15th Triennial World Congress, Barcelona, Spain

A MULTIPLE-LOOP SLIDING MODE CONTROL SYSTEM WITH SECOND-ORDER BOUNDARY LAYER DYNAMICS[1]

I. A. Shkolnikov and Y. B. Shtessel,

Department of Electrical and Computer Engineering,
University of Alabama in Huntsville,
Huntsville, AL 35899, USA,
shtessel@ece.uah.edu

Abstract: Presented is a method of continuous sliding mode control design to provide for the second-order sliding mode on the selected sliding surfaces in a three-loop control system, where two outer-loop virtual control signals must be smooth enough to be tracked in the inner loops. The control law in the vicinity of the sliding surface is a nonlinear dynamic feedback that in absence of unknown disturbances provides for finite-time convergence of the second-order reaching phase dynamics. The controller with a second-order disturbance observer in a combination with the proposed continuous dynamic feedback attracts the system trajectories to boundary layers around two sliding surfaces of the outer control loops with the second-order sliding accuracy in presence of unknown disturbances and the discrete-time control update.

Keywords: sliding mode control, multiloop control, dynamic control, tracking systems.

1. INTRODUCTION

In multiple-loop backstepping-type control systems, an important issue is to provide a so-called virtual control signal to be smooth in an outer loop, for it has to be tracked by inner cascades of a multi-loop system. Smoothness of a control signal is not easy achievable in control systems with sliding modes without degrading high precision and robustness.

The idea of this work is to use combination of both the sliding mode estimator for a plant disturbance and smooth sliding mode controller to design a smooth sliding mode controller that is robust to disturbances and provides finite time convergence to the custom made sliding surface. Two types of smooth sliding mode controllers are to be designed:
A first order smooth finite reaching time sliding mode controller with a traditional sliding mode observer for disturbance observation, (under the discrete-time control with a zero-order hold, the accuracy of holding the trajectories on the sliding surface is of the first order real sliding $O(T)$);
A second order smooth finite reaching time sliding mode controller with a second order sliding mode observer for disturbance observation, (under the discrete-time control with a zero-order hold, the accuracy of holding the trajectories on the sliding surface is of the second order real sliding $O(T^2)$).

For many control applications, Sliding Mode Control (SMC) has been proved the efficient technique to provide high-fidelity performance in different control problems for nonlinear systems with uncertainties in system parameters and external disturbances. Ideal sliding modes feature theoretically-infinite-frequency switching, while the real conventional sliding modes feature high finite frequency switching of an input signal (control). Such a mode might be unacceptable if the control signal has to be tracked by inner cascades of a multi-loop system. Trading the absolute

[1] This work was partially supported by the EPSRC grant GR/R5/667/01

robustness on the sliding surface for the system convergence to a small domain, the “boundary layer”, around it under a continuous control law, the methods of this group employ a high-gain saturation function or a sigmoid function (Utkin, 1992; Slotine and Li, 1991; Edwards and Spurgeon, 1998). In the continuous-time control systems with sampled-data measurements and/or discrete-time control action (zero-order hold with digital control), different types of closed-loop boundary-layer dynamics are employed to provide for a smooth control, varying from adaptive solutions (Zinober *et al*, 1999) to fixed-gain deadbeat controls with disturbance estimation using delayed-time data (Su *et al*, 2000). Another alternative (Kachroo, 1999) to the latter approach is to incorporate into the “boundary-layer” dynamics an exosystem model for disturbances (Francis and Wonham, 1976) avoiding the direct observer-based disturbance estimation.

The idea to hide discontinuity of control in its higher derivatives has been realized using higher order sliding modes (Levant, 1998; Bartolini *et al.*, 1998). The resulting higher-order sliding mode is of enhanced accuracy and robustness to disturbances. However, a drawback of the direct application of this approach to chattering attenuation is that it cannot tolerate unmodeled fast dynamics. Therefore, the designed continuous control cannot be, for instance, an outer-loop feedback in a multi-loop control system.

The idea of this paper is to use both the disturbance estimation and the higher-order sliding mode techniques to design a continuous sliding mode control, providing finite-time convergence to the sliding surface and establish the second-order sliding mode in absence of unknown disturbances. In case when disturbances are present, the disturbance observer determines the accuracy. Employing the second-order observer (Levant, 1998), the second-order sliding accuracy can be achieved. The main contribution of this paper is in further development of the approach presented in the work (Brown *et al*.,2000).

