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A DISCONTINUOUS CONTROL FOR ROBOTIC MANIPULATORS WITH COULOMB FRICTION

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Abstract: A nonsmooth controller design procedure for the regulation of a class of m-DOF mechanical manipulators with viscous and Coulomb friction is presented. Based on an invariance principle developed elsewhere, a discontinuous controller that uses only position measurement feedback is proposed. It is proved that the origin of the closed-loop system is globally and asymptotically stable. A discontinuous observer is also proposed to estimate the velocity and improve the performance of the controlled system. This kind of nonsmooth observer is important to achieve the global asymptotic stability of the closed-loop system. It is shown that the combination controller-observer can tolerate uncertainty in the friction coefficients if a bound on these parameters is known. *Copyright 2000 IFAC*

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1. INTRODUCTION

There are many important systems with nonsmooth dynamics. Among others, we can mention systems with Coulomb friction, contact interactions, variable structure systems, or systems where the control inputs are discontinuous. An important contribution for the analysis of these systems was given by Filippov (Filippov, 1988), who developed a solution concept for differential equations with Lebesgue measurable right-hand sides. The Lyapunov functions considered to study the stability of equilibria were smooth. Other authors extended the analysis using nonsmooth Lipschitz continuous Lyapunov functions, requiring that the trajectories were absolutely continuous (Shevitz and Paden, 1994). This allowed to prove the stability of equilibria of some nonsmooth systems using nonsmooth Lyapunov functions, which are natural for nonsmooth dynamics.

The control of discontinuous nonlinear systems is a nontrivial task. It is known that on a compact simply-connected manifold, a nonlinear system cannot be globally stabilized by a continuous, static feedback (Nikitin, 1999). Taking into account this fact, in this paper we propose a nonsmooth controller design procedure for the regulation of a class of m-DOF robotic manipulators with viscous and Coulomb friction terms. The proposed procedure is based on an invariance principle developed elsewhere (Alvarez *et al.*, 2000), restricted to a class of discontinuous dynamic systems whose trajectories are unambiguously defined (in the sense of Filippov). The proposed discontinuous controller uses only position measurement feedback. We prove that the origin of the closed-loop system is globally and asymptotically stable.

A discontinuous observer is also proposed to estimate the velocity and improve the performance of the controlled system. This kind of nonsmooth observer is important to achieve the global asymptotic stability of the closed-loop system. We also show that the controller-observer structure can tolerate uncertainty in the friction coefficients if a bound on these parameters is known. We illustrate the technique with numerical simulations of 1 and 2-DOF robotic manipulators.

2. DISCONTINUOUS DYNAMIC SYSTEMS

We consider discontinuous dynamic systems governed by differential equations of the form

$$\dot{x} = f(x), \quad (1)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise smooth function having discontinuities on a set $\mathcal{N} \in \mathbb{R}^n$ of measure zero. A solution of (1) is defined in the Filippov sense (Filippov, 1988).

Definition 1. For each $x \in \mathbb{R}^n$, let $F(x)$ be the smallest convex closed set containing all the limit values of $f(x^*)$ for $x^* \in \mathbb{R}^n \setminus \mathcal{N}$, $x^* \to x$. An absolutely continuous function x, defined on an interval I, is a solution of (1) if the differential inclusion $\dot{x} \in F(x)$ holds for $x(t)$ almost everywhere on I.

Because system (1) may have a non-unique solution for arbitrary initial conditions (Filippov, 1988), we must restrict our analysis to systems having at least right-uniqueness solutions. For that, let us assume that there exists a positive definite continuous function $V(x)$, nonincreasing along the trajectories of (1). Then, all the trajectories of (1) are bounded and, in accordance with (Filippov, 1988), they are globally defined in the direction of increasing t. Relating to continuous dynamic systems, the invariance principle ensures the convergence of the state trajectories $x(t)$ to the largest invariant subset Ω of the manifold $\mathcal{M} = \{x \in \mathbb{R}^n : D_t V(x(t)) = 0\}$, where

$$D_t V(x(t)) = \overline{\lim}_{h \to 0+} \frac{1}{h} \left[V(x(t+h)) - V(x(t)) \right] \quad (2)$$

is a fixed Dini derivative along the trajectories of the system. In general, the invariance principle does not admit an extension to dynamic systems governed by differential inclusions, and particularly to discontinuous dynamic systems like (1), possibly due to their ambiguous behavior (see (Michel and Wang, 1995)).

