15th Triennial World Congress, Barcelona, Spain

ROBUST FUZZY LQ/H_∞ CONTROL FOR UNCERTAIN NONLINEAR SYSTEMS WITH DELAY

Fei Liu[+], **Wenbo Xu**[+] **and Jian Chu**[++]

[+] *College of Information and Control, Southern Yangtze University, Wuxi, Jiangsu, 214036, P. R. China*
[++] *National Laboratory of Industrial Control Technology, Zhejiang University, Hangzhou, 310027, P. R. China.*

Abstract: This paper addresses robust fuzzy mixed LQ/H_∞ control design for nonlinear discrete-time systems in the presence of norm-bounded time-varying uncertainties and state delays. Based on Takagi-Sugeno fuzzy models, the mixed LQ/H_∞ index is introduced to nonlinear systems, which simultaneously guarantees an upper bound of LQ cost and its robustness for unknown external disturbance in the sense of induced H_∞ norm. Applying PDC techniques, the gains of robust state feedback controllers of subsystems are derived by the numerical solutions of a set of coupled LMIs, and then the overall LQ/H_∞ controller is constructed by fuzzy blending of the local linear controllers. *Copyright © 2002 IFAC*

Keywords: Fuzzy control, Robustness, Quadratic performance indices, H-infinity, Multiobjective optimisations, Nonlinear systems, Time delay, Lyapunov stability, Discrete-time systems.

1. INTRODUCTION

In the control engineering practice, modern control theory should take one thing with another. For example, in general, classical control theories depend on the precision of plant models and the restriction of disturbances, but almost all control systems operate in highly uncertain environments and suffer unknown disturbances. Also, extensive engineering systems subjecting to nonlinearities and time-delays, in which the well-known linear control theory does not hold. On the other hand, most engineering problems are multiobjective, thus often demand that the controller design should be multiobjective. Today, around all above challenging subjects, there have already been a great deal of involved works.

In the last two decades, many researchers have worked on robust linear quadratic (LQ) problem in the attempt to guarantee robust stability and robust performance in the presence of plant uncertainties. The H_∞ control, against unknown disturbances, has also attracted much interest in the literatures. Over the last decade, mixed H_2/H_∞ control problems, which combine H_2 performance index with H_∞ constraint, have been proposed for both linear continuous-time systems (Bernstein and Haddad, 1989; Khargonekar and Rotea,1991; Rotea and Khargonekar, 1991; Zhou *et al*., 1990) and discrete-time systems (Haddad *et al*., 1991; Kaminer *et al*.,1993). But the robust mixed H_2/H_∞ control for uncertain systems has turned out to be very difficult by far (e.g. Doyle *et al*., 1994; Zhou *et al*., 1994).

In the meantime, the notion of guaranteed cost control for uncertain systems (Petersen and McFarlane, 1994; Fishman *et al*., 1996) was used to establish an upper bound on the closed-loop value of a LQ cost. The guaranteed cost control was then extended to uncertain time-delay systems for continuous-time cases (Moheimani and Petersen, 1997) and discrete-time cases (Guan, 1999;

Mahmoud, 2000). It should be noted that, as a relaxed case of the H_2/H_∞ problem, a kind of so-called mixed LQ/H_∞ control might be realistic with the guaranteed cost instead of the optimal LQ cost. Recently, Kogar (2000) showed the robust H_∞ and guaranteed cost control design can be reformulated as minimax control problems for an auxiliary dynamic game. However there exist fewer results on robust mixed LQ/H_∞ control for time-delay systems.

For more complex engineering systems, especially with nonlinear dynamics, fuzzy control has emerged as one of the most powerful and successful approaches instead of classical control. In the past years, a number of fuzzy control approaches, which mainly based on the human experience or knowledge (Zadeh, 1973; Nakanishi *et al*., 1993), have been carried out via input fuzzifier, fuzzy inference and output defuzzifer. Accompanying with its theoretical intuitivity and practical acceptability, this kind of approaches has proved extremely difficult to do basic stability analysis, and hard to form a general design methodology for conventional fuzzy control systems.

