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PROPERTIES OF MGPC DESIGNED IN STATE SPACE [1]

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Abstract: This paper presents some properties regarding the Multivariable Generalized Predictive Controller (MGPC) designed in state space. The authors proposed the design of such a controller under the condition that it gives the same results as the input/output (I/O) formulation of the GPC proposed originally by Clarke. The main reason for redesigning the controller in state space is the power of the analysis tools available. These tools allow some properties to be treated more easily than under (I/O) formulation. These properties include: stability, robustness, closed loop representation, specifications, etc. This paper presents results related to: observability and controllability of the CARIMA model used, closed loop representation, uniqueness and the existence of control law without constraints and closed loop stability without constraints. *Copyright © 2002 IFAC*

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1. INTRODUCTION

To formulate the Multivariable Generalized Predictive Controller (MGPC) in space state, a model of the process described through a CARIMA model (Salcedo, Martinez, Blasco and Sanchis, 2001; Salcedo, 2001; Gambier and Unbehauen, 1999; Grimble, 1994; Ling and Lim, 1996) is assumed. The deterministic part of the CARIMA model can be represented according to the following state space model:

$$\bar{\boldsymbol{x}}(k+1) = \boldsymbol{A}\bar{\boldsymbol{x}}(k) + \boldsymbol{B}\bar{\boldsymbol{u}}(k) \;\; ; \;\; \bar{\boldsymbol{y}}(k) = \boldsymbol{C}\bar{\boldsymbol{x}}(k) \qquad (1)$$

This is a model consisting of n outputs, m inputs and r states.

To obtain a complete CARIMA model, it is necessary to add to the former deterministic model the $\xi_i(k)$ noise variables and their associated states which are called noise states $x_i^*(k)$. These states are nothing more than the accumulation of such inputs:

$$x_i^*(k+1) = x_i^*(k) + \pi_{2i}\xi_i(k) \quad i = 1,2,\ldots,n \qquad (2)$$

When these additional states and inputs are incorporated into the deterministic model given by equation (1), the following CARIMA model is obtained:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \bar{\boldsymbol{A}}\boldsymbol{x}(k) + \bar{\boldsymbol{B}}\bar{\boldsymbol{u}}(k) + \Pi\bar{\xi}(k) \\ \bar{\boldsymbol{y}}(k) &= \bar{\boldsymbol{C}}\boldsymbol{x}(k) + \Lambda\bar{\xi}(k) \end{aligned} \qquad (3)$$

Being:

$$\boldsymbol{x}(k) = \begin{pmatrix} \bar{\boldsymbol{x}}(k) \\ \boldsymbol{x}^*(k) \end{pmatrix} \; ; \; \bar{\xi}(k) = \begin{pmatrix} \xi_1(k) \\ \vdots \\ \xi_n(k) \end{pmatrix} \qquad (4)$$

$$\bar{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \Sigma_{rxn} \\ \boldsymbol{0} & \boldsymbol{I}_n \end{bmatrix} \; ; \; \bar{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} \qquad (5)$$

$$\Pi = \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} \; ; \; \bar{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{C} & \Omega \end{bmatrix} \qquad (6)$$

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This model is a state space CARIMA model equivalent to that used in the I/O case (Sanchis, 1997; Salcedo, 2001; Camacho and Bordons, 1995). Matrices Σ, Π_1, Π_2, Ω and Λ can be freely chosen to establish different noise models for the process. This means an increase in the complexity of the choice of noise model parameters with respect to the I/O formulation, in which only the $T_i(z^{-1})$ filter polynomials had to be chosen.

To estimate the model states, a full rank observer, known as CARIMA observer (Salcedo et al., 2001), is employed:

$$\boldsymbol{x}(k+1|k) = \left[\bar{\boldsymbol{A}} - \Pi\Lambda^{-1}\bar{\boldsymbol{C}}\right]\boldsymbol{x}(k|k-1) + \\ +\bar{\boldsymbol{B}}\bar{\boldsymbol{u}}(k) + \Pi\Lambda^{-1}\bar{\boldsymbol{y}}(k) \quad (7)$$

On the MGPC design a quadratic cost index, similar to the cost index used in (Ordys and Clarke, 1993), is proposed:

$$J_k(\hat{\boldsymbol{u}}) = E\Bigg[\sum_{i=N_1}^{N_2}(\bar{\boldsymbol{y}}(k+i) - \bar{\boldsymbol{w}}(k+i))^T\boldsymbol{Q}_i(\bar{\boldsymbol{y}}(k+i) - \\ -\bar{\boldsymbol{w}}(k+i)) + \sum_{i=1}^{N_u}\Delta\bar{\boldsymbol{u}}^T(k+i-1)\boldsymbol{R}_i\Delta\bar{\boldsymbol{u}}(k+i-1)\Bigg] \\ \boldsymbol{Q}_i \in \mathbb{R}^{n\times n}\ ;\ \boldsymbol{R}_i \in \mathbb{R}^{m\times m} \quad (8)$$

Where:

- $\bar{\boldsymbol{w}}(k+i)$ is the vector of the desired references in instant $k+i$.
- N_1, N_2 represents the limits of the prediction horizon.
- N_u is the control horizon.
- $\boldsymbol{Q}_i$ is the pondering matrix of the error in instant i inside the prediction horizon.
- $\boldsymbol{R}_j$ is the weighting matrix of the control action increment in instant j inside the control horizon.