We confine our analysis to systems like (1) for which right uniqueness of solutions holds; that is, we assume that any solution of (1) is uniquely continuable to the right. Some sufficient right uniqueness conditions for solutions of system (1) can be found in (Filippov, 1988), where the continuous dependence of the solutions on their initial data is also shown.

Supose that there exists a positive definite function $V(x)$ satisfying a Lipschitz condition in a neighborhood of any $x \in \mathbb{R}^n$. Then, for any solution of (1), $V(x(t))$ is absolutely continuous, and

$$\dot{V}(x(t)) = \frac{d}{dh} V(x(t) + h\dot{x}(t)) \bigg|_{h=0} \quad (3)$$

almost everywhere (Filippov, 1988). The next result is proved in (Alvarez *et al.*, 2000).

Theorem 2. Suppose there exists a positive definite, Lipschitz-continuous function $V(x)$ such that

$$\dot{V}(x(t)) \le 0 \quad (4)$$

almost everywhere. Let Ω be the largest invariant subset of the manifold $\mathcal{M}$ where the strict equality holds, and denote $V(x) \to \infty$ as $\operatorname{dist}(x, \Omega) \to \infty$. Then all the trajectories $x(t)$ of (1) converge to Ω, that is, $\lim_{t\to\infty} \operatorname{dist}(x(t), \Omega) = 0$.

The next propositions allow one to simplify the verification of the conditions of this Theorem.

Proposition 3. Condition (4) of Theorem 2 is fulfilled if (3) is nonpositive at the points of the set $\mathcal{N}_V$ where the gradient ∇V of the function $V(x)$ does not exist and in the continuity domains of the function $f(x)$ where (3) is expressed in the standard form

$$\dot{V}(x) = \nabla V(x) \cdot f(x), \; x \in \mathbb{R}^n \setminus (\mathcal{N} \cup \mathcal{N}_V). \quad (5)$$

Proposition 4. Let no trajectory of (1) stays in $\mathcal{N}_V \cup \{0\}$ within a finite time interval. Then condition (4) of Theorem 2 is fulfilled almost everywhere if (5) is nonpositive for all $x \in \mathbb{R}^n \setminus (\mathcal{N} \cup \mathcal{N}_V)$.

We summarize now the main concepts related to solutions of differential equations with discontinuous right-hand sides presented in (Filippov, 1988), which play an important role in the design of the proposed controllers.

Suppose that the domain G where f is smooth can be decomposed into the union of open, disjoint sets G_i, $G = \cup_{i=1}^{l} G_i$, such that $\mathcal{N}$ defines the boundary between the sets G_i. We consider the case where $\mathcal{N}$ consists of a finite number of $(n-1)$-dimensional surfaces, denoted by S_j. In this case the trajectories of system (1) either cross the surfaces or stay in there, displaying the so-called sliding motion (Filippov, 1988).

For each point $x \in G$ we denote by $F_0(x)$ the simplest closed convex set containing all the limit values of $f(x^*)$ for $x^* \notin \mathcal{N}$, $x^* \to x$. A solution of (1) is a solution of the differential inclusion

$$\dot{x} \in F_0(x). \quad (6)$$

For $x \notin \mathcal{N}$, $F_0(x) = \{f(x)\}$, so the solution satisfies (1) in the usual sense. When $x \in \mathcal{N}$, $F_0(x)$ is a segment, a connected polygon, or a polyedhre with vertices $f_i(x)$, $i \le k$, where

$$f_i(x) = \lim_{x^* \in G_i, x^* \to x} f(x^*). \quad (7)$$

Let us consider the case where f is discontinuous in a smooth surface S, defined by $\varphi(x) = 0$. This surface divides the domain into the sets $G^{\pm}$. We define the vector fields $f^{\pm}$ as $f^{\pm}(x) = \lim_{x^* \in G^{\pm}, x^* \to x} f(x^*)$. Therefore, the set $F_0(x)$ is a linear segment joining the final points of vectors $f^{\pm}(x)$ starting at x. If this segment does not intersect the tangent plan P to the surface S at the point x, then the trajectories cross the surface; otherwise denote as $f^0(x)$ the intersection of this segment with the surface. If $f^0 \neq f^{\pm}$, then a sliding motion is generated, given by

$$\dot{x} = f^0(x). \quad (8)$$

A function $x(t)$ satisfying (8) is a solution of (1). A way to find if this situation occurs is given next.