Contrastively, Takagi-Sugeno fuzzy-model-based methods (Takagi and Sugeno, 1985; Tanaka and Sugeno, 1992) avoid the above drawback, in which the nonlinear dynamical systems are characterized by smoothly connecting a series of local linear models through selected membership functions. For each local linear system, some analysis and design tools in classical control theory are used. In particular, a kind of design methods named parallel-distributed compensation (PDC) was proposed based on Takagi-Sugeno model (Wang *et al*., 1996). Under this framework, the resulting overall controller, which is nonlinear and non-local in general, is a fuzzy blending of each local linear controller associated to the local model. More significantly, PDC integrating linear matrix inequality (LMI) offers a numerically tractable means for a great deal of control problems, such as stability analysis and synthesis (Wang *et al*., 1996; Tanaka *et al*.1998; Cao and Frank, 2000), H_∞ control (Hong and Langari, 1998) and multiobjective control (Li *et al*., 1999). The issues of robust stabilization based on Takagi-Sugeno fuzzy models have already obtained some results (Tanaka *et al*., 1996). Recently robust H_∞ control for uncertain fuzzy systems have been discussed, see, Lee *et al*. (2000, 2001). However there are a few publications on multiobjective (for example, mixed H_2/H_∞ or LQ/H_∞) problems. The fuzzy mixed H_2/H_∞ controller has been proposed (Chen *et al*., 2000) via observer-based output feedback, but uncertainties and time delays are not considered.

Motivated by the above observation, this paper considers nonlinear systems described by Takagi-Sugeno fuzzy models, which subject to state time-delays and time-varying but norm-bounded uncertainties. First, introduce the mixed LQ/H_∞ index, which implies that LQ cost has an upper bound and that the induced H_∞ norm, from external disturbances to LQ cost, is less than a prescribed level. Based on Lyapunov stability theory, applying PDC techniques, the gains of robust state feedback controllers of subsystems are derived by the numerical solution of a class of coupled LMIs, and then the overall mixed LQ/H_∞ controller is constructed by fuzzy blending of the local linear controllers. Note that, at present, the solution of LMI can be worked out numerically very efficiently (Boyd *et al*., 1994). Hence the LMI-based design approaches without any parameter tuning is easy to use in practice. It is needed to point that the research framework of this paper covers both continuous and discrete case of nonlinear systems. With restriction of paper space, here only the later is addressed. The notations used in this paper are standard.

2. PROBLEM STATEMENT AND ANALYSIS

Consider a class of nonlinear discrete-time systems with uncertainties and state delay described by the Takagi-Sugeno models, which consist of a family of fuzzy IF-THEN rules as following

Plant rule i:

$$\begin{aligned}&\text{IF } x_1(k) \text{ is } \mu_{i1} \text{ and} \cdots \text{and } x_g(k) \text{ is } \mu_{ig} \\ &\text{THEN } x(k+1) = (A_i + \Delta A_i(k))x(k) \\ &\qquad + (A_{di} + \Delta A_{di}(k))x(k-d) \\ &\qquad + (B_i + \Delta B_i(k))u(k) + B_{wi}w(k), \\ &\qquad\qquad i = 1,2,\cdots,r\end{aligned} \tag{1}$$

with initial condition

$$x(k) = 0,\ -d \le k < 0,\ x(0) = x_0 . \tag{2}$$

where μ_{ij} is the fuzzy set with $j = 1,2,\cdots,g \le n$, r is the number of fuzzy rules, $x(k) \in R^n$ is the state, $u(k) \in R^m$ is the control input, $w(k) \in l_2[0,\infty]$ is the disturbance input, $d > 0$ is the time delay constant and $\Delta A(k), \Delta A_d(k), \Delta B(k)$ are time-varying but norm-bounded uncertainties with the following form

$$\begin{aligned}&[\Delta A_i(k)\ \ \Delta B_i(k)] = H_i F(k)[E_i\ \ E_{Bi}], \\ &\Delta A_{di}(k) = H_{di} F_d(k) E_{di}\ , \quad i = 1,2,\cdots,r\end{aligned} \tag{3}$$

where $H, H_d, E_i, E_{Bi}, E_{di}$ are known constant matrices of appropriate dimensions, and $F(k)$, $F_d(k)$ are unknown real time-varying matrices with Lebesgue measurable elements satisfying $F^T(k)F(k) \le I$ and $F_d^T(k)F_d(k) \le I$.

Let $h_i(x(k))$ be the normalised membership functions of the inferred fuzzy set μ_i, i.e.

$$\begin{aligned}&\mu_i(x(k)) = \prod_{j=1}^{g} \mu_{ij}(x_j(k)), \\ &h_i(x(k)) = \frac{\mu_i(x(k))}{\sum_{i=1}^{r} \mu_i(x(k))}.\end{aligned} \tag{4}$$

For the sake of notation simplification, thereinafter, the time dependence on k in the variables will be omitted wherever no confusion arises. Assume for all time k, $\mu_i(x) \ge 0$, $\sum_{i=1}^{r} \mu_i(x) > 0$, it is obvious that

$$h_i(x) \geq 0,\ \sum_{i=1}^{r} h_i(x) = 1,\quad i = 1, \cdots, r \tag{5}$$

By using a center-average defuzzifer, the dynamic fuzzy models (1) can be represented by the following overall model (6), which combines all the local models through membership functions.