The index can adopt the following matrix form:

$$J_k(\hat{\boldsymbol{u}}) = (\hat{\boldsymbol{y}}(k) - \hat{\boldsymbol{w}}(k))^T\boldsymbol{Q}(\hat{\boldsymbol{y}}(k) - \hat{\boldsymbol{w}}(k)) + \\ +\hat{\boldsymbol{u}}^T(k)\boldsymbol{R}\hat{\boldsymbol{u}}(k) \quad (9)$$

being:

$$\hat{\boldsymbol{w}}(k) = \left(\bar{\boldsymbol{w}}(k+N_1)\ \cdots\ \bar{\boldsymbol{w}}(k+N_2)\right)^T \quad (10)$$

$$\boldsymbol{Q} = diag\left(\boldsymbol{Q}_{N_1}\ \boldsymbol{Q}_{N_1+1}\ \cdots\ \boldsymbol{Q}_{N_2}\right) \quad (11)$$

$$\boldsymbol{R} = diag\left(\boldsymbol{R}_1\ \boldsymbol{R}_2\ \cdots\ \boldsymbol{R}_{N_u}\right) \quad (12)$$

The methodology for the design of the MGPC (Clarke, Mohtadi and Tuffs, 1987a; Clarke, Mohtadi and Tuffs, 1987b; Clarke and Mohtadi, 1989) is as follows: on each sampling instant k, index (9) has to be optimised to determine the control actions to be applied to the process. To optimize such an index it is necessary to predict the n outputs of process (3) inside their corresponding prediction horizon, and according to:

- The values of the m input variables inside their control horizons (unknown). These are precisely the independent variables from which the quadratic index depends ($\hat{\boldsymbol{u}}(k)$).
- The ξ_i variables considered as white noise.
- The values (known) of the previous applied inputs, and the actual state.

From of the control action values obtained after optimising index (9), only the control actions corresponding to the first instant of each control horizon $u_1(k), u_2(k), \cdots u_m(k)$ are applied to the process. This technique is known as receding horizon. The process is then repeated for the following sampling period $k+1$.

2. RESULTS ABOUT CONTROLLABILITY AND OBSERVABILITY

In this section, some interesting results regarding observability and controllability are described. See (Salcedo, 2001) for a more exhaustive description.

Lemma 1. The deterministic part of state space CARIMA model (1), matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, have the following realisation:

$$\boldsymbol{A} = \text{diag}\left[\boldsymbol{A}_1\ \boldsymbol{A}_2\ \cdots\ \boldsymbol{A}_n\right] \\ \boldsymbol{B} = \left[\boldsymbol{B}_1^T\ \boldsymbol{B}_2^T\ \cdots\ \boldsymbol{B}_n^T\right]^T \\ \boldsymbol{C} = \text{diag}\left[\boldsymbol{C}_1\ \boldsymbol{C}_2\ \cdots\ \boldsymbol{C}_n\right] \quad (13)$$

where:

$$\boldsymbol{A}_r = \begin{bmatrix} 0\ 0\ \ldots\ 0 & -a_{0,r} \\ \boldsymbol{I}_{n-1} & \vdots \\ & -a_{n_r-1,r} \end{bmatrix} \quad (14)$$

$$B_r(j,i) = b_{j-1,r,i}\ ;\ \boldsymbol{C}_r = \left(0\ \cdots\ 0\ 1\right) \quad (15)$$

being $a_{j,r}$ the coefficients of $A_r(z)$ polynomials and $b_{j-1,r,i}$ the coefficients of $B_{ri}(z)$ polynomials:

$$A_r(z) = z^{n_r} + a_{n_r-1,r}z^{n_r-1} + \cdots + a_{1,r}z + a_{0,r} \\ B_{ri}(z) = b_{n_r-1,r,i}z^{n_r-1} + \cdots + b_{1,r,i}z + b_{0,r,i}$$

The polynomials $A_r(z)$ and $B_{ri}(z)$ are the polynomials of the deterministic part of the I/O CARIMA model.

Proposition 2. The pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable if matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ are built as in Lemma 1.

