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PERTURBATION ANALYSIS OF COUPLED MATRIX RICCATI EQUATIONS

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Abstract: Local and non-local perturbation bounds for real continuous-time coupled algebraic matrix Riccati equations are derived using the technique of Lyapunov majorants and fixed point principles. Equations of this type arise in the robust analysis and design of linear control systems.

Keywords: Perturbation analysis, Riccati equations, Robust control

1. INTRODUCTION AND NOTATION

In this paper we present a complete perturbation analysis of real continuous-time coupled algebraic matrix Riccati equations (CAMRE) of the form $F_i(X_1, X_2, P_i) = 0,\ i = 1, 2$, where F_i are matrix quadratic functions in the unknown matrices X_i, and P_i are collections of matrix coefficients.

Throughout the paper we use the following notation: $R^{m\times n}$ – the space of $m \times n$ real matrices; $R^m = R^{m\times 1}$; $R_+ = [0, \infty)$; $A^\top$ – the transpose of the matrix A; $\preceq$ – the component-wise order relation on $R^{m\times n}$; $\mathrm{vec}(A)$ – the column-wise vector representation of the matrix A; $\mathrm{Mat}(\mathbf{L}) \in R^{pq\times mn}$ – the matrix representation of the linear matrix operator $\mathbf{L} : R^{m\times n} \to R^{p\times q}$; I_n – the unit $n \times n$ matrix; Π_{n^2} – the $n^2 \times n^2$ vec-permutation matrix such that $\mathrm{vec}(A^\top) = \Pi_{n^2}\mathrm{vec}(A)$ for all $A \in R^{n\times n}$; $A \otimes B$ – the Kronecker product of the matrices A and B; $\|\cdot\|_2$ – the Euclidean norm in R^m or the spectral (or 2-) norm in $R^{m\times n}$; $\|\cdot\|_{\mathrm{F}}$ – the Frobenius (or F-) norm in $R^{m\times n}$; $\|\cdot\|$ – a replacement of either $\|\cdot\|_2$ or $\|\cdot\|_{\mathrm{F}}$; $\mathrm{rad}(A)$ – the spectral radius of the square matrix A; $\det(A)$ – the determinant of the square matrix A; $\|P\| = [\|E_1\|, \ldots, \|E_r\|]^\top \in R_+^r$ is the generalized norm of P, when $P = (E_1, \ldots, E_r)$ is a collection of r matrices.

We also denote $\mathcal{R} = R^{n\times n}$ and $\mathcal{S} = \{A \in \mathcal{R} : A = A^\top\} \subset \mathcal{R}$; $\mathcal{S}_+ = \{A \in \mathcal{S} : A \geq 0\}$; $\mathbf{Lin}(\mathcal{L}_1, \mathcal{L}_2)$ – the space of linear operators $\mathcal{L}_1 \to \mathcal{L}_2$, where $\mathcal{L}_1$, $\mathcal{L}_2$ are linear spaces. We also use the abbreviation $\mathbf{Lin} = \mathbf{Lin}(\mathcal{R}, \mathcal{R})$.

We identify the Cartesian product $R^{m\times n} \times R^{m\times n}$, endowed with the structure of a linear space, with any of the spaces $R^{m\times 2n}$, $R^{2m\times n}$ and R^{2mn}. In particular, the ordered pair $(A, B) \in R^{m\times n} \times R^{m\times n}$ and the matrix $[A, B] \in R^{m\times 2n}$ are considered as identical objects. Finally, we use the same notation P for an ordered matrix r-tuple $(E_1, \ldots, E_r)$ as well as for the collection $\{E_1, \ldots, E_r\}$.

2. PROBLEM STATEMENT

Consider the system of real continuous-time CAMRE arising in the robust control of linear time-invariant systems (see e.g. [1])

$$F_1(X_1, X_2, P_1) := (A_1 + B_1X_2)^\top X_1 \qquad (1)$$
$$+ X_1(A_1 + B_1X_2) + C_1 - X_1D_1X_1 = 0,$$

$$F_2(X_1, X_2, P_2) := (A_2 + X_1B_2)X_2$$
$$+ X_2(A_2 + X_1B_2)^\top + C_2 - X_2D_2X_2 = 0,$$

where $X_i \in \mathcal{R}$ are the unknown matrices, $A_i, B_i \in \mathcal{R}$, $C_i, D_i \in \mathcal{S}$, are given matrix coefficients and $P_i := (A_i, B_i, C_i, D_i) \in \mathcal{R}^4$.

We set

$$P := (P_1, P_2) = (A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2)$$
$$=: (E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8) \in \mathcal{R}^8.$$

The *generalized norm* of the matrix 8-tuple P is the vector $\|P\| := [\|E_1\|_{\mathrm{F}}, \ldots, \|E_8\|_{\mathrm{F}}]^\top \in R^8_+$.