2. TRACKING PROBLEM FORMULATION

Consider a MIMO plant with *n* states and *m* controls, where the “diagonalization method“ (Utkin, 1992) has been applied producing *m* independent dynamics for each input-output channel. Then, consider the following SISO nonlinear uncertain system that can represent any input-output channel (we assume relative degree is equal to 3, although the given approach can be generalized to r^{th} order system as well)

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(x_1) + f_1(x_1,t), \\ \dot{x}_2 &= x_3 + \varphi_2(x_1,x_2) + f_2(x_1,x_2,t), \\ \dot{x}_3 &= \varphi_3(x_1,x_2,x_3) + f_3(\cdot,t) + u, \\ y &= x_1 \end{aligned} \qquad (1)$$

where the $\varphi_i(\cdot)$-functions are known, $f_i(\cdot,t)$ are uncertain time-varying functions that are bounded in any bounded compact set of their arguments. The problem is to provide for the output tracking: $y(t) \to y^c(t)$. The design problem is to achieve this tracking in multiple-loop sliding modes using full state feedback and the backstepping design.

3. MULTIPLE SLIDING SURFACE DESIGN WITH 1ST-ORDER SLIDING MODES

A multiple sliding surface implementation of the back-stepping approach looks as follows (Swaroop *et al.*, 2000).

Step 1. Define the first sliding surface

$$S_1 = y^c - y = x_1^c - x_1 = 0\,, \qquad (2)$$

and the following desired closed-loop dynamics for the sliding quantity S_1

$$\dot{S}_1 = -\gamma_1(S_1) + S_2 + \tilde{e}_1 \qquad (3)$$

where $S_2 = x_2^c - x_2$, $x_2^c(t)$ is to be defined, $\tilde{e}_1 = \left(\dot{x}_1^c - f_1(\cdot,t)\right) - \left(\hat{\dot{x}}_1^c - \hat{f}_1\right)$, $x_1^c = y^c$, and $\gamma_1(\cdot)$ is such a function that the homogeneous part of the system (3) is a finite time convergent equation.

From (1)-(3) one can formally obtain

$$\begin{aligned} \dot{S}_1 &= \dot{y}^c - x_2 - \varphi_1(\cdot) - f_1(\cdot,t) = \\ &-\gamma_1(S_1) + x_2^c - x_2 + \left(\dot{x}_1^c - f_1(\cdot,t)\right) - \left(\hat{\dot{x}}_1^c - \hat{f}_1\right) \end{aligned} \qquad (4)$$

From (4) we derive the following reference command for the next step

$$x_2^c = \hat{\dot{x}}_1^c - \hat{f}_1 - \varphi_1(x_1) + \gamma_1(S_1) \qquad (5)$$

where $\hat{\dot{x}}_1^c$, $\hat{f}_1$ are the best estimates of the reference signal and uncertainty in the first loop that are based on dynamic observers.

Step 2. The second sliding surface is defined as

$$S_2 = x_2^c - x_2 = 0\,, \qquad (6)$$

Assuming existence of (6) and finite time convergence of (3) even with $\tilde{e}_1 \neq 0$, the sliding mode on the surface (2) will be achieved.

Sliding quantity S_2 is calculated using feedback on x_2 and equation (5) for x_2^c. An alternative way to produce S_2 using S_1 feedback only can be obtained as follows. Derive the following identity from (1),(2)

$$\dot{x}_1^c - f_1 = \dot{S}_1 + x_2 + \varphi_1(x_1)\,,$$

then one can estimate

$$\hat{\dot{x}}_1^c - \hat{f}_1 = \hat{\dot{S}}_1 + x_2 + \varphi_1(x_1)\,. \qquad (7)$$

From (5),(7) we obtain

$$S_2 = \hat{\dot{S}}_1 + \gamma_1(S_1)\,. \qquad (8)$$

There are many ways to obtain (8). One can estimate the derivative and calculate the nonlinear term, one can try to estimate them both at once; at last, one can apply the nonlinear DSM approach and enforce (8) in the system motion in an auxiliary DSM.

