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A GENERALIZED MARKOV STABILITY CRITERION FOR LINEAR SYSTEMS

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Abstract: This paper presents a generalized Markov stability criterion for linear systems by determining the Cauchy index in arbitrary segments. Based on the proposed result, several new algebraic criteria on Schur stability and strict aperiodicity for linear systems are obtained. *Copyright © 2002 IFAC*

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1. INTRODUCTION

Given a characteristic polynomial of a linear system

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, (a_0 > 0) \qquad (1)$$

It is said to be Hurwitz stable if the real part of all zeros of the polynomial (1) are negative; It is said to be Schur stable if all zeros of the polynomial (1) lie within unit circle. By matrix theory, Gantmacher (1964), algebraic criteria on Hurwitz stability for continuous linear systems can be directly derived from calculation of a Cauchy index of a rational fraction $R(x)$, namely $I_{-\infty}^{+\infty} R(x)$, in the real axis. For the discrete-time systems, Nour-Eldin (1971) has shown that a linear system is Schur stable if and only if the Cauchy index of a rational fraction in the interval $(-1,+1)$ equals the order of the denominator. Inspired by Nour-Eldin's work, some reduced criteria on Schur stability using combination of Markov parameters with some linear relationships of the system coefficients have been presented by Anderson et al (1976, 1990).

A relevant concept to stability of the linear systems is the strict aperiodicity. The polynomial (1) is said to show strict aperiodicity if its zeros are all real, positive and distinct in the interval $(0,1)$. Algebraic criteria on checking the strict aperiodicity have been established by Soh and Burger (1989).

This paper aims at developing a generalized Markov stability criterion through calculating the Cauchy index in arbitrary intervals. Combining Sturm theorem, Gantmacher (1964), and the proposed results, several new criteria on Schur stability and the strict aperiodicity are established.

2. MAIN RESULTS

Let $P_1(x)$ and $P_2(x)$ be two real polynomials of degree n and m respectively. Suppose $P_1(x)$ and $P_2(x)$ are coprime and the degrees $m \le n$. Then the irreducible rational fraction $P_2(x)/P_1(x)$ can be expanded in a series of decreasing power of x as follows:

$$\frac{P_2(x)}{P_1(x)} = s_{-1} + \frac{s_0}{x} + \frac{s_1}{x^2} + \cdots \qquad (2)$$

For the case of degree $m < n$, $(s_0, s_1, \cdots, s_{2n-1})$ are the Markov parameters of the polynomials pair $(P_1(x), P_2(x))$. Similarly, for the case of degree $m = n$, $(s_{-1}, s_0, \cdots, s_{2n-1})$ represent the Markov parameters. By using these Markov parameters, a Hankel matrix can be formed as follows:

$$S_n(P_1, P_2) = \begin{bmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{bmatrix} \qquad (3)$$

It is well known that $I_{-\infty}^{+\infty} P_2(x)/P_1(x) = n$ if and only if $S_n(P_1, P_2) > 0$ for the case $m < n$ and an additional condition $s_{-1} > 0$ is needed for the case $m = n$ (see Gantmacher (1964)), which means that all the roots of $P_1(x)$ and $P_2(x)$ are real and interlacing. This paper extends the above result to a general case, i.e., $I_a^b P_2(x)/P_1(x)=n$ for any real segments (a,b) in the real axis. We only consider the case $m<n$. This is because the results for the case $m=n$ can be easily obtained by making some slight modifications.

THEOREM 1 Let a and b $(a<b)$ be two real numbers. Then, $I_a^b P_2(x)/P_1(x)=n$ is equivalent to the conditions (4) and (5) below:

$$\begin{aligned} &I_{-\infty}^{+\infty}(x-a)P_2(x)/P_1(x)=n \\ &I_{-\infty}^{+\infty}(x-b)P_2(x)/P_1(x)=-n \end{aligned} \tag{4}$$

$$s_0>0, S_{n,a}(P_1,P_2)>0 \text{ and } S_{n,b}(P_1,P_2)<0 \tag{5}$$

where $S_{n,c}(P_1,P_2)\equiv S_n^*(P_1,P_2)-cS_n(P_1,P_2)$ is a Hankel matrix with a real parameter c and

$$S_n^*(P_1,P_2)=\begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ s_2 & s_3 & \cdots & s_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n-1} \end{bmatrix} \tag{6}$$

Proof. The proof is straightforward.

REMARK 1 The proposed result in Theorem 1 can be viewed as a generalization of a significant result on calculation of a Cauchy index of a rational fraction in Gantmacher (1964). If $a=-\infty$ and $b=+\infty$, condition (5) will be reduced to the condition $S_n(P_1,P_2)>0$. If $a=-\infty$, condition (5) becomes $S_n(P_1,P_2)>0$ and $S_{n,b}(P_1,P_2)<0$; If $b=+\infty$, condition (5) becomes $S_n(P_1,P_2)>0$ and $S_{n,a}(P_1,P_2)>0$. So, the condition $S_{n,b}(P_1,P_2)<0$ ($S_{n,a}(P_1,P_2)>0$) can be considered as an additional boundary condition on $I_{-\infty}^{b}P_2(x)/P_1(x)=n$ ($I_a^{+\infty}P_2(x)/P_1(x)=n$) besides the basic condition $S_n(P_1,P_2)>0$.