The previous results are only interesting in order to complete the proof of the following proposition, which characterises the observability of the state space CARIMA model, that is to say, the observability of the pair $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{C}})$:

Proposition 3. If noise matrices, Σ, Ω, Π_1, Π_2 and Λ, are designed in order to place the CARIMA observer poles (Salcedo et al., 2001), and no pole is located in 1, then the pair $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{C}})$ is observable.

In general, the pairs $(\boldsymbol{A}, \boldsymbol{B})$ and $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}})$ will not be controllable, however the pair $(\bar{\boldsymbol{A}}, [\bar{\boldsymbol{B}}\ \Pi])$ will be controllable under some smooth conditions:

Proposition 4. If noise matrices, Σ, Ω, Π_1, Π_2 and Λ, are designed in order to place the CARIMA observer poles, and the matrices $\boldsymbol{K}_j$:

$$\boldsymbol{K}_j = \left[\Pi_{jj}\ \boldsymbol{A}_j\Pi_{jj} + \Pi_{jj}\Omega_{jj}\ \cdots \right.$$
$$\left. \boldsymbol{A}_j^{s+n-1}\Pi_{jj} + \boldsymbol{A}_j^{s+n-2}\Pi_{jj}\Omega_{jj} + \cdots + \Pi_{jj}\Omega_{jj}\right]$$
$$j = 1, \cdots, n \tag{16}$$

have no empty row, then the pair $(\bar{\boldsymbol{A}}, [\bar{\boldsymbol{B}}\,\Pi])$ is controllable.

Proposition 5. (About matrices $\boldsymbol{K}_j$). The $\boldsymbol{K}_j$ matrix has no empty row if some of the following conditions hold:

(1) If the pair $(\boldsymbol{A}_j, \Pi_{jj})$ is controllable.
(2) If $\forall\, k\ \ \Gamma_{jj,k} \neq a_{k,j}$.
(3) If $\exists\, k:\ \Gamma_{jj,k} = a_{k,j}$, the matrix:

$$\boldsymbol{Z}^k = \begin{bmatrix} 0 \cdots 0 \underset{(k+1)}{1} 0 \cdots 0 \\ {}^{k+1}\boldsymbol{A}_j \\ \vdots \\ {}^{k+1}\boldsymbol{A}_j^{n_j-1} \end{bmatrix} \tag{17}$$

is non-singular and $\exists\, l:\ \Gamma_{jj,l} \neq a_{l,j}$. Moreover, $\exists\, s \in \mathbb{N}:\ {}^{k+1}\boldsymbol{A}_j^s\Pi_{jj} \neq 0$ being ${}^{k+1}\boldsymbol{A}_j^{s-1}\Pi_{jj} = 0$.

${}^{k+1}\boldsymbol{A}_j^l$ is the $(k+1)$-th row of the matrix $\boldsymbol{A}_j^l$

Finally, using the previous results, sufficient conditions can be presented under which the state space CARIMA model is a minimal realisation:

Proposition 6. The state space CARIMA model (3), under the hypothesis of propositions 3 and 4, is a minimal realisation.

3. EXISTENCE AND UNIQUENESS OF THE UNCONSTRAINED CONTROL LAW

It is important to emphasize that only the unconstrained case will be analysed in this paper, altough, the constrained case will analysed in future works.

In (Salcedo, 2001) was established that:

Proposition 7. If $\boldsymbol{Q}_i$ matrices are positive definite, the $\boldsymbol{R}_i$ matrices are non-negative definite and the $\boldsymbol{N}$ matrix has full column rank, then the unconstrained MGPC control law has a unique solution given by:

$$\hat{\boldsymbol{u}}(k) = -\left(\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N} + \boldsymbol{R}\right)^{-1}\boldsymbol{N}^T\boldsymbol{Q}^T\Big(\boldsymbol{M}\boldsymbol{x}(k) +$$
$$+\boldsymbol{O}\bar{\boldsymbol{u}}(k-1) + \boldsymbol{P}\bar{\xi}(k) - \hat{\omega}(k)\Big)$$
$$\hat{\boldsymbol{u}}(k) = -\left(\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N} + \boldsymbol{R}\right)^{-1}\boldsymbol{N}^T\boldsymbol{Q}^T\hat{\boldsymbol{e}}_c(k)$$

This expression is deducted from the linear system:

$$-\left(\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N} + \boldsymbol{R}\right)\hat{\boldsymbol{u}}(k) = \boldsymbol{N}^T\boldsymbol{Q}^T\hat{\boldsymbol{e}}_c(k) \tag{18}$$

This linear system has a unique solution if, and only if, the $\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N} + \boldsymbol{R}$ matrix is non-singular. By extension, it is non-negative definite, so it will only be non-singular if, and only if, it is positive definite.
In the case that such matrix is singular, it may be possible that the linear system has no solution. In order to guarantee the existence of the solutions:

$$\text{Rank}\left(\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N} + \boldsymbol{R}\right) =$$
$$= \text{Rank}\left[\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N} + \boldsymbol{R}\quad \boldsymbol{N}^T\boldsymbol{Q}^T\hat{\boldsymbol{e}}_c(k)\right] \tag{19}$$

condition which depends on the reference.