The closed-loop dynamics for the sliding quantity S_2 is selected in the form

$$\dot{S}_2 = -\gamma_2(S_2) + S_3 + \tilde{e}_2 \qquad (9)$$

where $S_3 = x_3^c - x_3$, $x_3^c(t)$ is to be defined, $\tilde{e}_2 = \left(\dot{x}_2^c - f_2(\cdot,t)\right) - \left(\hat{\dot{x}}_2^c - \hat{f}_2\right)$, and $\gamma_2(\cdot)$ is of the same class as $\gamma_1(\cdot)$. Similar to Eq.(3.5) the reference command $x_3^c(t)$ is obtained

$$x_3^c = \hat{x}_2^c - \hat{f}_2 - \varphi_2(x_1, x_2) + \gamma_2(S_2) . \qquad (10)$$

Step 3. Finally, the third sliding surface,

$$S_3 = x_3^c - x_3 = 0 , \qquad (11)$$

is achieved under the control

$$u = -\varphi_3(\cdot) + k_{1,3}S_3 + k_{0,3}\,\mathrm{sgn}(S_3) . \qquad (12)$$

The closed-loop system in the (S_1, S_2, S_3) state space is derived as

$$\begin{aligned} \dot{S}_1 &= -\gamma_1(S_1) + S_2 + \tilde{e}_1, \\ \dot{S}_2 &= -\gamma_2(S_2) + S_3 + \tilde{e}_2, \qquad (13) \\ \dot{S}_3 &= \dot{x}_3^c - f_3(\cdot, t) - k_{1,3}S_3 - k_{0,3}\,\mathrm{sgn}(S_3). \end{aligned}$$

Dynamics (13) plus the error dynamics in the dynamic observers that govern $(\tilde{e}_1, \tilde{e}_2)$ complete the total closed-loop dynamics of the plant.
In (Swaroop *et al*., 2000), it’s proved the closed-loop stability and convergence of (13) with first-order linear dynamic observers. These observers have served as integral filters to smooth out possibly non-Lipshitz behavior of $\dot{x}_1^c, \dot{x}_2^c, \dot{x}_3^c$. That’s why it was possible to define feedback terms in each compensated dynamics as $\gamma_i(S_i) = k_{i,1}S_i + k_{i,0}\,\mathrm{sgn}(S_i), i = \overline{1,3}$. These functions provide for finite time convergence in (13), but they are not differentiable. Linear smoothing filters in (Swaroop *et al.*, 2000) overcome the problem of “explosion of terms” trading robustness for stability. However, in numerical implementations this discontinuous form is rarely used, for even with filtering the term $\mathrm{sgn}(S)$ behavior in outer loops is not good for inner tracking loops. There is no finite time convergence and the true sliding mode (Edwards and Spurgeon, 1998) in this case. Another approach is developed in this work to establish finite time convergence in each loop to a dynamic integral-type sliding surface. Additionally, the integral part specifies time scale separation (Shtessel *et al*., 1999) between the loops to ensure overall stability of a backstepping-type tracking system.

4. MULTIPLE DYNAMIC SLIDING SURFACE DESIGN WITH 2ND-ORDER SLIDING MODES

4.1 Continuous Control for Finite Reaching Time Sliding Mode

The following finite-reaching-time continuous standard-sliding-mode controllers have been developed in (Brown *et al*., 2000) for the first order σ-dynamics

$$\dot{\sigma} = f(\sigma, t) + u . \qquad (14)$$

They provide for finite-time-convergence of the first-order closed-loop σ-dynamics. One of the forms in the work [16] is given by

$$\dot{\sigma} + \rho\frac{\sigma}{|\sigma|^{0.5}} = 0 . \qquad (15)$$

In absence of uncertainty in the function $f(\sigma, t)$, the control law

$$u(\sigma) = -f(\sigma, t) - \rho\frac{\sigma}{|\sigma|^{0.5}} , \qquad (16)$$

renders the closed-loop dynamics (15), as a finite time convergent nonlinear manifold. When the function $f(\sigma, t)$ is totally uncertain, the continuous control law

$$u(\sigma) = -\rho\frac{\sigma}{|\sigma|^{0.5}} , \qquad (17)$$

provides for convergence to the arbitrarily small domain of attraction, the boundary layer, around the sliding surface $\sigma = 0$ in a standard sliding mode, where the gain ρ and the uncertainty limit L determine the boundary layer thickness. The drawbacks of this controller are that the uncertainty limit defines the boundary layer, and even in absence of uncertainty the domain of attraction to $\sigma = 0$ is proportional to the discrete interval T under the discrete-time control (first-order sliding accuracy). An additional problem for a mupltiple loop system is that (17) is not smooth enough to be r times differentiable.