$$\begin{aligned} x(k+1) = \sum_{i=1}^{r} h_i(x)[\ (A_i + \Delta A_i(k))x(k) \\ + (A_{di} + \Delta A_{di}(k))x(k-d) \\ + (B_i + \Delta B_i(k))u(k) + B_{wi}w(k)]. \end{aligned} \tag{6}$$

Associated with systems (1), the local state feedback controllers are considered as

Control rule i:

IF x_1 is μ_{i1} and $\cdots$ and x_g is μ_{ig} (7)

THEN $u_i(k) = K_i x(k)$, $i = 1, 2, \cdots, r$

Accordingly, the final overall fuzzy controller follows

$$u(k) = \sum_{i=1}^{r} h_i(x) K_i x(k) \cdot \tag{8}$$

The resulting fuzzy closed-loop system is then described by

$$\begin{aligned} x(k+1) = (\overline{A}_C + \overline{H}F(k)\overline{E}_C)x(k) \\ + (\overline{A}_d + \overline{H}_d F_d(k)\overline{E}_d)x(k-d) + \overline{B}_w w(k) \end{aligned} \tag{9}$$

where

$$\overline{A}_C = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(x)h_j(x)(A_i + B_i K_j),$$

$$\overline{E}_C = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(x)h_j(x)(E_i + E_{Bi} K_j),$$

$$\overline{H} = \sum_{i=1}^{r} h_i(x)H_i,\ \overline{H}_d = \sum_{i=1}^{r} h_i(x)H_{di},$$

$$\overline{E}_d = \sum_{i=1}^{r} h_i(x)E_{di},\ \overline{B}_w = \sum_{i=1}^{r} h_i(x)B_{wi} \cdot$$

Similar to the well-known linear quadratic control theory, in the following, the LQ cost is defined for a class of fuzzy discrete-time systems.

Definition 2.1: Associated with the system (6) and controller (8), or with the closed-loop system (9), the linear quadratic (LQ) cost is defined as

$$J = \sum_{k=0}^{\infty}[\ x^T(k)Qx(k) + u^T(k)Ru(k)] \tag{10}$$

where $0 \leq Q = Q^T \in R^{n\times n}$ and $0 \leq R = R^T \in R^{m\times m}$ are given weighting matrices of state and control input respectively.

Remark 2.2: By introducing the auxiliary output $z(k) = Cx(k) + Du(k)$ with $C = [\ Q^{1/2}\ \ 0\ ]^T$ and $D = [\ 0\ \ R^{1/2}\ ]^T$, the above LQ cost can be rewritten as $J = \sum_{k=0}^{\infty} z^T(k)z(k) = \|z\|_2^2$. Here $\|\cdot\|_2$ denotes the l_2 norm. Substituting controller (8) in, the auxiliary output of the closed-loop system is given as $z(k) = \overline{C}_C x(k)$ with $\overline{C}_C = \sum_{i=1}^{r} h_i(x)(C + DK_i)$ ·

Remark 2.3: In general, it is very difficult to get an optimal LQ cost for the fuzzy systems with parametric uncertainties. In this paper, a sub-optimal LQ is used instead of optimal LQ, which is actually an upper bound α on LQ, i.e. $J \leq \alpha$. In other words, the notation of LQ here is in line with that of what is often called guaranteed cost problem with respect to robust control of linear uncertain systems in references.

While taking into account the external disturbance w, the H_∞ norm index γ (or H_∞ constraint level) may be introduced by notation $\|z\|_2 \leq \gamma\|w\|_2$, where γ characters the impact of external disturbance w on LQ cost J.

Definition 2.4: Associated with the closed-loop system (9), the mixed LQ/H_∞ index is defined

$$J = \|z\|_2^2 \leq \gamma^2\|w\|_2^2 + \alpha \tag{11}$$

where $\alpha > 0$ is the upper bound on LQ and $\gamma > 0$ is the attenuation level of disturbance.

Obviously, while external disturbance $w(k) = 0$, index (11) reduces to $J \leq \alpha$, i.e. LQ cost has an upper bound.

Definition 2.5: The controller, to make the resulting fuzzy closed-loop system satisfying above mixed LQ/H_∞ index for all admissible uncertainties and time delays, is said to be a robust fuzzy LQ/H_∞ controller, for short, a LQ/H_∞ controller.

Now, following theorem investigates the existence condition of LQ/H_∞ controller and indicates that the closed-loop system satisfying the LQ/H_∞ index is always robust stable. Before the proof of theorem, a useful lemma is borrowed.

Lemma 2.6 (Deng 1997): For any vectors $a, b \in R^n$ and any positive-definite matrix $P \in R^{n\times n}$, inequality $2a^T Pb \leq a^T Pa + b^T Pb$ holds.

