completed following similar lines to those of the proof of (Khalil 1996, Lemma 5.6). ■

[2] For notational simplicity we have omitted the arguments of $F_1(k, x_k)$ and $G(k, x_k, z_k)$.

[3] Notice that both, $\zeta(\cdot)$ and $u(\cdot)$ are always nonnegative and $u(\cdot)$ is piecewise constant so we write "max" instead of "sup" and we avoid the use of "$|\cdot|$" to avoid a cumbersome notation.