4.2 Conditions on Smoothness of a Virtual Control

In order for the control of form (17) to be a virtual control to be followed by inner cascades of total order r of a multi-loop system, it has to be r times continuously differentiable. In (Brown *et al*., 2000), finite time convergence has been proved for the closed-loop dynamics

$$\dot{\sigma} + \rho|\sigma|^{\alpha}\,\mathrm{sgn}(\sigma) = 0 , \qquad (18)$$

where $\alpha \in (0,1)$. Similar to (17), it gives us the virtual control in the form

$$u_{(1)} = -\rho|\sigma|^{\alpha}\,\mathrm{sgn}(\sigma) . \qquad (19)$$

For $u_{(1)}$ to be followed by a first order tracking system is has to be one time continuously differentiable. From (18),(19) we have

$$\dot{u}_{(1)} = -\rho^2\alpha|\sigma|^{2\alpha-1}\,\mathrm{sgn}(\sigma) . \qquad (20)$$

Eq. (3.20) gives the following condition on smoothness of the virtual control at the origin

$$\alpha > \tfrac{1}{2} .$$

In case of an r^{th} order tracking system, we have

$$u^{(r)}{}_{(r)} \sim -|\sigma|^{(r+1)\alpha - r}\,\mathrm{sgn}(\sigma) . \qquad (21)$$

Finally, combining conditions of finite time convergence of the compensated dynamics and smoothness of the virtual control, we obtain the following condition for the r^{th} order tracking system.
Condition on Smoothness. Terminal sliding mode can be enforced on the surface (18) by an r^{th} order tracking system in all loops if

$$\frac{r}{r+1} < \alpha < 1 . \qquad (22)$$

Applying this condition to system (1) we must have $\tfrac{2}{3} < \alpha < 1$ if we select $\gamma_1(S_1) = |S_1|^{\alpha}\,\mathrm{sgn}(S_1)$ for

(3). One should not worry about overall stability of system (3.13) and convergence rate of the estimation error $(\tilde{e}_1, \tilde{e}_2)$-dynamics if time scale is applied in each loop. Multiple time scale is achieved in dynamic sliding surfaces designed in the next section. Second order sliding modes are established on dynamic sliding surfaces that govern compensated time scaled dynamics in each loop.

4.3 Multiple Time Scale Dynamic Sliding Surfaces

The first (outer) loop design Consider the tracking problem for system (1). We introduce the following dynamic sliding surface for the compensated dynamics in the outer loop of the expected 3-loop system

$$\begin{aligned}&\sigma_1 + \rho_1 \int |\sigma_1|^{\beta_{1,1}} \operatorname{sgn}(\sigma_1) dt = \\ &e_1 + c_1 \int |e_1|^{\alpha_1} \operatorname{sgn}(e_1) dt\end{aligned}, \qquad (23)$$

where e_1 is the input and the dynamic sliding surface quantity σ_1 is the output. If the sliding mode exists on the surface $\sigma_1 = 0$, then from (23) we obtain in sliding mode

$$e_1 + c_1 \int |e_1|^{\alpha_1} \operatorname{sgn}(e_1) dt = const, \ \sigma_1 = 0,$$

or

$$\dot{e}_1 = -c_1 |e_1|^{\alpha_1} \operatorname{sgn}(e_1), \qquad (24)$$

which is a finite time convergent system, where the terminal time is a function of $e_1(0), \alpha_1, c_1$.

To enforce convergence to $\sigma_1 = 0$, we consider σ_1-dynamics using Eq.(1) and introducing the virtual control x_2^c, and the tracking error $e_2 = x_2^c - x_2$ of the inner loop

$$\begin{aligned}&\dot{\sigma}_1 + \rho_1 |\sigma_1|^{\beta_{1,1}} \operatorname{sgn}(\sigma_1) = \dot{y}_c - \varphi_1(x_1) - \\ &f_1(.,t) - (x_2^c - e_2) + c_1 |e_1|^{\alpha_1} \operatorname{sgn}(e_1)\end{aligned}. \qquad (25)$$

The virtual control x_2^c is designed as

$$\begin{aligned}&x_2^c = \hat{\dot{y}}_c - \varphi_1(x_1) - \hat{f}_1(.,t) + \\ &c_1 |e_1|^{\alpha_1} \operatorname{sgn}(e_1) + \rho_0 \int |\sigma_1|^{\beta_{1,0}} \operatorname{sgn}(\sigma_1) dt\end{aligned}, \qquad (26)$$