In this work we are interested only in symmetric solutions of system (1).

The solution pair $(X_1, X_2) \in \mathcal{S}^2$ is called *stabilizing* if the matrices $G_1 := A_1 + B_1X_2 - D_1X_1$ and $G_2 := A_2 + X_1B_2 - X_2D_2$ are stable.

Note that F_i as defined by (1) are functions from $\mathcal{R} \times \mathcal{R} \times \mathcal{R}^4 = \mathcal{R}^6$ to $\mathcal{R}$. It will be convenient to write the system of CAMRE as one matrix equation. For this purpose we denote

$$X := (X_1, X_2),\ F := (F_1, F_2).$$

Then the system (1) may be written as

$$F(X, P) = 0. \qquad (2)$$

Here F is considered as a mapping $\mathcal{R}^{10} \to \mathcal{R}^2$, or equivalently, as a mapping $R^{n\times 2n} \times \mathcal{R}^8 \to R^{n\times 2n}$.

We assume that system (1) has a solution $X = (X_1, X_2) \in \mathcal{S}^2$ such that the partial Fréchet derivative $F_X(X, P)(\cdot)$ of F in X at the point (X, P) is invertible.

A direct calculation gives

$$\begin{aligned}
F_{1,X_1}(X, P_1)(Z) &= G_1^\top Z + ZG_1, \\
F_{1,X_2}(X, P_1)(Z) &= X_1B_1Z + Z^\top B_1^\top X_1, \qquad (3) \\
F_{2,X_1}(X, P_2)(Z) &= X_2B_2^\top Z^\top + ZB_2X_2, \\
F_{2,X_2}(X, P_2)(Z) &= G_2Z + ZG_2^\top.
\end{aligned}$$

Further on we set

$$\begin{aligned}
\mathbf{L}(\cdot) &:= F_X(X, P)(\cdot) \in \mathbf{Lin}(\mathcal{R}^2, \mathcal{R}^2), \\
\mathbf{L}_i(\cdot) &:= F_{i,X}(X, P_i)(\cdot) \in \mathbf{Lin}(\mathcal{R}^2, \mathcal{R}), \\
\mathbf{L}_{ij}(\cdot) &:= F_{i,X_j}(X, P_i)(\cdot) \in \mathbf{Lin}(\mathcal{R}, \mathcal{R}).
\end{aligned}$$

Thus

$$F_X(X, P)(Y) = (\mathbf{L}_1(Y), \mathbf{L}_2(Y))$$
$$= (\mathbf{L}_{11}(Y_1) + \mathbf{L}_{12}(Y_2), \mathbf{L}_{21}(Y_1) + \mathbf{L}_{22}(Y_2)).$$

Applying the vec operation to $F_X(X, P)(Y)$ and using the equality $(A \otimes B)\Pi_{n^2} = \Pi_{n^2}(B \otimes A)$ we find the matrix representation of the operator $\mathbf{L}(\cdot)$

$$L := \mathrm{Mat}(\mathbf{L}(\cdot)) = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \in R^{2n^2\times 2n^2},$$

where

$$\begin{aligned}
L_{11} &:= I_n \otimes G_1^\top + G_1^\top \otimes I_n, \\
L_{12} &:= (I_{n^2} + \Pi_{n^2})(I_n \otimes (X_1B_1)), \\
L_{21} &:= (I_{n^2} + \Pi_{n^2})\left((B_2X_2)^\top \otimes I_n\right), \\
L_{22} &:= I_n \otimes G_2 + G_2 \otimes I_n.
\end{aligned}$$

Here $L_{ij} \in R^{n^2\times n^2}$ is the matrix representation of the operator $\mathbf{L}_{ij}(\cdot)$, $i, j = 1, 2$.

Let the matrices from P_i be perturbed as $A_i \mapsto A_i + \delta A_i$, etc. We assume that the perturbations δC_i and δD_i are symmetric in order to ensure that the perturbed equation also has a solution in $\mathcal{S}^2$. Symmetric perturbations in C_i and D_i arise naturally in many applications.

Denote by $P_i + \delta P_i$ the perturbed collection P_i, in which each matrix $Z \in P_i$ is replaced by $Z + \delta Z$ and let $\delta P = (\delta P_1, \delta P_2)$. Then the perturbed version of equation (2) is

$$F(X + \delta X, P + \delta P) = 0. \qquad (4)$$

Equation (4) has a unique isolated solution $Y = X + \delta X \in \mathcal{S}^2$ in the neighbourhood of X if the perturbation δP is sufficiently small.