Let us consider the system

$$\dot{x} = f\left(x, u_1\left(x\right), \ldots, u_r\left(x\right)\right), \tag{9}$$

where $x \in \mathbb{R}^n$, $f\left(x, u_1, \ldots, u_r\right)$ is continuous in all its arguments, each scalar function $u_i(x)$, $i = 1, \ldots, r$ is discontinuous only in a smooth surface S_i defined by $\varphi_i(x) = 0$. For each point x of discontinuity of the function u_i a closed set $U_i(x)$ must be given, defining a set of possible values of u_i. For $i \neq j$ it is supposed that u_i and u_j vary independently in the sets $U_i(x)$ and $U_j(x)$, respectively. $U_i(x)$ must contain all the limit points for any sequence with the form $v_k \in U_i\left(x_k\right)$, where $x_k \to x$, $k = 1, 2, \ldots$. Let us define the set $F_1(x)$ as

$$F_1(x) = f\left(x, U_1\left(x\right), \ldots, U_r\left(x\right)\right). \tag{10}$$

Therefore, a solution of (9) is a solution of the differential inclusion

$$\dot{x} \in F_1(x). \tag{11}$$

Let us consider now the system

$$\dot{x} = f\left(x, u_1^{eq}\left(x\right), \ldots, u_m^{eq}\left(x\right), u_{m+1}(x), \ldots, u_r(x)\right), \tag{12}$$

where u_1^{eq}, $\ldots$, u_m^{eq}, called equivalent controls, are such that the vector field f in (12) is tangent to the surfaces S_1, $\ldots$, S_m $(1 \leq m \leq r)$ and $u_i^{eq}(x) \in \left[u_i^-(x), u_i^+(x)\right]$, where $u_i^{\pm}(x)$ are limit values of u_i in both sides of the surface S_i, $i = 1, \ldots, m$. The functions $u_i^{eq}(x)$, $i = 1, \ldots, m$, are determined by the equations

$$\nabla\varphi_i(x) \cdot f\left(x, u_1^{eq}, \ldots, u_m^{eq}, u_{m+1}, \ldots, u_r\right) = 0$$

for $i = 1, \ldots, m$. A solution is an absolutely continuous function that, out of the surfaces S_i satisfies (9), and on the surfaces and their intersections satisfies (12) (Filippov, 1988).

We consider systems such that f is linear in $u_1, \ldots, u_r$, and where all the surfaces S_i are different and such that at the intersection points the normal vectors to these surfaces are linearly independent. Therefore, the sets F_0 and F_1 and the two definitions coincide (Filippov, 1988).

Now consider the system (9) such that the all the vectors $p_i = \nabla\varphi_i(x)$ are linearly independent for all $x \in S$. Therefore, near the surfaces $S_1, \ldots, S_m$ and their intersection S, (9) has the form

$$\dot{x} = f_0(x) + B_0(x)u(x), \tag{13}$$

where u is the vector $\left(u_1, \ldots, u_m\right)^T$, $1 \leq m \leq r$, and f_0 and the $(n \times m)$-matrix B_0 are clearly identified. The other functions $u_{m+1}, \ldots, u_r$ are continuous in S.

Let us consider the matrix G such that its rows are given by $p_i = \nabla\varphi_i(x)$. Therefore, if GB_0 is not singular, the motion along the intersection of $S_1, \ldots, S_m$, is given by $G\dot{x} = G\left(f_0 + B_0u^{eq}\right) = 0$; then

$$u^{eq}(x) = -\left[G(x)B_0(x)\right]^{-1} G(x)f_0(x). \tag{14}$$

If each component of the vector u^{eq} satisfies

$$u_i^-(x) \leq u_i^{eq}(x) \leq u_i^+(x),\ (i = 1, \ldots m), \tag{15}$$
$$\left\{\text{or } u_i^+(x) \leq u_i^{eq}(x) \leq u_i^-(x)\right\},$$

then a sliding motion is produced along S and the velocity vector of this motion is given by

$$\dot{x} = f_0(x) - B_0(x)\left[G(x)B_0(x)\right]^{-1} G(x)f_0(x). \tag{16}$$

An important fact is that, if at least one inequality of (15) is not satisfied, then there is not (sliding) motion along the surface S.