Given (26) and estimation error $\hat{e}_1 = (\dot{y}_c - f_1(.,t)) - (\hat{\dot{y}}_c - \hat{f}_1(.,t))$, the outer loop σ_1-dynamics closed under control (26) is obtained

$$\begin{aligned}&\dot{\sigma}_1 + \rho_1 |\sigma_1|^{\beta_{1,1}} \operatorname{sgn}(\sigma_1) + \\ &\rho_0 \int |\sigma_1|^{\beta_{1,0}} \operatorname{sgn}(\sigma_1) dt = \hat{e}_1 + e_2\end{aligned}. \qquad (27)$$

Assuming exact estimation, i.e., $\hat{e}_1 = 0$, and fast convergence of e_2 to zero (time scale of the inner loop dynamics), we have the second order σ_1-dynamics

$$\ddot{\sigma}_1 + \rho_1 \frac{\dot{\sigma}_1}{|\sigma_1|^{1-\beta_{1,1}} \operatorname{sgn}(\sigma_1)} + \rho_0 |\sigma_1|^{\beta_{1,0}} \operatorname{sgn}(\sigma_1) = 0, \qquad (28)$$

which is a finite time convergent differential equation given conditions formulated in the following lemma.

<u>Lemma 1</u> Consider the following σ-dynamics

$$\dot{\sigma} = -\alpha_1 |\sigma|^{1/2} \operatorname{sgn}(\sigma) - \alpha_0 \int |\sigma|^{1/3} \operatorname{sgn}(\sigma) d\tau, \qquad (29)$$

$\alpha_1 > 0, \alpha_0 > 0$, which can be equivalently presented by the system of two first-order equations

$$\begin{cases}\dot{x}_1 = x_2 - \alpha_1 |x_1|^{1/2} \operatorname{sgn}(x_1), \\ \dot{x}_2 = -\alpha_0 |x_1|^{1/3} \operatorname{sgn}(x_1),\end{cases} \qquad (30)$$

where $x_1 = \sigma$. The system (29) is asymptotically stable.

<u>Proof</u>: It's not difficult to prove asymptotic stability of system (29), let a Liapunov function candidate be

$$V(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} \alpha_0 |z|^{1/3} \operatorname{sgn}(z) dz,$$

$V(x) > 0$, if $x \in \Re^2 \setminus \{0\}$, then the Liapunov function derivative will be

$$\dot{V} = \frac{\partial V}{\partial x} \cdot \begin{bmatrix} x_2 - \alpha_1 |x_1|^{1/2} \operatorname{sgn}(x_1) \\ -\alpha_0 |x_1|^{1/3} \operatorname{sgn}(x_1) \end{bmatrix} = -\alpha_0 \alpha_1 |x_1|^{\frac{1}{3} + \frac{1}{2}} < 0,$$

if $x \in \Re^2 \setminus \{0\}$.

Moreover, system (29) approaches the origin in a finite time. However, in order to prove the finite time convergence to the origin of system (29), one has to apply special topics of Liapunov analysis of the finite time convergent differential equations.

To avoid explosion of terms, the virtual control (26), must be continuously differentiable twice. This gives us design condition on α_1

$$\tfrac{2}{3} < \alpha_1 < 1, \qquad (31)$$

and the following condition must be checked for $\beta_{1,0}$. For the second derivative of x_2^c be bounded, i.e. since $\ddot{x}_2^c \sim \frac{\dot{\sigma}_1}{|\sigma_1|^{1-\beta_{1,0}}}$, we must have $\left| \frac{\dot{\sigma}_1}{|\sigma_1|^{1-\beta_{1,0}}} \right| < \infty$. From (28) we obtain that

$$\dot{\sigma}_1 \sim \sigma_1^{1-\beta_{1,1}} \sigma_1^{\beta_{1,0}},$$

then

$$\ddot{x}_2^c \sim \frac{\dot{\sigma}_1}{|\sigma_1|^{1-\beta_{1,0}}} \sim \frac{\sigma_1^{1-\beta_{1,1}} \sigma_1^{\beta_{1,0}}}{|\sigma_1|^{1-\beta_{1,0}}} \sim \sigma_1^{2\beta_{1,0} - \beta_{1,1}}.$$

So, to avoid explosion of terms one should have $2\beta_{1,0} - \beta_{1,1} \geq 0$. If $\beta_{1,1} = 1/2, \beta_{1,0} = 1/3$, this condition is satisfied.