Theorem 2.7: For system (6) and a prescribed real number $\gamma > 0$, there exists a fuzzy LQ/H_∞ controller (8) with a set of gain matrices $K_i \in R^{m\times n}$, $i = 1, 2, \cdots, r$, if there exist positive-definite symmetric matrices $P, S \in R^{n\times n}$ satisfying condition

$$\begin{aligned} x^T(k)[3(\overline{A}_C + \overline{H}F(k)\overline{E}_C)^T P(\overline{A}_C + \overline{H}F(k)\overline{E}_C) \\ - P + S + \overline{C}_C^T\overline{C}_C]x(k) + x^T(k-d)[\ 3(\overline{A}_d \\ + \overline{H}_d F_d(k)\overline{E}_d)^T P(\overline{A}_d + \overline{H}_d F_d(k)\overline{E}_d) - S\ ]x(k-d) \\ + w^T(k)(\ 3\overline{B}_w^T P\overline{B}_w - \gamma^2 I)w(k) < 0. \end{aligned} \tag{12}$$

Agreeing with the condition, the resulting closed-loop system (9) is robust stable and satisfies the LQ/H_∞ index (11).

Proof: For positive-definite symmetric matrices P, S, let the following Lyapunov function candidate for the closed-loop system (9)

$$V(x,k) = x^T(k)Px(k) + \sum_{l=k-d}^{k-1} x^T(l)Sx(l) \cdot \tag{13}$$

Along the trajectories of system (9), the first-forward

difference of the Lyapunov function (13) is given by
$\Delta V(x,k) = V(x,k+1) - V(x,k) =$

$$
\begin{aligned}
& x^T(k)[(\overline{A}_C + \overline{H}F(k)\overline{E}_C)^T P(\overline{A}_C + \overline{H}F(k)\overline{E}_C) - P + S]x(k) \\
& + x^T(k)(\overline{A}_C + \overline{H}F(k)\overline{E}_C)^T P(\overline{A}_d + \overline{H}_d F_d(k)\overline{E}_d)x(k-d) \\
& + x^T(k)(\overline{A}_C + \overline{H}F(k)\overline{E}_C)^T P\overline{B}_w w(k) \\
& + w^T(k)\overline{B}_w^T P(\overline{A}_C + \overline{H}F(k)\overline{E}_C)x(k) \\
& + x^T(k-d)[(\overline{A}_d + \overline{H}_d F_d(k)\overline{E}_d)^T P(\overline{A}_d + \overline{H}_d F_d(k)\overline{E}_d) \\
& - S]x(k-d) + x^T(k-d)(\overline{A}_d + \overline{H}_d F_d(k)\overline{E}_d)^T P\overline{B}_w w(k) \\
& + x^T(k-d)(\overline{A}_d + \overline{H}_d F_d(k)\overline{E}_d)^T P(\overline{A}_C + \overline{H}F(k)\overline{E}_C)x(k) \\
& + w^T(k)\overline{B}_w^T P(\overline{A}_d + \overline{H}_d F_d(k)\overline{E}_d)x(k-d) \\
& + w^T(k)\overline{B}_w^T P\overline{B}_w w(k).
\end{aligned}
$$

Applying Lemma 2.6 to the right-hand side of above equality, it follows

$$
\begin{aligned}
& \Delta V(x,k) \\
& \le x^T(k)[3(\overline{A}_C + \overline{H}F(k)\overline{E}_C)^T P(A_C + \overline{H}F(k)\overline{E}_C) \\
& \quad - P + S]x(k) \\
& \quad + x^T(k-d)[\,3(\overline{A}_d + \overline{H}_d F_d(k)\overline{E}_d)^T P(\overline{A}_d \\
& \quad + \overline{H}_d F_d(k)\overline{E}_d) - S\,]x(k-d) \\
& \quad + 3w^T(k)\overline{B}_w^T P\overline{B}_w w(k).
\end{aligned} \tag{14}
$$

Considering the condition (12), and letting $w(k) = 0$, the following (15) holds

$$
\begin{aligned}
\Delta V(x,k) &= V(x,k+1) - V(x,k) \\
&< -x^T(k)\overline{C}_C^T\overline{C}_C x(k) < 0
\end{aligned} \tag{15}
$$

and then, the robust quadratic stability of the closed-loop system (9) is guaranteed. Furthermore, reusing the condition (12) with $w(k) \neq 0$ and recalling $z(k) = \overline{C}_C x(k)$, obviously

$$
\begin{aligned}
& V(x,k+1) - V(x,k) \\
& \le -z^T(k)z(k) + \gamma^2 w^T(k)w(k).
\end{aligned} \tag{16}
$$

Substituting (13) into (16), it follows

$$
\begin{aligned}
& x^T(k+1)Px(k+1) - x^T(k)Px(k) + x^T(k)Sx(k) \\
& - x^T(k-d)Sx(k-d) + z^T(k)z(k) \\
& \le \gamma^2 w^T(k)w(k).
\end{aligned}
$$