3. DISCONTINUOUS CONTROL OF M-DOF ROBOTIC MANIPULATORS

Let us consider an m-DOF robotic manipulator, described by the classical equation

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + F(\dot{q}) = \tau, \tag{17}$$

where $q \in \mathbb{R}^m$ is the position, $\tau \in \mathbb{R}^m$ is the control input, $M = M^T > 0$, C, and g are matrices with smooth inputs and proper dimensions, denoting the classical, physical meaning (Spong and Vidyasagar, 1989). Furthermore, the friction force

$$F(\dot{q}) = F_v\dot{q} + F_c\text{sgn}\left(\dot{q}\right) \tag{18}$$

is composed by viscous and Coulomb friction terms, $\text{sgn}\left(\dot{q}\right) = \left(\text{sgn}\left(\dot{q}_1\right), \ldots, \text{sgn}\left(\dot{q}_n\right)\right)^T$, and F_v, F_c are diagonal, positive definite matrices $F_v = \text{diag}\left\{f_{v_i}\right\}_{i=1}^m$, $F_c = \text{diag}\left\{f_{c_i}\right\}_{i=1}^m$. Note that $\dot{q}^T F\left(\dot{q}\right) > 0$ for all $\dot{q} \neq 0$.

Denote by $\tilde{q} = q - q_d$ the position error, where q_d is a constant, desired position. System (17) can then be expressed as

$$\begin{pmatrix} \dot{\tilde{q}} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ M^{-1}(q)\left[\tau - C\left(q,\dot{q}\right)\dot{q} - g(q) - F_v\dot{q}\right] \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(q) \end{pmatrix}\left[-F_c\text{sgn}\left(\dot{q}\right)\right], \tag{19}$$

where $q = \tilde{q} + q_d$.

Suppose that the system is not controlled, that is, $\tau = 0$. Then system (19) has the form $\dot{x} = f(x) + B(x)u(x)$, where $x = (\tilde{q}^T, \dot{q}^T)^T$ and f, B, and u are clearly identified. The functions $u_i = -f_{c_i}\text{sgn}(\dot{q}_i)$ are discontinuous on the surfaces S_i defined by $\varphi_i(x) = \dot{q}_i = 0$, $i = 1, \ldots, m$. Then $p_i = \nabla\varphi_i(x) = e_{m+i}$, where $e_j = (0, \ldots, 1, 0, \ldots, 0)^T$, with the "1" placed at the j-th position, and $\{p_1, \ldots, p_m\}$ is a linearly independent set of vectors in the intersection surface S of the discontinuity surfaces $S_1, \ldots, S_m$. It is then possible to obtain an expression for u^{eq}.

Let us consider the positive definite function

$$V(\tilde{q}, \dot{q}) = U(q) - U(q_d) + \frac{1}{2}\dot{q}^T M(q)\dot{q}, \tag{20}$$

where $U(q)$ is the potential energy of (17) such that $g(q) = \partial U(q)/\partial q$. The derivative $\dot{V}$ along the trajectories of the discontinuous system is given by

$$\dot{V}(\tilde{q},\dot{q}) = \dot{q}^T \frac{\partial U}{\partial q} + \dot{q}^T\left[-C(q,\dot{q})\dot{q} - g(q) - F(\dot{q})\right] + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q}.$$

Since $\dot{q}^T\left(\dot{M}/2 - C\right)\dot{q} = 0$ (Spong and Vidyasagar, 1989), then $\dot{V}(\tilde{q},\dot{q}) = -\dot{q}^T F(\dot{q})$, which is negative for any $\dot{q} \neq 0$ and $\dot{V}(0,0) = 0$, which just defines the discontinuity surfaces S_i. We now have the conditions to apply theorem 2.