The second loop design Dynamics of the tracking error $e_2 = x_2^c - x_2$ is to be enforced in the second loop on the dynamic sliding surface

$$\sigma_2 + \rho_1 \int |\sigma_2|^{\beta_{2,1}} \operatorname{sgn}(\sigma_2) dt = e_2 + c_2 \int |e_2|^{\alpha_2} \operatorname{sgn}(e_2) dt \quad (32)$$

which is similar to the first one (23) with one difference, it must be faster enough to enforce sufficient time scale to consider $e_2 = 0$ in the outer loop.

To enforce convergence to $\sigma_2 = 0$, we consider σ_2-dynamics using Eq.(1) and introducing the virtual control x_3^c, and the tracking error $e_3 = x_3^c - x_3$ of the inner loop

$$\dot{\sigma}_2 + \rho_{2,1} |\sigma_2|^{\beta_{2,1}} \operatorname{sgn}(\sigma_2) = \dot{x}_2^c - \varphi_2(x_1, x_2) - f_2(.,t) - (x_3^c - e_3) + c_2 |e_2|^{\alpha_2} \operatorname{sgn}(e_2) \quad (33)$$

The virtual control x_3^c is designed as

$$x_3^c = \hat{\dot{x}}_2^c - \varphi_2(x_1, x_2) - \hat{f}_2(.,t) + c_2 |e_2|^{\alpha_2} \operatorname{sgn}(e_2) + \rho_{2,0} \int |\sigma_2|^{\beta_{2,0}} \operatorname{sgn}(\sigma_2) dt \quad (34)$$

Given (33) and estimation error $\hat{e}_2 = (\dot{x}_2^c - f_2(.,t)) - (\hat{\dot{x}}_2^c - \hat{f}_2(.,t))$, the second loop σ_2-dynamics closed under control (34) is obtained

$$\dot{\sigma}_2 + \rho_{2,1} |\sigma_2|^{\beta_{2,1}} \operatorname{sgn}(\sigma_2) + \rho_{2,0} \int |\sigma_2|^{\beta_{2,0}} \operatorname{sgn}(\sigma_2) dt = \hat{e}_2 + e_3 \quad (35)$$

Assuming exact estimation, i.e., $\hat{e}_2 = 0$, and fast convergence of e_3 to zero (time scale of the inner loop dynamics), we have the second order finite time convergent σ_2-dynamics.

The third loop design The very inner loop is to be designed for actual control u to stabilize the tracking error e_3-dynamics

$$\dot{e}_3 = \dot{x}_3^c - \varphi_3(x_1, x_2, x_3) - f_3(.,t) - u . \quad (36)$$

The sliding mode control is designed in a standard way (Utkin, 1992)

$$u = \varphi_3(x_1, x_2, x_3) + \rho \operatorname{sgn}(e_3), \quad \rho > |\dot{x}_3^c| + |f_3| . \quad (37)$$

Additional requirements must exist for α_2 in order for $\dot{x}_3^c$ to be bounded. From the obtained condition on smoothness we have $\frac{1}{2} < \alpha_2 < 1$. The closed loop dynamics in the third loop,

$$\dot{e}_3 = \dot{x}_3^c - f_3(.,t) - \rho \operatorname{sgn}(e_3) . \quad (38)$$

is a finite time convergent system.

5. SIMULATION EXAMPLE

To illustrate the disturbance cancellation characteristics of the developed method, we consider the simplified model of a ballistic interceptor missile. A simplified numerical model in the pitch plane is given by (Shtessel *et. al*, 1998)

$$\dot{v}_x = -\frac{1}{70}\left((u_a + 1000)\sin\theta + \psi_x\right),$$
$$\dot{v}_z = \frac{1}{70}\left((u_a + 1000)\cos\theta + \psi_z\right) - 9.81,$$
$$\dot{\theta} = q,$$
$$\dot{q} = -0.1(u_a + \psi_a),$$
$$\alpha = \theta - \gamma,$$
$$\gamma = \tan^{-1}\left(\frac{v_z}{v_x}\right),$$
$$V = \sqrt{v_x^2 + v_z^2},$$
$$\psi_x = -3 \cdot 10^{-4} V^2 \alpha \sin\gamma, \; \psi_z = 3 \cdot 10^{-4} V^2 \alpha \cos\gamma,$$
$$v_x(0) = 2000 m/s, v_z(0) = 0 m/s, \theta(0) = 0.3 rad,$$
$$q(0) = 0 rad/s.$$