Summing up above inequality over the time dependence section $k = 0 \to \infty$, keeping in mind $\lim_{k\to\infty} x(k) = 0$ and initial condition (2), it is not difficult to have

$$J = \|z\|_2^2 \le \gamma^2 \|w\|_2^2 + x^T(0)Px(0). \tag{17}$$

By imposing a cost upper bound α on $x^T(0)Px(0)$, obviously, the resulting closed-loop system (9) satisfies the LQ/H$_\infty$ index. □

Remark 2.8: In above proof, the LQ cost is related to initial condition. In detail, the upper bound of LQ cost is given by

$$\alpha \ge x^T(0)Px(0) + \sum_{l=0}^{d-1} x^T(l-d)Sx(l-d).$$

For the simple case of initial condition (2), it is easy to get $\alpha \ge x_0^T P x_0$, and then $J \le \gamma^2 \|w\|_2^2 + \alpha$. While focusing on external disturbance, from (17) with zero initial condition, the attenuation level of disturbance in the sense of induced H$_\infty$ norm is given by γ.

3. FUZZY CONTROLLER SYNTHESIS

In this section, the main results are presented on how to design the robust fuzzy LQ/H$_\infty$ controller defined in last section. Using inequality manipulations and parallel distributed compensation (PDC) techniques, the condition (12) in Theorem 2.7 is reduced to a set of coupled LMIs. The local state feedback controllers are then derived by the numerical solutions of the coupled LMIs and the overall robust fuzzy LQ/H$_\infty$ controller is constructed by fuzzy blending of the local linear controllers.

Lemma 3.1(Li 1997): Let A, D, E and F be real matrices of appropriate dimensions and $\|F\| \le 1$. For any matrix $P = P^T > 0$ and scale $\varepsilon > 0$ such that $P - \varepsilon DD^T > 0$ and

$$
\begin{aligned}
& (A + DFE)^T P^{-1}(A + DFE) \\
& \quad \le A^T(P - \varepsilon DD^T)^{-1}A + \varepsilon^{-1}E^T E.
\end{aligned}
$$

Theorem 3.2: Given system (6) and a prescribed H$_\infty$ norm level $\gamma > 0$, there exists a robust fuzzy LQ/H$_\infty$ controller if there exist positive-definite symmetric matrices $X, W \in R^{n\times n}$, $Y_i \in R^{m\times n}$, $i = 1,2,\cdots,r$, and positive scalars λ_1, λ_2 such that the following coupled LMIs (18)-(19) are feasible. Moreover a set of local controllers is given by $u_i(k) = Y_i X^{-1} x(k)$ and the overall robust fuzzy LQ/H$_\infty$ controller is constructed by (8) with LQ cost $\alpha \ge x_0^T X^{-1} x_0$.

$$\Pi_{ii} < 0, \quad i = 1,2,\cdots,r \tag{18}$$

$$\Pi_{ij} + \Pi_{ji} < 0, \quad i < j \le r \tag{19}$$

where

$$
\Pi_{ij} = \begin{bmatrix}
-X+W & 0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & -W & 0 & 0 & 0 & 0 & * & * & 0 \\
0 & 0 & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & * \\
A_i X + B_i Y_j & 0 & 0 & -\frac{1}{3}X + \lambda_1 H_i H_i^T & 0 & 0 & 0 & 0 & 0 \\
E_i X + E_{Bi} Y_j & 0 & 0 & 0 & -\lambda_1 I & 0 & 0 & 0 & 0 \\
CX + DY_i & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\
0 & A_{di} X & 0 & 0 & 0 & 0 & -\frac{1}{3}X + \lambda_2 H_{di} H_{di}^T & 0 & 0 \\
0 & E_{di} X & 0 & 0 & 0 & 0 & 0 & -\lambda_2 I & 0 \\
0 & 0 & B_{wi} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}X
\end{bmatrix}
$$

in which $*$ represent blocks that are readily inferred by symmetry.

Proof: Assume there exist positive-definite symmetric matrices $P, S \in R^{n\times n}$, real matrices $K_i \in R^{m\times n}$, $i = 1,2,\cdots,r$ and positive scalars $\varepsilon_1, \varepsilon_2$, applying Theorem 2.7 and Lemma 3.1, the condition