Denote by

$$\delta := \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \in R^8_+,$$

where $\delta_i := [\delta_{A_i}, \delta_{B_i}, \delta_{C_i}, \delta_{D_i}]^\top \in R^4_+$, the vector of absolute Frobenius norm perturbations $\delta_Z := \|\delta Z\|_F$ in the data matrices $Z \in P$.

The perturbation problem for CAMRE (1) is to find bounds

$$\delta_{X_i} \le f_i(\delta),\ \delta \in \Omega \subset R^8_+,\ i = 1, 2, \qquad (5)$$

for the perturbations $\delta_{X_i} := \|\delta X_i\|_{\mathrm{F}}$. Here Ω is a certain set and f_i are continuous functions, non-decreasing in each of their arguments and satisfying $f_i(0) = 0$. The inclusion $\delta \in \Omega$ guarantees that the perturbed CAMRE (4) has a unique solution $Y = X + \delta X$ in a neighbourhood of the unperturbed solution X such that the elements of δX_1, δX_2 are analytic functions of the elements of the matrices δZ, $Z \in P$, provided δ is in the interior of Ω.

First order local bounds

$$\delta_{X_i} \leq \mathrm{est}_i(\delta) + O(\|\delta\|^2), \delta \to 0,\ i = 1,2, \quad (6)$$

are first derived with $\mathrm{est}_i(\delta) = O(\|\delta\|)$, $\delta \to 0$, which are then incorporated in the non-local bounds (5).

3. LOCAL PERTURBATION ANALYSIS

Consider first the conditioning of the CAMRE (1).

The perturbed equations may be written as

$$F_i(X + \delta X_i, P_i + \delta P_i) = \sum_{j=1}^{2} \mathbf{L}_{ij}(\delta X_j) + \sum_{Z \in P_i} F_{i,Z}(\delta Z) + H_i(\delta X, \delta P_i) = 0,$$

where $F_{i,Z}(\cdot) := F_{i,Z}(X, P_i)(.) \in \mathbf{Lin}$, $Z \in P_i$, are the Fréchet derivatives of $F_i(X, P_i)$ in the matrix argument Z, evaluated at the point (X, P_i). The matrix expressions $H_i(\delta X, \delta P_i) = O\left(\|[\delta X, \delta P_i]\|^2\right)$, $\delta X \to 0$, $\delta P_i \to 0$, contain second and higher order terms in δX, δP_i. In fact, for $Y = (Y_1, Y_2) \in \mathcal{S}^2$, we have

$$\begin{aligned} H_1(Y, \delta P_1) = {} & (\delta B_1 Y_2 - \delta D_1 Y_1)^\top X_1 \\ & + X_1(\delta B_1 Y_2 - \delta D_1 Y_1) \\ & + Y_1 \delta B_1 X_2 + X_2 \delta B_1^\top Y_1 \\ & - Y_1(D_1 + \delta D_1)Y_1 + Y_1 \delta A_1 + \delta A_1^\top Y_1 \\ & + Y_1(B_1 + \delta B_1)Y_2 + Y_2(B_1 + \delta B_1)^\top Y_1 \end{aligned} \quad (7)$$

and

$$\begin{aligned} H_2(Y, \delta P_2) = {} & X_2 \left(Y_1 \delta B_2 - Y_2 \delta D_2\right)^\top \\ & + \left(Y_1 \delta B_2 - Y_2 \delta D_2\right) X_2 \\ & + X_1 \delta B_2 Y_2 + Y_2 \delta B_2^\top X_1 \\ & - Y_2(D_2 + \delta D_2)Y_2 + \delta A_2 Y_2 + Y_2 \delta A_2^\top \\ & + Y_2(B_2 + \delta B_2)^\top Y_1 + Y_1(B_2 + \delta B_2)Y_2. \end{aligned} \quad (8)$$

We also have, for $(X_1, X_2) \in \mathcal{S}^2$,

$$\begin{aligned} F_{1,A_1}(Z) &= X_1 Z + Z^\top X_1, \\ F_{1,B_1}(Z) &= X_1 Z X_2 + X_2 Z^\top X_1, \\ F_{1,C_1}(Z) &= Z, \quad F_{1,D_1}(Z) = -X_1 Z X_1, \\ F_{2,A_2}(Z) &= Z X_2 + X_2 Z^\top, \\ F_{2,B_2}(Z) &= X_1 Z X_2 + X_2 Z^\top X_1, \\ F_{2,C_2}(Z) &= Z, \quad F_{2,D_2}(Z) = -X_2 Z X_2. \end{aligned}$$