Evidently, only those cases where the unconstrained control law has a unique solution are interesting, and so the $\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N} + \boldsymbol{R}$ matrix is positive definite. There are three possibilities:

(1) The $\boldsymbol{R}_i$ matrices are positive definite. In such case $\boldsymbol{R}$ will also be positive definite too, as well as $\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N} + \boldsymbol{R}$. This situation is not frequent in predictive control.
(2) The conditions of proposition 7 hold. The condition related to the full column rank of matrix $\boldsymbol{N}$ seems artificial, however, the following proposition gives smooth conditions that guarantee such a property:

Proposition 8. (Rank of $\boldsymbol{N}$). If the matrix $\bar{\boldsymbol{B}}$ has full column rank, $\text{Rank}B = m$, and the pair $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{C}})$ is observable, then the $\boldsymbol{N}$ matrix has full column rank when $N_2 - N_u \geq n + \sum_{j=1}^{n} n_j - 1$.

See (Salcedo, 2001) for the proof of this proposition.
(3) Neither of the previous hypotheses is verified, that is to say, matrices $\boldsymbol{R}$ and $\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N}$ are non-negative definite, but its sum is positive definite. The following proposition guarantees when this situation will happen:

Proposition 9. Suppose that the matrices $\boldsymbol{R}$ and $\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N}$ are non-negative definite, $\exists\ \boldsymbol{x} \neq \boldsymbol{0}, \boldsymbol{y} \neq \boldsymbol{0}\ :\ \ \boldsymbol{x}^T\boldsymbol{R}\boldsymbol{x} = 0\ ,\ \boldsymbol{y}^T\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N}\boldsymbol{y} = 0$, if the null spaces of $\boldsymbol{R}^{1/2}$ and $\boldsymbol{N}$ has no common vectors (except $\boldsymbol{0}$), then the $\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N} + \boldsymbol{R}$ matrix is positive definite.

See (Salcedo, 2001) for the proof of this proposition.

4. CLOSED LOOP REPRESENTATION

The next section contains the state space closed loop representation of the plant + MGPC given by the following proposition (Salcedo, 2001):

Proposition 10. If there are no discrepancies between the plant and the model, all the CARIMA observer poles are inside the unit disk and $\boldsymbol{R} = \boldsymbol{0}$, then the state space closed loop representation of the plant + MGPC can be reduced to:

$$\bar{\boldsymbol{x}}(k+1) = \boldsymbol{A}_{BC}\bar{\boldsymbol{x}}(k) + \boldsymbol{B}_{BC}\hat{\omega}(k)$$
$$\bar{\boldsymbol{y}}(k) = \boldsymbol{C}_{BC}\bar{\boldsymbol{x}}(k) \tag{20}$$
$$\boldsymbol{A}_{BC} = \boldsymbol{A} - \boldsymbol{B}\sigma\boldsymbol{M}\boldsymbol{W}, \quad \boldsymbol{B}_{BC} = \boldsymbol{B}\sigma$$
$$\boldsymbol{C}_{BC} = \boldsymbol{C} \tag{21}$$

5. STABILITY RESULTS

In order to perform the stability analyse, the cost index optimum value will be used as a Lyapunov function.

Proposition 11. The cost index optimum value for the unconstrained MGPC is given by:

$$J_k^* = \hat{\boldsymbol{e}}_c(k)^T \mathscr{K} \hat{\boldsymbol{e}}_c(k) \tag{22}$$

$$\mathscr{K} = \left[\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{N}\left(\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N} + \boldsymbol{R}\right)^{-1}\boldsymbol{N}^T\boldsymbol{Q}^T\right] \tag{23}$$

particulary, when $\boldsymbol{R} = \boldsymbol{0}$, the references are constant and there is no differences between the model and the plant then:

$$J_k^* = \begin{pmatrix} \overline{\boldsymbol{x}}(k-1) \\ \overline{\omega}(k-1) \end{pmatrix}^T \bar{\mathscr{T}}^T \mathscr{K} \bar{\mathscr{T}} \begin{pmatrix} \overline{\boldsymbol{x}}(k-1) \\ \overline{\omega}(k-1) \end{pmatrix} \quad \forall\, k \geq 1 \tag{24}$$

$$\bar{\mathscr{T}} = \left[\mathscr{T} \;\; \left([\boldsymbol{MWB} + \boldsymbol{O}]\,\sigma - \boldsymbol{I}\right)\boldsymbol{U}\right] \tag{25}$$

$$\mathscr{T} = [\boldsymbol{MW}\,(\boldsymbol{A} - \boldsymbol{B}\sigma\boldsymbol{MW}) - \boldsymbol{O}\sigma\boldsymbol{MW}] \tag{26}$$

See (Salcedo, 2001) for the proof of this proposition.