To find the largest invariant set contained in $\mathcal{M} = \left\{(\tilde{q},\dot{q}) \in \mathbb{R}^{2m} : \dot{V}(\tilde{q},\dot{q}) = 0\right\} = \left\{(\tilde{q},0) \in \mathbb{R}^{2m}\right\}$ we use the equivalent control approach. From (14) and (19) we find that

$$u^{eq} = g(q). \tag{21}$$

Now from (15) and (21) we conclude that there will be a sliding motion in the intersection S of the discontinuity surfaces S_i if all the inequalities

$$|g_i(q)| \leq f_{c_i},\ (i = 1,\ldots,m) \tag{22}$$

are satisfied. In this case the largest invariant set is given by $\Omega = \left\{(\tilde{q},0) \in \mathbb{R}^{2m}\right\}$ and the system trajectories converge to this surface.

Example 5. Let us consider the simple pendulum

$$\begin{pmatrix}\dot{\tilde{q}}\\ \ddot{q}\end{pmatrix} = \begin{pmatrix}\dot{q}\\ -\sin q - f_v\dot{q}\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}\left[-f_c \mathrm{sgn}\left(\dot{q}\right)\right]$$

From (21-22), the largest invariant set contained in $\mathcal{M}$ is given by $\Omega = \left\{(\tilde{q},\dot{q}) \in \mathbb{R}^2 : \dot{q} = 0\right\} = \mathcal{M}$ if $|\sin(\tilde{q} + q_d)| \leq f_c$. In this case, the trajectories converge to Ω, if not, then $\Omega = \{(0,0)\}$, which is then an asymptotically stable equilibrium point. Figure 1 shows the phase portraits for $f_v = 0.5$, and some values of f_c.

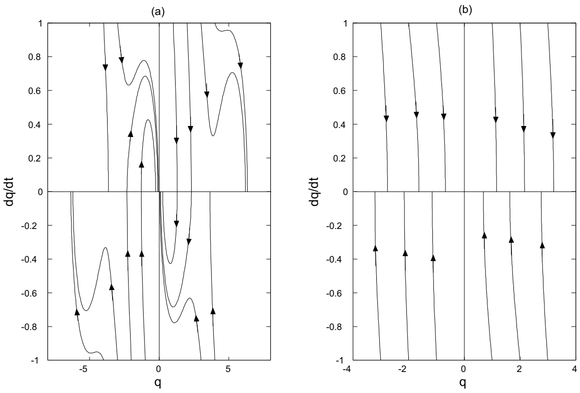

Fig. 1. Phase portraits of the uncontrolled pendulum with viscous and Coulomb friction. (a) $f_c = 0.5$. (b) $f_c = 2$.

Now consider a control law with the form

$$\tau = k_g g(q) - \partial V_1(\tilde{q}),$$

where $k_g \in \{0,1\}$ is a constant used to take or not into account the gravity compensation, V_1 is a positive definite scalar function, and ∂ denotes the generalized gradient. Note that this is a static control law that depends only on the position. Then system (17) acquires the form

$$\begin{pmatrix}\dot{\tilde{q}}\\ \ddot{q}\end{pmatrix} = \begin{pmatrix}\dot{q}\\ M^{-1}\left[(k_g - 1)g - \partial V_1 - C\dot{q} - F_v\dot{q}\right]\end{pmatrix} + \begin{pmatrix}0\\ M^{-1}(q)\end{pmatrix}\left[-F_c \mathrm{sgn}\left(\dot{q}\right)\right], \tag{23}$$

A candidate Lyapunov function can be given by

$$V = V_1(\tilde{q}) + (1 - k_g)\left[U(q) - U(q_d)\right] + \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

whose derivative $\dot{V}$ along the system trajectories is (remember that $g = \partial U$) $\dot{V} = -\dot{q}^T F\dot{q} \leq 0$, for which we can apply theorem 2. Because the system converges to the set $\left\{(\tilde{q},0) \in \mathbb{R}^{2m}\right\}$, we must find the largest invariant set in there. For system (23) the equivalent control is $u^{eq} = (1 - k_g)g(q) + \partial V_1(\tilde{q})$. Therefore, there exists a sliding motion in this set if

$$|(1 - k_g)g_i(q) + \partial V_{1_i}(\tilde{q})| \leq f_{c_i}$$

for all $i = 1,\ldots m$. In this case the system will converge to a point $(\tilde{q},0)$. If these inequalities are not satisfied, then the system will converge to $(0,0)$.