(12) is guaranteed if

$$\begin{aligned} & x^T(k)[\,3\overline{A}_C^T(P^{-1}-\varepsilon_1\overline{H}\overline{H}^T)^{-1}\overline{A}_C+3\varepsilon_1^{-1}\overline{E}_C^T\overline{E}_C \\ & \quad -P+S+\overline{C}_C^T\overline{C}_C]x(k) \\ & \quad +x^T(k-d)[\,3\overline{A}_d^T(P^{-1}-\varepsilon_2\overline{H}_d\overline{H}_d^T)^{-1}\overline{A}_d \\ & \quad +3\varepsilon_2^{-1}\overline{E}_d^T\overline{E}_d-S]x(k-d) \\ & \quad +w^T(k)(3\overline{B}_w^TP\overline{B}_w-\gamma^2I)w(k)<0 \end{aligned} \tag{20}$$

and

$$P^{-1}-\varepsilon_1\overline{H}\overline{H}^T>0\,,\;\; P^{-1}-\varepsilon_2\overline{H}_d\overline{H}_d^T>0\,. \tag{21}$$

For any $\begin{bmatrix}x^T(k) & x^T(k-d) & w^T(k)\end{bmatrix}^T \neq 0$, above inequality (20) holds if

$$diag\{\overline{M}_1 \quad \overline{M}_2 \quad \overline{M}_3\}<0 \tag{22}$$

where

$$\begin{aligned} \overline{M}_1 &= 3\overline{A}_C^T(P^{-1}-\varepsilon_1\overline{H}\overline{H}^T)^{-1}\overline{A}_C \\ &\quad +3\varepsilon_1^{-1}\overline{E}_C^T\overline{E}_C-P+S+\overline{C}_C^T\overline{C}_C, \\ \overline{M}_2 &= 3\overline{A}_d^T(P^{-1}-\varepsilon_2\overline{H}_d\overline{H}_d^T)^{-1}\overline{A}_d \\ &\quad +3\varepsilon_2^{-1}\overline{E}_d^T\overline{E}_d-S, \\ \overline{M}_3 &= 3\overline{B}_w^TP\overline{B}_w-\gamma^2I. \end{aligned}$$

Furthermore, the inequality (22) is equivalent to

$$\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(x)h_j(x)diag\{M_{1ij} \quad M_{2i} \quad M_{3i}\}<0 \tag{23}$$

where

$$\begin{aligned} M_{1ij} &= 3(A_i+B_iK_j)^T(P^{-1}-\varepsilon_1H_iH_i^T)^{-1}(A_i+B_iK_j) \\ &\quad +3\varepsilon_1^{-1}(E_i+E_{Bi}K_j)^T(E_i+E_{Bi}K_j)-P+S \\ &\quad +(C+DK_i)^T(C+DK_i), \end{aligned}$$

$M_{2i}=3A_{di}^T(P^{-1}-\varepsilon_2H_{di}H_{di}^T)^{-1}A_{di}+3\varepsilon_2^{-1}E_{di}^TE_{di}-S$,

$M_{3i}=3B_{wi}^TPB_{wi}-\gamma^2I$.

Considering (5), above (23) can be rewritten as

$$\begin{aligned} & \sum_{i=1}^{r}h_i(x)h_i(x)diag\{M_{1ii} \;\; M_{2i} \;\; M_{3i}\} \\ & +\sum_{i<j}^{r}h_i(x)h_j(x)diag\{M_{1ij}+M_{1ji} \;\; M_{2i}+M_{2j} \;\; M_{3i}+M_{3j}\} \\ & <0 \end{aligned} \tag{24}$$

and in turn it is guaranteed by the following two class of inequalities

$$diag\{M_{1ii} \quad M_{2i} \quad M_{3i}\}<0 \quad i=1,2,\cdots,r, \tag{25}$$

$$diag\{M_{1ij}+M_{1ji} \quad M_{2i}+M_{2j} \quad M_{3i}+M_{3j}\}<0, \quad i<j\le r. \tag{26}$$

On the other hand, obviously, the inequalities (21) are guaranteed if

$$P^{-1}-\varepsilon_1H_iH_i^T>0\,,\;\; P^{-1}-\varepsilon_2H_{di}H_{di}^T>0\,, \quad i=1,2,\cdots,r \tag{27}$$

By combining (27) and applying Schur complements to inequalities (25) and (26), furthermore by using some matrix manipulations and setting $X=P^{-1}$, $W=P^{-1}SP^{-1}$, $Y_i=K_iP^{-1}$, $\lambda_1=\frac{1}{3}\varepsilon_1$ and $\lambda_2=\frac{1}{3}\varepsilon_2$, respectively the inequalities (18) and (19) can be deduced from (25) and (26), in which the conditions (27) are embedded. Thus the proof is completed. □

In addition, as an extension of Theorem 3.2, following corollary can be used to search the minimal upper bound on LQ, which is often referred to the optimal guaranteed cost control problem.