The inverse operator

$$\mathbf{M}(\cdot) := \mathbf{L}(\cdot)^{-1} \in \mathbf{Lin}(\mathcal{R}^2, \mathcal{R}^2)$$

of the operator $\mathbf{L} = F_X(X, P)(\cdot)$ may be represented as $\mathbf{L}^{-1}(\cdot) = (\mathbf{M}_1(\cdot), \mathbf{M}_2(\cdot))$, where, for $Z := (Z_1, Z_2) \in \mathcal{R}^2$,

$$\mathbf{M}_i(Z) = \mathbf{M}_{i1}(Z_1) + \mathbf{M}_{i2}(Z_2),\ M_{ij}(\cdot) \in \mathbf{Lin}.$$

Hence

$$\delta X = -\mathbf{M}(W_1(\delta X, \delta P_1), W_2(\delta X, \delta P_2)), \quad (9)$$

where $W_i(Y, \delta P_i) := \sum_{Z \in P_i} F_{i,Z}(\delta Z) + H_i(Y, \delta P_i)$. In this way

$$\delta X_i = -\sum_{j=1}^{2} \mathbf{M}_{ij}(W_j(\delta X, \delta P_j)),$$

which gives

$$\begin{aligned} \delta X_i = & -\sum_{j=1}^{2} \sum_{Z \in P_j} \mathbf{M}_{ij} \circ F_{j,Z}(\delta Z) \\ & -\sum_{j=1}^{2} \mathbf{M}_{ij}(H_j(\delta X, \delta P_j)). \end{aligned} \quad (10)$$

Therefore

$$\delta_{X_i} \leq \sum_{j=1}^{2} \sum_{Z \in P_j} K_{ij,Z} \delta_Z + O(\|\delta\|^2),\ \delta \to 0,$$

where the quantity $K_{ij,Z} := \|\mathbf{M}_{ij} \circ F_{j,Z}\|_{\mathbf{Lin}}$, is the *absolute condition number* of the solution component X_i with respect to the matrix coefficient $Z \in P_j$. Here $\|.\|_{\mathbf{Lin}}$ is the induced norm in the space $\mathbf{Lin}$ of linear operators $\mathcal{R} \to \mathcal{R}$.

If $X_i \neq 0$, estimates in terms of relative perturbations are

$$\varepsilon_{X_i} \leq \sum_{j=1}^{2} \sum_{Z \in P_i} k_{ij,Z} \varepsilon_Z + O(\|\delta\|^2),\ \delta \to 0,$$

where the quantity $k_{ij,Z} := K_{ij,Z} \frac{\|Z\|_{\mathrm{F}}}{\|X_i\|_{\mathrm{F}}}$, is the *relative condition number* of the solution component X_i with respect to the matrix coefficient $0 \neq Z \in P_j$.

The calculation of the condition numbers $K_{ij,Z}$ is staightforward for the Frobenius. Denote by $L_{i,Z} \in R^{n^2 \times n^2}$ the matrix of the operator $F_{i,Z} \in \mathbf{Lin}$. We have

$$\begin{aligned} L_{1,A_1} &= (\Pi_{n^2} + I_{n^2})(I_n \otimes X_1), \\ L_{2,A_2} &= (\Pi_{n^2} + I_{n^2})(X_2 \otimes I_n), \\ L_{i,B_i} &= (\Pi_{n^2} + I_{n^2})(X_2 \otimes X_1), \\ L_{1,C_1} &= I_{n^2},\ \ L_{2,C_2} = I_{n^2}, \\ L_{1,D_1} &= -X_1 \otimes X_1,\ \ L_{2,D_2} = -X_2 \otimes X_2. \end{aligned}$$

Let the matrix representation of the operator

$$\mathbf{M}(\cdot) = F_X^{-1}(X, P)(\cdot) \in \mathbf{Lin}(\mathcal{R}^2, \mathcal{R}^2)$$

be denoted as

$$M := \mathrm{Mat}(\mathbf{M}) = L^{-1} := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

$$M_{ij} \in R^{n^2 \times n^2}.$$

Having in mind the expressions for $L_{i,Z}$ the absolute condition numbers are calculated from

$$K_{ij,Z} = \|M_{ij}L_{j,Z}\|_2,\ Z \in P_j,\ i,j = 1,2.$$

Rewrite equations (10) in vectorized form as

$$\mathrm{vec}(\delta X_i) = \sum_{j=1}^{2}\sum_{Z\in P_j} N_{i,Z}\mathrm{vec}(\delta Z) \qquad (11)$$
$$-\sum_{j=1}^{2} M_{ij}\mathrm{vec}(H_j(\delta X, \delta P_j)),$$

where

$$N_{i,Z} := -M_{ij}L_{j,Z} \in R^{n^2\times n^2},\ Z \in P_j.$$

The condition number based perturbation bounds may be derived as an immediate consequence of (11),

$$\delta_{X_i} = \|\delta X_i\|_{\mathrm{F}} = \|\mathrm{vec}(\delta X_i)\|_2 \le \mathrm{est}_i^{(1)}(\delta) + O(\|\delta\|^2),$$
$$\mathrm{est}_i^{(1)}(\delta) := \sum_{j=1}^{2}\sum_{Z\in P_j} \|N_{i,Z}\|_2\delta_Z.$$