The nomenclature is as in (Shtessel *et. al*, 1998).
The plant output to be stabilized is angle of attack, α. The attitude control trust is actuated via a first order actuator

$$\dot{u}_a = -20(u_a - u),$$

so the actual control signal is *u*, and we have a third order input-output dynamics

$$\dot{\alpha} = q - \frac{A_n}{V},$$
$$\dot{q} = -0.1(u_a + \psi_a),$$
$$\dot{u}_a = -20(u_a - u),$$

where we consider $A_n = -\dot{v}_x \sin\gamma + \dot{v}_z \cos\gamma$ and ψ_a as disturbances in the first and second loops of a 3-loop control system to be designed.
The stabilizing controller is designed as follows. The first dynamic sliding surface is selected as

$$\sigma_1 + 0.5 \int |\sigma_1|^{1/2} \operatorname{sgn}(\sigma_1) dt = \alpha + 0.05 \int |\alpha|^{2/3} \operatorname{sgn}(\alpha) dt$$

Then, σ_1-dynamics are identified, introducing $e_q = q_c - q$,

$$\dot{\sigma}_1 = -0.5|\sigma_1|^{1/2} \operatorname{sgn}(\sigma_1) + 0.05|\alpha|^{2/3} \operatorname{sgn}(\alpha) + q_c - e_q - \frac{A_n}{V}.$$

Virtual control in the first loop is designed

$$q_c = -0.05|\alpha|^{2/3} \operatorname{sgn}(\alpha) - 1.5 \int |\sigma_1|^{1/3} \operatorname{sgn}(\sigma_1) dt .$$

The second dynamic sliding surface with appropriate time scale is selected as

$$\sigma_2 + 3 \int |\sigma_2|^{1/2} \operatorname{sgn}(\sigma_2) dt = e_q + 0.5 \int |e_q|^{1/2} \operatorname{sgn}(e_q) dt$$

Then, σ_2-dynamics are identified, introducing $e_u = u_{ac} - u_a$,

$$\dot{\sigma}_2 = -3|\sigma_2|^{1/2} \operatorname{sgn}(\sigma_2) + 0.5|e_q|^{1/2} \operatorname{sgn}(e_q) + \dot{q}_c + 0.1(u_{ac} - e_u) - 0.1\psi_a .$$

Virtual control in the second loop is designed

$$u_{ac} = 10 \cdot \left(-\dot{q}_c - 0.5|e_q|^{1/2} \operatorname{sgn}(e_q) - 10 \int |\sigma_2|^{1/3} \operatorname{sgn}(\sigma_2) dt \right)$$

Tracking error dynamics in the third loop are identified

$$\dot{e}_u = \dot{u}_{ac} + 20 u_a - 20 u .$$

Actual discontinuous control is designed

$$u = 20 \operatorname{sgn}(e_u) .$$

Finite time convergent multiple sliding surfaces dynamics are obtained

$$\dot{\sigma}_1 + 0.5|\sigma_1|^{1/2}\operatorname{sgn}(\sigma_1) + 1.5\int|\sigma_1|^{1/3}\operatorname{sgn}(\sigma_1)dt = -e_q - \frac{A_n}{V},$$

$$\dot{\sigma}_2 + 3|\sigma_2|^{1/2}\operatorname{sgn}(\sigma_2) + 10\int|\sigma_2|^{1/3}\operatorname{sgn}(\sigma_2)dt = -e_u - 0.1\psi_a,$$

$$\dot{e}_u = \dot{u}_{ac} + 20u_a - 400\operatorname{sgn}(e_u).$$

Analyzing the last system, one can conclude that e_u reaches zero in a finite time robustly to bounded behavior of $\dot{u}_{ac}$, u_a. When $e_u = 0$, σ_2 approaches in a finite time a small domain around zero attenuating the disturbance ψ_a. When $\sigma_2 \approx 0$, e_q goes to zero in a finite time according to $\dot{e}_q = -0.5|e_q|^{0.5}\operatorname{sgn}(e_q)$. When $e_q \approx 0$, σ_1 approaches in a finite time a small domain around zero attenuating the disturbance $\frac{A_n}{V}$. When $\sigma_1 \approx 0$, α goes to zero in a finite time according to $\dot{\alpha} = -0.05|\alpha|^{2/3}\operatorname{sgn}(\alpha)$. Results of a simulation are given in Figs.1-4.