Since this analysis deals with linear systems without constraints, the stability does not depend on reference nor on initial conditions. In order to simplify, it will be assumed that the reference is zero and the linear system starts from a generic initial condition $\overline{\boldsymbol{x}}(0)$. Under these conditions the cost index optimum value is now:

$$J_k^* = \overline{\boldsymbol{x}}(k-1)^T \mathscr{T}^T \mathscr{K} \mathscr{T} \overline{\boldsymbol{x}}(k-1) \quad \forall\, k \geq 1 \tag{27}$$

Using this result, it is obtained:

Proposition 12. Once selected the $\boldsymbol{Q}_i$ matrices, for those pairs of N_2 and N_u such that the matrix:

$$\mathscr{L} = \boldsymbol{A}_{BC}^T \mathscr{T}^T \mathscr{K} \mathscr{T} \boldsymbol{A}_{BC} - \mathscr{T}^T \mathscr{K} \mathscr{T} \tag{28}$$

is negative definite, then the closed loop of the plant + MGPC is stable.

PROOF. It will be established that if the matrix $\mathscr{L}$ is negative definite then the cost index optimum value is a Laypunov function for the closed loop. By extension:

$$J_k^* = \overline{\boldsymbol{x}}(k-1)^T \mathscr{T}^T \mathscr{K} \mathscr{T} \overline{\boldsymbol{x}}(k-1) \geq 0 \tag{29}$$

Now it can be seen if this optimum value decreases with k:

$$\begin{aligned} J_{k+1}^* - J_k^* = {} & \overline{\boldsymbol{x}}(k)^T \mathscr{T}^T \mathscr{K} \mathscr{T} \overline{\boldsymbol{x}}(k) - \\ & - \overline{\boldsymbol{x}}(k-1)^T \mathscr{T}^T \mathscr{K} \mathscr{T} \overline{\boldsymbol{x}}(k-1) \end{aligned} \tag{30}$$

using (20):

$$\overline{\boldsymbol{x}}(k+1) = \boldsymbol{A}_{BC}\overline{\boldsymbol{x}}(k) + \boldsymbol{B}_{BC}\underbrace{\hat{\omega}(k)}_{\boldsymbol{0}}$$

$$\overline{\boldsymbol{x}}(k) = \boldsymbol{A}_{BC}\overline{\boldsymbol{x}}(k-1) \tag{31}$$

it is obtained:

$$\begin{aligned} J_{k+1}^* - J_k^* = {} & \overline{\boldsymbol{x}}(k-1)^T \boldsymbol{A}_{BC}^T \mathscr{T}^T \mathscr{K} \mathscr{T} \boldsymbol{A}_{BC} \overline{\boldsymbol{x}}(k-1) - \\ & - \overline{\boldsymbol{x}}(k-1)^T \mathscr{T}^T \mathscr{K} \mathscr{T} \overline{\boldsymbol{x}}(k-1) \end{aligned} \tag{32}$$

$$J_{k+1}^* - J_k^* = \overline{\boldsymbol{x}}(k-1)^T \mathscr{L} \overline{\boldsymbol{x}}(k-1)^T \tag{33}$$

As $\mathscr{L}$ is negative definite:

$$J_{k+1}^* - J_k^* < 0 \tag{34}$$

and so the closed loop of the plant + MGPC is stable. □

This stability result can be presented with an alternative proposition, following the ideas shown in (Nevistić, 1997; Primbs and Nevistić, 2000):

Proposition 13. (Second form). Let be $\psi^* = \max\{\psi : -\overline{\boldsymbol{x}}(k-1)^T \mathscr{L} \overline{\boldsymbol{x}}(k-1) \geq \psi J_k^* \quad \forall\, k\}$, then it is verified that:

$$J_{k+1}^* \leq (1 - \psi^*) J_k^* \tag{35}$$

Once selected the $\boldsymbol{Q}_i$ matrices, for those pairs of N_2 and N_u such that $(1 - \psi^*) < 1$, then the closed loop of the plant + MGPC is stable.