Relations (11) also give a second perturbation bound

$\delta_{X_i} \le \mathrm{est}_i^{(2)}(\delta) + O(\|\delta\|^2),\ \mathrm{est}_i^{(2)}(\delta) := \|N_i\|_2\|\delta\|_2$

where

$$N_i := [N_{i,1}, N_{i,2}] \in R^{n^2\times 8n^2},$$
$$N_{i,j} := [N_{i,A_j}, N_{i,B_j}, N_{i,C_j}, N_{i,D_j}] \in R^{n^2\times 4n^2}.$$

The bounds $\mathrm{est}_i^{(1)}(\delta)$ and $\mathrm{est}_i^{(2)}(\delta)$ are alternative, i.e., which one is smaller depends on the particular value of δ.

There is a third bound, which is always less or equal to $\mathrm{est}_1^{(1)}(\delta)$. Indeed, we have

$$\delta_{X_i} \le \mathrm{est}_i^{(3)}(\delta) + O(\|\delta\|^2),\ \mathrm{est}_i^{(3)}(\delta) := \sqrt{\delta^\top \widehat{N}_i\delta},$$

where $\widehat{N}_i = [n_{i,pq}] \in R_+^{8\times 8}$ is a matrtix with elements $n_{i,pq} := \left\|\widehat{N}_{i,p}^\top \widehat{N}_{i,q}\right\|_2,\ p,q = 1,\ldots,8$ where $\widehat{N}_{i,k},\ k = 1,\ldots,8$ denote the $n^2\times n^2$ blocks of N_i,i.e., $N_i = \left[\widehat{N}_{i,1}, \widehat{N}_{i,2}, \ldots, \widehat{N}_{i,8}\right]$, $\widehat{N}_{i,k} \in R^{n^2\times n^2}$, and $\widehat{N}_{i,1} := N_{i,A_1}, \widehat{N}_{i,2} := N_{i,B_1}, \ldots, \widehat{N}_{i,8} := N_{i,D_2}$.

Since $\left\|\widehat{N}_{i,p}^\top \widehat{N}_{i,q}\right\|_2 \le \left\|\widehat{N}_{i,p}\right\|_2 \left\|\widehat{N}_{i,q}\right\|_2$ then $\mathrm{est}_i^{(3)}(\delta) \le \mathrm{est}_i^{(1)}(\delta)$ and we have the overall estimates

$$\delta_{X_i} = \mathrm{est}_i(\delta) + O(\|\delta\|^2), \qquad (12)$$

where

$$\mathrm{est}_i(\delta) := \min\left\{\mathrm{est}_i^{(2)}(\delta), \mathrm{est}_i^{(3)}(\delta)\right\}. \qquad (13)$$

The local bounds are continuous, first order homogeneous, non-linear functions in δ.

The bounds $\mathrm{est}_i^{(k)}$ are majorants for the solution of a complicated optimization problem, defining the conditioning of the problem as follows. Set $\xi_i := \mathrm{vec}(\delta X_i),\ i = 1,2$ and $\delta := [\delta_1,\ldots,\delta_8]^\top := [\delta_{A_1},\ldots,\delta_{D_2}]^\top \in R_+^8$. Then we have

$$\xi_i = \sum_{k=1}^{8} \widehat{N}_{i,k}\eta_k + O(\|\delta\|^2),\ \delta \to 0$$

and $\delta_{X_i} = \|\xi_i\|_2 \le K_i(\delta) + O(\|\delta\|^2),\ \delta \to 0$. Here

$$K_i(\delta) := \max\left\{\left\|\sum_{k=1}^{8}\widehat{N}_{i,k}\eta_k\right\|_2 : \|\eta_k\| \le \delta_k,\right.$$
$$\left. k = 1,\ldots,8\right\}$$

is the exact upper bound for the first order term in the perturbation bound for the solution component X_i (note that $K_i(\delta)$ is well defined, since the minimization in η is carried out over a compact set). The calculation of $K_i(\delta)$ is a difficult task. Instead, one can use a bound above such as $\mathrm{est}_i(\delta) \ge K_i(\delta)$.