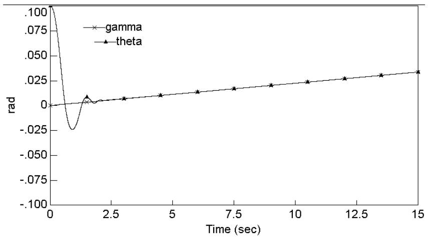

Fig.1 γ, θ versus time

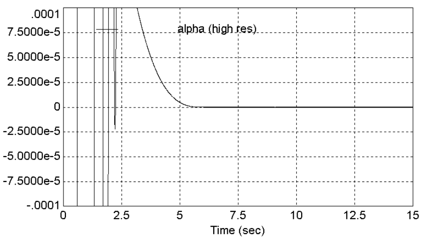

Fig.2 angle of attack vs.time

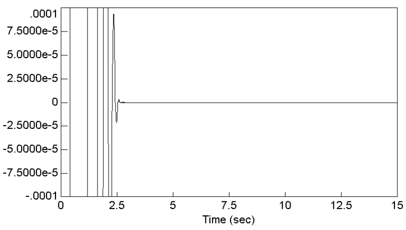

Fig.3 σ_1 versus time

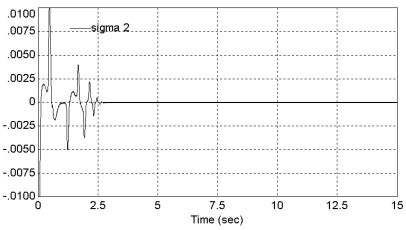

Fig.4 σ_2 versus time

CONCLUSIONS

A three-loop tracking control system has been designed for a third order uncertain SISO system. Tracking error dynamics is enforced in each loop in terminal dynamic sliding surfaces. The second order sliding performance is provided for sliding modes on dynamic sliding surfaces in the outer and the inner loops under virtual controls. Time scale separation ensures overall stability of the system. When $\hat{e}_1 = \hat{e}_2 = 0$, exact tracking $y = y_c$ is achieved in a finite time. A signal differentiator and a disturbance observer must accompany the presented design to ensure $\hat{e}_1 \to 0, \hat{e}_2 \to 0$ as close as possible.

REFERENCES

Bartolini, G., A. Ferrara and E. Usai (1998). Chattering avoidance by second-order sliding mode control. *IEEE Transactions on Automatic Control*, **43**, 241-246.

Brown, M. D., Y. B. Shtessel and J. M. Buffington (2000). Finite reaching time continuous sliding mode control with enhanced robustness. In: *Proceedings of AIAA Guidance, Navigation and Control Conference*, (Denver, CO, August 14-18), Paper 2000-3964.

Edwards C., and S. K. Spurgeon (1998). *Sliding Mode Control: Theory and Applications*. Taylor and Francis, London.

Francis, B. A. and W. M. Wonhan (1976). The internal model principle of control theory. *Automatica*, **12**, 457-465.

Kachroo, P. (1999). Existence of solutions to a class of nonlinear convergent chattering-free sliding mode control systems. *IEEE Transactions on Automatic Control*, **44**, 1620-1624.

Levant, A. (1998). Robust exact differentiation via sliding mode technique. *Automatica*, **34**, 379-384.

Slotine, J.-J. E. and W. Li (1991). *Applied Nonlinear Control*. Prentice Hall, Englewood Cliffs.

Shtessel, Y., M. Brown, K. Moore, R. Toomey and K. Cook (1998). Sliding mode controller design for kinetic energy kill vehicles. In: *Proc. 7th Annual AIAA/BMDO Technology Readiness Conference*, (Ft. Carson, Colorado, August 3-6), Paper AIAA-09-05.

Shtessel, Y., J. Buffington, and S. Banda (1999). Multiple time scale flight control using re-configurable sliding modes. *AIAA Journal on Guidance, Control, and Dynamics*, **22**, 873-883.

Su, W.-C., S. V. Drakunov and Ü. Özgüner (2000). An O(T^2) boundary layer in sliding mode for sampled-data systems. *IEEE Transactions on Automatic Control*, **45**, 482-485.

Swaroop, D., J. K Hedrick, P. P Yip and J. C. Gerdes (2000). Dynamic surface control for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, **45**, 1893-1899.

Utkin, V. I. (1992). Sliding Modes in Control Optimization. Springer, Berlin.

Zarchan, P. (1990) Tactical and Strategic Missile Guidance. Progress in Astronautics and Aeronautics, Vol. 124, AIAA.

Zinober, A. S. I., J. C. Scarratt, A. Ferrara, L. Giacomini and M. Rios-Bolívar (1999). In: *Proc. of 7th Mediterranean Conference on Control and Automation*, (Haifa, Israel, June 28-30), pp. 2360-2375.