Corollary 3.3: The optimal upper bound α^* with respect to the controller in Theorem 3.2 is given by solving following convex optimization problem with a new decision variable α:

minimize α

subject to LMIs (18), (19) and

$$\begin{bmatrix}-\alpha & x_0^T \\ x_0 & -X\end{bmatrix}<0\,. \tag{28}$$

Proof: Recalling Remark 2.8 and Schur complements, it is easy to get LMI (28). The other is just the same as proof in Theorem 2.7. □

Example 3.4: Consider the uncertain discrete-time fuzzy system (1) with following parameter matrices

$$A_1=\begin{bmatrix}1.5 & 1 \\ 0.5 & 1.2\end{bmatrix}, A_2=\begin{bmatrix}0.8 & 0.1 \\ 0.1 & 0\end{bmatrix},\; B_1=B_2=\begin{bmatrix}1.0 \\ 0.8\end{bmatrix},$$

$$A_{d1}=\begin{bmatrix}0 & 0.2 \\ 0.18 & 0.15\end{bmatrix}, A_{d2}=\begin{bmatrix}0 & 0 \\ 0.08 & 0.02\end{bmatrix},$$

$$B_{w1}=B_{w2}=\begin{bmatrix}0.06 \\ 0.05\end{bmatrix}, H_1=H_2=\begin{bmatrix}0.12 & 0.10 \\ 0.10 & 0.10\end{bmatrix},$$

$$E_{B1}=E_{B2}=\begin{bmatrix}0.15 \\ 0.10\end{bmatrix}, H_{d1}=H_{d2}=\begin{bmatrix}0.05 & 0.03 \\ 0.05 & 0.05\end{bmatrix},$$

$$E_1=E_2=\begin{bmatrix}0.20 & 0.15 \\ 0.15 & 0.10\end{bmatrix},\; E_{d1}=E_{d2}=\begin{bmatrix}0.05 & 0.05 \\ 0.05 & 0.03\end{bmatrix}$$

and $x_0=[1 \;\; 1]^T$. Applying Theorem 3.2 with $Q=I$, $R=0.5$, for attenuation level $\gamma=1$, the robust fuzzy LQ/H$_\infty$ controller (8) is obtained with gains of the local controllers as $K_1=[-1.4175 \;\; -1.0398]$ and $K_2=[-0.7376 \;\; -0.0911]$, and with LQ cost $\alpha=14.57$. Furthermore, by Corollary 3.3, the optimal LQ upper bound $\alpha^*=11.3221$ is ensured by the controllers $K_1=[-1.3939 \;\; -1.0503]$ and $K_2=[-0.7204 \;\; -0.0887]$.

Remark 3.5: Based on Theorem 3.2, let $\beta=\gamma^2$, the so-called optimal H$_\infty$ controller with minimal attenuation level $\gamma^*=\sqrt{\beta}$ can be obtained by following optimization problem: minimize β subjecting to LMIs (18) and (19). Reconsider Example 3.4 by adding new decision variable β, the result shows $\gamma^*=\sqrt{\beta}=0.34$.

4. CONCLUSION

Based on Lyapunov stability theory, applying PDC techniques, a systemic framework to analysis and synthesis the robust fuzzy LQ/H$_\infty$ controllers for nonlinear systems is presented via Takagi-Sugeno models. All the issues about such controllers, including existence condition, design approaches and the optimization of LQ cost, are cast to feasible problem of LMIs. The numerical examples illustrate the results. An analysis of the conservativeness of the proposed results will be the subject of further investigations.

REFERENCES

Bernstein, D.S. and W.M. Haddad (1989). LQG control with an H_∞ performance bound: a Riccati equation approach, *IEEE Trans. Automat. Contr.*, **34**, 293-305.

Boyd, S., L.El., Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear matrix inequalities in systems and control theory*, SIAM, Philadelphia.

Cao, Y.Y. and P.M. Frank (2000). Analysis and synthesis of nonlinear time-delay systems via fuzzy control approach, *IEEE Trans. Fuzzy Systems*, **8**(2), 200-211.

Chen, B-S., C-S. Tseng and H-J. Uang (2000). Mixed H_2/H_∞ fuzzy output feedback control design for nonlinear dynamic systems: an LMI approach, *IEEE Trans. Fuzzy Systems*, **8**(3), 249 –265.

Deng, F.Q., Z.S. Feng and Y.Q. Liu (1997). A sign method of robust controllers of linear delay systems based on Riccati-Ito equations. *Control Theory Appl. (Chinese)*, **14**(6), 887-890.

Doyle, J., K. Zhou, K. Glover and B. Bodenheimer (1994). Mixed H_2 and H_∞ performance objectives II: optimal control, *IEEE Trans. Automat. Contr.*, **39**(8), 1575-1587.

Fishman, A., J. M. Dion, L. Dugard and A. T. Neto (1996). A linear matrix inquality approach for guaranteed cost control. *Proceedings of 13th IFAC World Congress*, San Francisco, USA, Vol.**4**, pp. 197-202.