Let $\gamma \in R_+^8$ be a given vector and let $X_i \ne 0$. Then the relative condition number of X_i with respect to γ is $\kappa_i(\gamma) := K_i(\gamma)/\|X_i\|_{\mathrm{F}}$. If $|||P|||$ is the generalized norm of P, then $\kappa_i(|||P|||)$ is the *relative norm-wise condition number* of X_i.

4. NON-LOCAL PERTURBATION ANALYSIS

Local bounds of the type considered in Section 3 are valid only asymptoticaly, for $\delta \to 0$. But in practice they are usually used simply neglecting terms of order $O(\|\delta\|^2)$.

The disadvantages of the local estimates may be overcome using the techniques of non-linear perturbation analysis. As a result, we find bounds (5). The estimate (5) is rigorous. The perturbed equation $F(X+\delta X, P+\delta P) = 0$ may be rewritten as an operator equation for the perturbation δX

$$\delta X = \Pi(\delta X, \delta P),\ \Pi = (\Pi_1, \Pi_2), \qquad (14)$$

where $\Pi(Y, \delta P) := -\mathbf{M}(F_P(X,P)(\delta P) + H(Y, \delta P))$.

Here $H(Y,\delta P) := (H_1(Y,\delta P_1), H_2(Y,\delta P_2))$ contains second and third order terms in Y and δP, see (7), (8).

Equation (14) comprizes two equations, namely

$$\delta X_i = \Pi_i(\delta X, \delta P_i),\ i = 1,2, \qquad (15)$$

where the right-hand side of (15) is defined by relations (10). Setting

$$\xi_i := \mathrm{vec}(\delta X_i) \in R^{n^2}, \;\; \xi := \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in R^{2n^2},$$

we obtain the vector operator equation

$$\xi = \pi(\xi, \eta), \tag{16}$$

in R^{2n^2}, which is reduced to two coupled vector equations

$$\xi_i = \pi_i(\xi, \eta), \;\; i = 1, 2,$$

in R^{n^2}, respectively.

The vectorizations of the matrices $H_i(Y, \delta P_i)$ are

$$\begin{aligned} \mathrm{vec}(H_1(Y, \delta P_1)) &= (I_n \otimes X_1)\,(I_{n^2} + \Pi_{n^2}) \\ &\times \mathrm{vec}(\delta B_1 Y_2 - \delta D_1 Y_1) \\ &+ (X_2 \otimes I_n)\,(I_{n^2} + \Pi_{n^2})\,\mathrm{vec}(Y_1 \delta B_1) \\ &- \mathrm{vec}(Y_1(D_1 + \delta D_1)Y_1) + \mathrm{vec}\left(Y_1 \delta A_1 + \delta A_1^\top Y_1\right) \\ &+ \mathrm{vec}\left(Y_1(B_1 + \delta B_1)Y_2 + Y_2(B_1 + \delta B_1)^\top Y_1\right) \end{aligned} \tag{17}$$

and

$$\begin{aligned} \mathrm{vec}(H_2(Y, \delta P_2)) &= (X_2 \otimes I_n)\,(I_{n^2} + \Pi_{n^2}) \\ &\times \mathrm{vec}\,(Y_1 \delta B_2 - Y_2 \delta D_2) \\ &+ (I_n \otimes X_1)\,(I_{n^2} + \Pi_{n^2})\,\mathrm{vec}(\delta B_2 Y_2) \\ &- \mathrm{vec}(Y_2(D_2 + \delta D_2)Y_2) + \mathrm{vec}\left(\delta A_2 Y_2 + Y_2 \delta A_2^\top\right) \\ &+ \mathrm{vec}\left(Y_2(B_2 + \delta B_2)^\top Y_1 + Y_1(B_2 + \delta B_2)Y_2\right). \end{aligned} \tag{18}$$

Let $\|Y_i\|_{\mathrm{F}} \leq \rho_i, \;\; i = 1, 2,$ where ρ_i are non-negative constants. Then it follows from (17), (18) that

$$\begin{aligned} \|\pi_i(\xi, \eta)\|_2 &\leq \mathrm{est}_i(\delta) + \left\| \sum_{j=1}^{2} M_{ij} \mathrm{vec}(H_j(Y, \delta P_j)) \right\|_2 \\ &\leq \mathrm{est}_i(\delta) + \sum_{j=1}^{2} \|M_{ij} \mathrm{vec}(H_j(Y, \delta P_j))\|_2 \\ &\leq h_i(\rho, \delta), \end{aligned}$$

where

$$\rho = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \in R_+^2$$

and

$$\begin{aligned} h_i(\rho_1, \rho_2, \delta) &:= \mathrm{est}_i(\delta) + a_{i1}(\delta)\rho_1 + a_{i2}(\delta)\rho_2 \\ &\quad + 2b_i(\delta)\rho_1\rho_2 + c_{i1}(\delta)\rho_1^2 + c_{i2}(\delta)\rho_2^2. \end{aligned}$$