PROOF. First, the expression for ψ^* is calculated:

$$-\overline{\boldsymbol{x}}(k-1)^T \mathscr{L} \overline{\boldsymbol{x}}(k-1) \geq \psi J_k^* \quad \forall\, k \tag{36}$$

$$\iff -\mathscr{L} \geq \psi \mathscr{T}^T \mathscr{K} \mathscr{T} \tag{37}$$

$$\iff \mathscr{L} + \psi \mathscr{T}^T \mathscr{K} \mathscr{T} \leq \boldsymbol{0} \tag{38}$$

$$\iff \psi \boldsymbol{I} + \mathscr{L}\left(\mathscr{T}^T \mathscr{K} \mathscr{T}\right)^{-1} \leq \boldsymbol{0} \tag{39}$$

$$\iff \psi \boldsymbol{I} \leq -\mathscr{L}\left(\mathscr{T}^T \mathscr{K} \mathscr{T}\right)^{-1} \tag{40}$$

$$\iff \psi \leq \underline{\lambda}\left(-\mathscr{L}\left(\mathscr{T}^T \mathscr{K} \mathscr{T}\right)^{-1}\right) \tag{41}$$

$\underline{\lambda}(\cdot)$ represents the minor eigenvalue of the matrix. Consequently, it is deducted that:

$$\psi^* = \underline{\lambda}\left(-\mathscr{L}\left(\mathscr{T}^T \mathscr{K} \mathscr{T}\right)^{-1}\right) \tag{42}$$

At the previous proposition it was established that:

$$J_{k+1}^* - J_k^* = \overline{\boldsymbol{x}}(k-1)^T \mathscr{L} \overline{\boldsymbol{x}}(k-1)^T \tag{43}$$

Applying the definition equation of ψ^*:

$$J_k^* - J_{k+1}^* = -\overline{\boldsymbol{x}}(k-1)^T \mathscr{L} \overline{\boldsymbol{x}}(k-1)^T \tag{44}$$

$$J_k^* - J_{k+1}^* \geq \psi^* J_k^* \tag{45}$$

$$J_{k+1}^* \leq (1 - \psi^*) J_k^* \tag{46}$$

if $(1 - \psi^*) < 1$ then J_k^* is a Lypunov function, and so the closed loop of the plant + MGPC is stable. □

5.1 *When J_k^* is not a Lyapunov function*

There are cases in which J_k^* is not a Lyapunov function, but however, the closed loop is stable. As a consequence, it is necessary to have alternative methodologies to analyse the closed loop stability.

It is possible to propose an alternative quadratic function which is a Lyapunov function of the closed loop, with the following condition:

$$I_k = \overline{\boldsymbol{x}}(k-1)^T \boldsymbol{Z} \overline{\boldsymbol{x}}(k-1) \quad : \; I_k \geq J_k^* \quad \forall\, k \tag{47}$$

$$\iff \boldsymbol{Z} \geq \mathscr{T}^T \mathscr{K} \mathscr{T} \tag{48}$$

As this is a Lyapunov function it must decrease with k:

$$I_{k+1} - I_k < 0 \quad \forall\, k \neq 0 \tag{49}$$

operating:

$$I_{k+1} - I_k = \overline{\boldsymbol{x}}(k)^T \boldsymbol{Z} \overline{\boldsymbol{x}}(k) - \overline{\boldsymbol{x}}(k-1)^T \boldsymbol{Z} \overline{\boldsymbol{x}}(k-1) \quad (50)$$

$$I_{k+1} - I_k = \overline{\boldsymbol{x}}(k-1)^T \boldsymbol{A}_{BC}^T \boldsymbol{Z} \boldsymbol{A}_{BC} \overline{\boldsymbol{x}}(k-1) - \overline{\boldsymbol{x}}(k-1)^T \boldsymbol{Z} \overline{\boldsymbol{x}}(k-1) \quad (51)$$

being $\boldsymbol{A}_{BC} = \boldsymbol{A} - \boldsymbol{B}\sigma\boldsymbol{M}\boldsymbol{W}$. As a consequence, because this is a Lyapunov function, the following matrix has to be negative definite:

$$\boldsymbol{A}_{BC}^T \boldsymbol{Z} \boldsymbol{A}_{BC} - \boldsymbol{Z} < \boldsymbol{0} \quad (52)$$

In order to build an adequate $\boldsymbol{Z}$ matrix it is necessary to verify the conditions (48) and (52). This can be achieved solving the following discrete Lyapunov equation:

$$\boldsymbol{A}_{BC}^T \boldsymbol{Z} \boldsymbol{A}_{BC} - \boldsymbol{Z} = \boldsymbol{L} \quad (53)$$

choosing $\boldsymbol{L}$ so that the another condition is verified.
If $\boldsymbol{L}$ is symmetric and negative definite then the symmetric solution of the discrete Lyapunov equation verifies:

- If the closed loop is stable, all the eigenvalues of $\boldsymbol{A}_{BC}$ are inside the unit disk, the solution is positive definite.
- If there is some eigenvalue on the outside of the unit disk, with some remaining inside, the solution is indefinite.
- If all the eigenvalues are outside the unit disk, the solution is negative definite.