THEOREM 2 Without loss of generality, suppose that the coefficient of the highest order term of the polynomial $P_1(x)$ be positive, a and b be two real numbers $(a<b)$. Then, $I_a^b P_2(x)/P_1(x)=n$ if and only if

$$S_n(P_1,P_2)>0 \tag{7}$$

$$(-1)^{n+k}P_1^{(k)}(a)>0 \text{ and } P_1^{(k)}(b)>0, \tag{8}$$

for $k=0,1,\cdots,n-1$.

Proof. <u>Necessity</u>. Note that

$$I_{-\infty}^{+\infty}\frac{P_2(x)}{P_1(x)}=I_a^b\frac{P_2(x)}{P_1(x)}=n \tag{9}$$

It implies $S_n(P_1,P_2)>0$, and the polynomial $P_1(x)$ has only simple real roots in the interval (a,b). Consider the following polynomial sequence

$$P_1(x), P_1^{'}(x), \ldots, P_1^{(n)}(x) \tag{10}$$

It can be easily proved that the sequence (10) is a Sturm chain in the interval (a,b). By Sturm Theorem, we have

$$Var(a)-Var(b)=n \tag{11}$$

where $Var(x)$ represents the number variations of sign in the Sturm chain for a fixed value x.

Note that *Var(a)* and *Var(b)* are two positive integers, hence we must have both *Var(a)=n* and *Var(b)=0*, simultaneously. So the linear conditions come immediately by noticing $sign(P_1^{(n)}(x)>0$.

<u>Sufficiency</u>. Let $S_n(P_1,P_2)>0$. Then, all of the roots of $P_1(x)$ and $P_2(x)$ must be real and interlacing. Furthermore, the polynomial sequence (10) is a Sturm chain in the interval (a,b). So the linear conditions (9) imply (11), that is, excluding the possibility of real roots of $P_1(x)$ lying on the real axis outside the interval (a,b). This ensures $I_a^b P_2(x)/P_1(x)=n$. This completes the proof.

REMARK 2 From the conclusions in Theorems 1 and Theorem 2, some equivalent relationships between Markov parameters and the polynomial coefficients can be established. As $S_n(P_1,P_2)>0$, for example, the condition $S_{n,a}(P_1,P_2)>0$ is equivalent to the conditions $(-1)^{n+k}P_1^{(k)}(a)>0$ for $k=0,1,\cdots,n-1$.

Now, we apply for the above results in studying Schur stability and the strict aperiodicity. Let $h(z)$ and $g(z)$ denote the symmetric and antisymmetric parts of $f(z)$ in (1), respectively. The projection of $h(z)$ and $g(z)$ for $n=2q$ (even) from unit circle onto the real line $(-1,+1)$ can be expressed by

$$P(x)=\sum_{i=0}^{q-1}\alpha_i T_{q-i}(x)+\frac{\alpha_q}{2} \tag{12}$$

$$Q(x)=\sum_{i=0}^{q-1}\beta_i U_{q-i}(x) \tag{13}$$

where $T_k(x)$ is the *k-th* Chebyshev polynomial of the first kind and $U_k(x)$ is the *k-th* Chebyshev polynomial of the second kind, respectively. For details about (12) and (13), readers may refer to the papers by Nour-Eldin (1971), Mansour and Anderson (1990).

THEOREM 3 Consider the polynomial $f(z)$ of (1), for which is constructed $g(z), h(z), P(x)$ and $Q(x)$. Then, the polynomial $f(z)$ is Schur stable if and only if for $k = 0,1,\cdots,q-1$, we have

$$S_q(P,Q) > 0\,, \tag{14}$$

$$(-1)^{q+k}P^{(k)}(-1) > 0 \text{ and } P^{(k)}(1) > 0\,. \tag{15}$$

REMARK 3 The linear conditions in Theorem 3 are not equivalent to those given by Mansour and Anderson (1990). It should be pointed out that the bilinear transformation used by Mansour and Anderson (1990) is not necessary because of knowing all zeros of $P(x)$ be real. If n is an odd, the polynomial $f(z)$ can be replaced by the polynomial $zf(z)$.

Soh and Berger (1989) studied the problem of strict aperiodicity and established some algebraic criteria on it. We here directly apply the results in Theorem 2 above to give two new algebraic criteria on strict aperiodicity.

THEOREM 4 For discrete-time systems, the polynomial $f(z)$ is strictly aperiodic if and only if $S_n(f, f') > S_n^*(f, f') > 0$; For continuous-time systems, the polynomial $f(z)$ is strictly aperiodic if and only if $S_n(f, f') > 0$ and $S_n^*(f, f') < 0$.

THEOREM 5 For discrete-time systems, the polynomial $f(z)$ is strictly aperiodic if and only if

$$a_{odd} < 0, a_{even} > 0\,, \tag{16}$$

$$S_n(f, f') > 0\,, S_n(f, f') > 0\,, \text{ and } f^{(k)}(1) > 0 \tag{17}$$

for $k = 0,1,\cdots,n-1$; For continuous-time systems, the polynomial $f(z)$ is strictly aperiodic if and only if $S_n(f, f') > 0$, $a_k > 0$ for $k = 0,1,\cdots,n-1$.

3. CONCLUSION

By using Markov parameters and the Sturm chain, this paper presents some necessary and sufficient conditions on determining the Cauchy index in finite intervals. New criteria on Schur stability and strict aperiodicity for linear systems are obtained.

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