Here

$$\begin{aligned} a_{i1}(\delta) &:= 2\|M_{i1}\|_2 \delta_{A_1} + \nu_{i1}(\delta_{B_1} + \delta_{D_1}) + \; \nu_{i2}\delta_{B_2}, \\ a_{i2}(\delta) &:= 2\|M_{i2}\|_2 \delta_{A_2} + \nu_{i2}(\delta_{B_2} + \delta_{D_2}) + \; \nu_{i1}\delta_{B_2}, \\ b_i(\delta) &:= \|M_{i1}\|_2(\|B_1\|_2 + \delta_{B_1}) \\ &\quad + \|M_{i2}\|_2(\|B_2\|_2 + \delta_{B_2}), \\ c_{i1}(\delta) &:= \|M_{i1}\|_2(\|D_1\|_2 + \delta_{D_1}), \\ c_{i2}(\delta) &:= \|M_{i2}\|_2(\|D_2\|_2 + \delta_{D_2}), \end{aligned}$$

and

$$\begin{aligned} \nu_{i1} &:= \|M_{i1}(I_n \otimes X_1)\,(I_{n^2} + \Pi_{n^2})\|_2\,, \\ \nu_{i2} &:= \|M_{i2}(X_2 \otimes I_n)\,(I_{n^2} + \Pi_{n^2})\|_2\,. \end{aligned}$$

The function $h : R_+^2 \times R_+^8 \to R_+^2$, constructed above, is a vector Lyapunov majorant for the operator equation (16), see [2].

Consider the majorant system of two scalar quadratic equations

$$\rho_i = h_i(\rho_1, \rho_2, \delta), \;\; i = 1, 2, \tag{19}$$

which may also be written in vector form as $\rho = h(\rho, \delta)$, where

$$h(\rho, \delta) := \begin{bmatrix} h_1(\rho, \delta) \\ h_2(\rho, \delta) \end{bmatrix}.$$

We have

$$h(0, \delta) = \begin{bmatrix} \mathrm{est}_1(\delta) \\ \mathrm{est}_2(\delta) \end{bmatrix}.$$

Hence $h(0,0) = 0, \;\; h_\rho(0,0) = 0$. Therefore, for δ sufficiently small, the system (19) has a solution

$$\rho = f(\delta) = \begin{bmatrix} f_1(\delta) \\ f_2(\delta)) \end{bmatrix},$$

which is continuous, real analytic in $\delta \neq 0$ and satisfies $\rho(0) = 0$. The function $f(\cdot)$ is defined in a domain $\Omega \subset R_+^8$ whose boundary $\partial\Omega$ may be obtained by excluding ρ from the system of equations

$$\rho = h(\rho, \delta), \;\; \det(I_2 - h_\rho(\rho, \delta)) = 0. \tag{20}$$

The second equation in (20) is equivalent to

$$\begin{aligned} \omega(\rho, \delta) &:= 1 - \varepsilon(\delta) + \alpha_1(\delta)\rho_1 + \alpha_2(\delta)\rho_2 \\ &\quad + 2\beta(\delta)\rho_1\rho_2 + \gamma_1(\delta)\rho_1^2 + \gamma_2(\delta)\rho_2^2 = 0, \end{aligned}$$

where

$$\begin{aligned} \varepsilon(\delta) &:= a_{11}(\delta) + a_{22}(\delta) - a_{11}(\delta)a_{22}(\delta) \\ &\quad + a_{12}(\delta)a_{21}(\delta), \\ \alpha_1(\delta) &:= -2(c_{11}(\delta)(1 - a_{22}(\delta)) + b_2(\delta)(1 - a_{11}(\delta)) \\ &\quad + a_{12}(\delta)c_{21}(\delta) + b_1(\delta)a_{21}(\delta)), \\ \alpha_2(\delta) &:= -2(c_{22}(\delta)(1 - a_{11}(\delta)) + b_1(\delta)(1 - a_{22}(\delta)) \\ &\quad + a_{21}(\delta)c_{12}(\delta) + b_2(\delta)a_{12}(\delta)), \\ \beta(\delta) &:= 4(c_{11}(\delta)c_{22}(\delta) - c_{12}(\delta)c_{21}(\delta)), \\ \gamma_1(\delta) &:= 4(b_2(\delta)c_{11}(\delta) - b_1(\delta)c_{21}(\delta)), \\ \gamma_2(\delta) &:= 4(b_1(\delta)c_{22}(\delta) - b_2(\delta)c_{12}(\delta)). \end{aligned}$$

As a result, we have the non-local non-linear perturbation bounds

$$\delta_{X_i} \leq f_i(\delta), \;\; \delta \in \Omega. \tag{21}$$

We can find a new Lyapunov majorant g, such that $h(\rho, \delta) \preceq g(\rho, \delta)$ and for which the equation

$$\rho = g(\rho, \delta) \tag{22}$$

has an explicit form solution. This can be done in many ways. Three of them are described below.