Here a particular method for designing the matrix $\boldsymbol{Z}$ is analysed:

$$A_{BC}^T \boldsymbol{Z} \boldsymbol{A}_{BC} - \boldsymbol{Z} = \boldsymbol{L} \text{ ,with } \boldsymbol{L} = -l\boldsymbol{I} \quad l > 0 \quad (54)$$

$$A_{BC}^T \boldsymbol{Z} \boldsymbol{A}_{BC} - \boldsymbol{Z} = -l\boldsymbol{I} \quad (55)$$

$$A_{BC}^T \underbrace{\boldsymbol{Z}/l}_{\boldsymbol{Z}_I} \boldsymbol{A}_{BC} - \underbrace{\boldsymbol{Z}/l}_{\boldsymbol{Z}_I} = -\boldsymbol{I} \quad (56)$$

$\boldsymbol{Z}_I$ is the solution of the discrete Lyapunov equation when $\boldsymbol{L} = -\boldsymbol{I}$. Now l is selected so that the another condition is verified:

$$\boldsymbol{Z} \geq \mathscr{T}^T \mathscr{K} \mathscr{T} \quad (57)$$

$$\iff l\boldsymbol{Z}_I \geq \mathscr{T}^T \mathscr{K} \mathscr{T} \quad (58)$$

$$\iff l\boldsymbol{Z}_I - \mathscr{T}^T \mathscr{K} \mathscr{T} \geq \boldsymbol{0} \quad (59)$$

accepting that $\boldsymbol{Z}_I$ is positive definite, that is to say, stable closed loop:

$$l\boldsymbol{I} - \mathscr{T}^T \mathscr{K} \mathscr{T} \boldsymbol{Z}_I^{-1} \geq \boldsymbol{0} \quad (60)$$

$$\iff l \geq \overline{\lambda}\left(\mathscr{T}^T \mathscr{K} \mathscr{T} \boldsymbol{Z}_I^{-1}\right) \quad (61)$$

where $\overline{\lambda}(\cdot)$ represents the bigger eigenvalue of the matrix.

These results give the proposition:

Proposition 14. Once selected the $\boldsymbol{Q}_i$ matrices, for each pair of N_2 and N_u, if the matrix $\boldsymbol{Z}_I$, solution of the discrete Lyapunov equation:

$$A_{BC}^T \boldsymbol{Z}_I \boldsymbol{A}_{BC} - \boldsymbol{Z}_I = -\boldsymbol{I} \quad (62)$$

verifies that $\underline{\lambda}(\boldsymbol{Z}_I) > 0$, then the closed loop of the plant + MGPC is stable. Moreover, the quadratic function:

$$I_k = \overline{\boldsymbol{x}}(k-1)^T \boldsymbol{Z} \overline{\boldsymbol{x}}(k-1)$$

$$\boldsymbol{Z} = l\boldsymbol{Z}_I \quad , \; l \geq \overline{\lambda}\left(\mathscr{T}^T \mathscr{K} \mathscr{T} \boldsymbol{Z}_I^{-1}\right) \quad (63)$$

is a Lyapunov function of the closed loop, and it is a upper bound for the cost index optimum value in all the sampling periods:

$$I_k \geq J_k^* \quad \forall\, k \quad (64)$$

Example 15. (Stirred tank reactor). The following application example is based on the model of a stirred tank reactor (Camacho and Bordons, 1995)(page 113) given by the transfer matrix:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+0.1s} & \dfrac{5}{1+s} \\ \dfrac{1}{1+0.5s} & \dfrac{2}{1+0.4s} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \quad (65)$$

For this process, a state space MGPC controller is designed with the following parameters:

- Sampling period: $T = 0.05$ minutes. $N_1 = 1$. Error pondering matrices: $\boldsymbol{Q}_i = \boldsymbol{I} \quad i = N_1, \cdots, N_2$. Control action increment pondering matrices: $\boldsymbol{R}_i = \boldsymbol{0} \quad i = 1, \cdots, N_u$. All the CARIMA observer poles are located in 0 (maximum speed). References are constant. This design operates with the process model without any kind of decoupling.

This example only refers to the stability analyse, and so, the simulations of the controlled process will not presented.

Three different cases will be analysed:

(1) $N_2 = 20$ and $N_u = 1$. In this case the closed loop is unstable since $\overline{\lambda}(\boldsymbol{A}_{BC}) = 1.0134$.
Proposition 12 gives $\overline{\lambda}(\boldsymbol{L}) = 4.5656$ and so $\boldsymbol{L}$ is not negative definite, consequently this criteria does not give information about stability.
Proposition 13 gives $1 - \psi^* = 1.24$, the same situation as before.
Proposition 14 gives $\underline{\lambda}(\boldsymbol{Z}_I) = -182.761$, and so the closed loop is unstable.
(2) $N_2 = 50$ and $N_u = 1$. In this case the closed loop is stable since $\overline{\lambda}(\boldsymbol{A}_{BC}) = 0.9582$.
Proposition 12 gives $\overline{\lambda}(\boldsymbol{L}) = 0.2857$ and so $\boldsymbol{L}$ is not negative definite, consequently this criteria does not give information about stability.
Proposition 13 gives $1 - \psi^* = 1.0321$, the same situation as before.
Proposition 14 gives $\underline{\lambda}(\boldsymbol{Z}_I) = 1.4658$, and so the closed loop is stable.
Figure 1 compares the cost index optimum value and the alternative quadratic function, and as can be seen, the alternative quadratic function is always an upper bound for the optimum value. In this case, J_k^* is not a Lyapunov function.