Guan, X., Z. Lin and G. Duan (1999). Robust guaranteed cost control for discrete-time uncertain systems with delay. *IEE Proc. Control Theory and Application*. **146**(6), 598-602.

Haddad, W.M., D.S. Bernstein and D. Mustafa (1991). Mixed H_2 and H_∞ regulation and estimation: the discrete-time case. *Systems and Control Letters*, **16**, 235-247.

Hong, S.K. and R. Langari (1998). Synthesis of an LMI-based fuzzy control system with guaranteed optimal H_∞ performance, *Proceedings of IEEE International Conference on Fuzzy Systems*, Anchorage, AK, vol.**1**, pp. 422-427.

Kaminer, I., P.P. Khargonekar and M.A. Rotea (1993). Mixed H_2/H_∞ control for discrete-time system via convex optimization. *Automatica*, **29**, 57-70.

Khargonekar, P.P. and M. A. Rotea (1991). Mixed H_2/H_∞ control : a convex optimization approach. *IEEE Trans. Automat. Cortr.*, **36**, 824-837.

Kogan, M.M. (2000). Robust H_∞ suboptimal and guaranteed cost state feedbacks as solutions to linear-quadratic dynamic games under uncertainty. *International Journal of control*, **73**(3), 219-224.

Lee, K.R., E.T. Jeung and H.B. Park (2001). Robust fuzzy H_∞ control for uncertain nonlinear systems via state feedback: an LMI approach, *Fuzzy Sets and Systems*, **120**, 123-134.

Lee, K.R., J.H. Kim, E.T. Jeung and H.B. Park (2000). Output feedback robust H_∞ control of uncertain fuzzy dynamic systems with time-varying delay, *IEEE Trans. Fuzzy Systems*, **8**(6), 657-664.

Li, J, D. Niemann, H. O. Wang and K. Tanaka (1999) Parallel distributed compensation for Takagi-Sugeno Fuzzy models: multiobjective controller design. *Proceedings of the American Control Conference*, San Diego, Califomia, pp.1832-1836.

Li, X. and C.E.D. Souza (1997). Criteria for robust stability and stabilization of uncertain linear systems with state delay. *Automatica*, **33**(9), 1657-1662.

Mahmoud, M.S. and L. Xie (2000). Guaranteed cost control of uncertain discrete systems with delays. *International Journal of control*, **73**(2), 105-114.

Moheimani, S.O.R and I.R. Petersen (1997). Optimal guaranteed cost control of a class of uncertain time-delay system. *IEE Proc. Control Theory Appl.*, **144**, 183-188.

Nakanishi, H., I. B. Turksen and M. Sugeno.(1993) A review and comparison of six reasoning methods, *Fuzzy Sets and Systems*., **57**, 257-294.

Petersen, I.R. and D.C. McFarlane (1994). Optimal guaranteed cost control and filtering for uncertain linear systems. *IEEE Trans. Automat. Contr.*, **39**(9), 1971-1977.

Rotea, M.A. and P.P. Khargonekar (1991). H_2–optimal control with an H_∞-constraint, the state-feedback case, *Automatica*, **27**(2), 307-316.

Takagi, T. and M. Sugeno (1985). Fuzzy identification of systems and its applications to modeling and control, *IEEE Trans. Systems Man Cybernet*, **15**(1), 116-132.

Tanaka, K and M. Sugeno (1992). Stability analysis and design of fuzzy control systems. *Fuzzy Sets and Systems*, **45**(2), 135-156.

Tanaka, K, T. Ikeda and H.O. Wang (1996). Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: quadratic stabilizability, H_∞ control theory, and linear matrix inequalities, *IEEE Trans. Fuzzy Systems*, **4**(1), 1-13.

Tanaka, K, T. Ikeda and H.O. Wang (1998). Fuzzy regulators and fuzzy observers: relaxed stability conditions and LMI-based design, *IEEE Trans. Fuzzy Systems*, **6**(2), 250-265.

Wang, H. O., K. Tanaka and M.F. Griffin (1996). An approach to fuzzy control of nonlinear systems: stability and design issues, *IEEE Trans. Fuzzy Systems*, **4**(1), 14-23.

Zadeh, L.A. (1973). Outline of a new approach to the analysis of complex systems and decision processes, *IEEE Trans. Systems Man Cybernet*, **3**, 28-44.

Zhou, K., J.C. Doyle, K. Glover and B. Bodenheimer (1990). Mixed H_2 and H_∞ control., *Proceedings of the American Control Conference*, San Diego,CA, pp. 2502-2507.

Zhou, K., K.Glover, B. Bodenheimer and J.C. Doyle (1994). Mixed H_2 and H_∞ performance objectives I: robust performance analysis, *IEEE Trans. Automat. Contr.*, **39**(8), 1564-1574.