Let

$$\begin{aligned}\mathrm{est}(\delta) &:= \max\{\mathrm{est}_1(\delta), \mathrm{est}_2(\delta)\},\\ a_i(\delta) &:= \max\{a_{1i}(\delta), a_{2i}(\delta)\},\\ b(\delta) &:= \max\{b_1(\delta), b_2(\delta)\},\\ c_i(\delta) &:= \max\{c_{1i}(\delta), c_{2i}(\delta)\}.\end{aligned}$$

Hereinafter, in order to simplify the notation, we set $a_{ij} := a_{ij}(\delta)$, $a_i := a_i(\delta)$, $b = b(\delta)$, $c_i := c_i(\delta)$, $e_i := \mathrm{est}_i(\delta)$, $e := \mathrm{est}(\delta)$.

Consider the function g with components

$$\begin{aligned}&g_1(\rho,\delta) = g_2(\rho,\delta)\\ &= e + a_1\rho_1 + a_2\rho_2 + 2b\rho_1\rho_2 + c_1\rho_1^2 + c_2\rho_2^2.\end{aligned}$$

Now the majorant equation (22) has solutions with $\rho_1 = \rho_2$, where

$$e - (1 - a_1 - a_2)\rho_1 + (2b + c_1 + c_2)\rho_1^2 = 0.$$

Hence, if

$$\begin{aligned}\delta \in \Theta := \Big\{&\delta \in R_+^8 : a_1 + a_2\\ &+ \ 2\sqrt{e(2b + c_1 + c_2)} \le 1\Big\}\end{aligned}$$

then

$$\begin{aligned}\delta_{X_1}, \delta_{X_2} \le (2e)/(1 - a_1 - a_2 \quad (23)\\ + \sqrt{(1 - a_1 - a_2)^2 - 4e(2b + c_1 + c_2)}\ ).\end{aligned}$$

However, in this approach one of the bounds (23) is not asymptotically sharp unless $e_1 = e_2$. We next derive another explicit bound that is asymptotically sharp in the sense that its first order term is equal to $\mathrm{est}_i(\delta)$.

Consider the function k with components

$$k_i(\delta,\rho) := e_i + a_1\rho_1 + a_2\rho_2 + 2b\rho_1\rho_2 + c_1\rho_1^2 + c_2\rho_2^2.$$

It is easy to see that k is again a Lyapunov majorant. Since $h(\rho,\delta) \preceq k(\rho,\delta) \preceq g(\rho,\delta)$ the solution of the majorant system $\rho = k(\rho,\delta)$ will majorize the solution of the system $\rho = h(\rho,\delta)$ thus producing less sharp bounds, but will give tighter bounds than these based on the majorant g. However, this solution is easily computable. Indeed, here we have $\rho_1 = \rho_2 + e_1 - e_2$. Substituting this expression in any of the equations $\rho_i = k_i(\rho,\delta)$ we obtain quadratic equations for ρ_i. Choosing the smaller solutions, we find the bounds

$$\delta_{X_i} \le \rho_i = \frac{2\left(a_j e_j + (1 - a_j)e_i + c_j(e_1 - e_2)^2\right)}{1 - a_1 - a_2 + 2(b + c_j)(e_i - e_j) + \sqrt{d}}, \tag{24}$$

where

$$\begin{aligned}d = d(\delta) &:= (1 - a_1 - a_2 + 2(b + c_j)(e_i - e_j))^2\\ &\quad - 4(2b + c_1 + c_2)\left(a_j e_j + (1 - a_j)e_i\right.\\ &\quad + \left. c_j(e_1 - e_2)^2\right)\\ &= (1 - a_1 - a_2)^2 - 4(a_1(b + c_2)\\ &\quad + (1 - a_2)(b + c_1))e_1\\ &\quad - 4(a_2(b + c_1) + (1 - a_1)(b + c_2))e_2\\ &\quad + 4(b^2 - c_1c_2)(e_1 - e_2)^2\end{aligned}$$

and $j \ne i$. These bounds hold provided $d(\delta) \ge 0$.

5. CONCLUSIONS

In this paper we have presented a complete local and non-local perturbation analysis of coupled continuous-time matrix Riccati equations, arising in the theory of H_∞ control. The results obtained may be extended to other more general systems of matrix quadratic equations.

6. REFERENCES

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