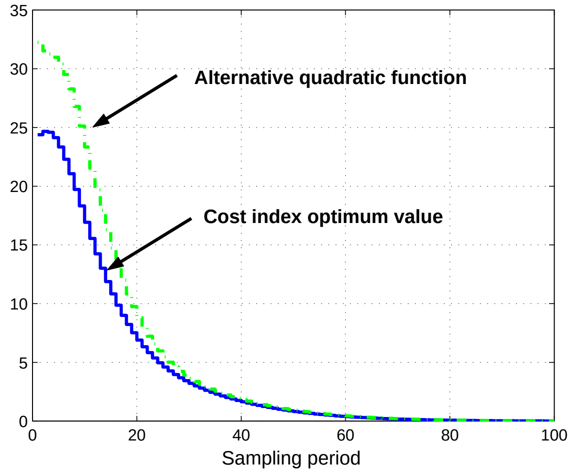

Fig. 1. Comparison between J_k^* and I_k

(3) $N_2 = 50$ and $N_u = 2$. In this case, the closed loop is stable since $\overline{\lambda}(\boldsymbol{A}_{BC}) = 0.9014 \pm 0.0064j$.
Proposition 12 gives $\overline{\lambda}(\boldsymbol{L}) = -0.0076$ and so $\boldsymbol{L}$ is negative definite, consequently under this criteria the closed loop is stable too. In this case, J_k^* is a Lyapunov function.
Proposition 13 gives $1 - \psi^* = 0.9618$, the same situation as before.
Proposition 14 gives $\underline{\lambda}(\boldsymbol{Z}_I) = 1.0002$, and so the closed loop is also stable with this criteria.

6. CONCLUSIONS

(1) The state space CARIMA model is a minimal realisation under smooth conditions.
(2) The existence and uniqueness of the unconstrained control law has been analysed in depth.
(3) Some results about stability have been presented. All are related with the unconstrained case and $\boldsymbol{R} = \boldsymbol{0}$.
(4) A more powerful stability result, not based on the property that J_k^* is a Lyapunov function, has been developed.

7. REFERENCES

Camacho, E. and Bordons, C. (1995). *Model Predictive Control in the Process Industry*, Springer.

Clarke, D. and Mohtadi, C. (1989). Properties of generalized predictive control, *Automatica* **25**(6): 859–875.

Clarke, D., Mohtadi, C. and Tuffs, P. (1987a). Generalized predictive control-part I, *Automatica* **23**(2): 137–148.

Clarke, D., Mohtadi, C. and Tuffs, P. (1987b). Generalized predictive control-part II. Extensions and Interpretations, *Automatica* **23**(2): 149–160.

Gambier, A. and Unbehauen, H. (1999). Multivariable generalized state-space receding horizon control in a real-time environment, *Automatica* **35**: 1787–1797.

Grimble, M. (1994). State-Space Approach to LQG Multivariable Predictive and Feedforward Optimal Control, *Journal of Dynamic Systems, Measurement and Control* **116**: 610–617.

Ling, K. and Lim, K. (1996). A state space GPC with extensions to multirate control, *Automatica* **32**(7): 1067–1071.

Nevistić, V. (1997). *Constrained Control of Nonlinear Systems*, PhD thesis, Swiss Federal Institute of Technology.

Ordys, A. W. and Clarke, D. W. (1993). A state-space description for GPC controllers, *Int. J. Systems Sci.* **24**: 1727–1744.

Primbs, J. and Nevistić, V. (2000). A New Approach to Stability Analysis for Constrained Finite Receding Horizon Control Without End Constraints, *IEEE Transactions on Automatic Control* **45**(8): 1507–1512.

Salcedo, J. (2001). Design and properties of multivariable generalized predictive controllers in state space., *Technical report*, Dept. of Systems Engineering and Control, Technical University of Valencia, Spain. (In spanish).

Salcedo, J., Martinez, M., Blasco, X. and Sanchis, J. (2001). Design of GPC's in state space, *9th IEEE Mediterranean Conference on Control and Automation*, Zagreb (CROATIA).

Sanchis, J. (1997). GPC design for MIMO processes. Constraint analysis at the actuators and Application to real processes, *Technical report*, Dept. of Systems Engineering and Control, Technical Unversity of Valencia, Spain. (In spanish).