Example 6. Suppose that $V_1(\tilde{q}) = \sum_{i=1}^{m} k_{c_i}|\tilde{q}_i|$, and that there is no gravity compensation, that is, $k_g = 0$. Therefore, the system will converge to the origin if $|g_i(q) + k_{c_i}\mathrm{sgn}\left(\tilde{q}_i\right)| > f_{c_i}$, which is satisfied depending on the gravity term g. In this case the control input has the form $\tau = -K_c \mathrm{sgn}\left(\tilde{q}\right)$. Figure 2 shows some time responses for this case (solid lines).

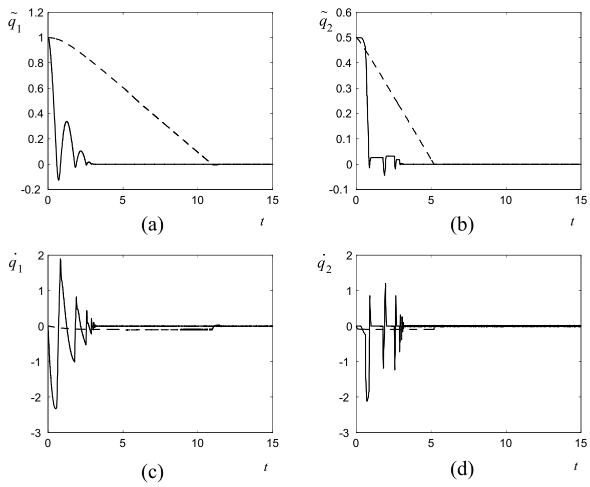

Fig. 2. Typical response of a 2-DOF robot with Coulomb friction controlled with $\tau = -K_c\mathrm{sgn}(\tilde{q})$ (solid lines) and with gravity compensation $\tau = g(q) - K_c\mathrm{sgn}(\tilde{q})$ (dashed lines).

Now consider a gravity term in the control ($k_g = 1$). The origin will be asymptotically stable if, for some i, $k_{c_i} > f_{c_i}$ is satisfied. The same figure 2 shows some responses obtained for this case (dashed lines).

Other choices of V_1 can give a better response, for example $V_1 = \sum_{i=1}^m k_{c_i} |\tilde{q}_i| + \tilde{q}^T K_p \tilde{q} + \int \tilde{q}^T K_I \tilde{q}\, dt$, where K_p, $K_I > 0$.

4. A DISCONTINUOUS OBSERVER

Although the control law proposed in the previous section can stabilize the system around a point $(q_d, 0)$, a better performance can be attained using also velocity feedback. However, if this variable is not available, an observer must be included. In this section we propose an observer for robot manipulators with Coulomb friction, and prove that the connection controller-observer leads to an asymptotically stable behavior of the system.

Let us suppose that the only measured variable is the position, that is, $y = q \in \mathbb{R}^m$, and consider the following model,

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + h_D \tilde{y}, \\ \dot{\hat{x}}_2 &= M^{-1}(y)[-C(y, \dot{\hat{x}}_1)\dot{\hat{x}}_1 - g(y) - F(\dot{\hat{x}}_1) \\ &\quad + u + H_P \tilde{y} + K_{cc}\mathrm{sgn}(\tilde{y})], \\ \hat{y} &= \hat{x}_1, \end{aligned} \tag{24}$$

where $h_D > 0$ is a constant, $H_P > 0$ is a matrix, $K_{cc} = \mathrm{diag}\{k_{cc_i}\}_{i=1}^m > 0$ is a diagonal matrix, and $\tilde{y} = y - \hat{y} = q - \hat{x}_1$ is the output estimation error, whose dynamics are given by

$$\begin{aligned} M\ddot{\tilde{y}} + C(y,\dot{y})\dot{\tilde{y}} + F_v\dot{\tilde{y}} &= -H_P\tilde{y} - h_D M\dot{\tilde{y}} \\ -C(y,\dot{\tilde{y}})\dot{\tilde{y}} - F_c &\left[\mathrm{sgn}(\dot{y}) - \mathrm{sgn}(\dot{\hat{y}})\right] - K_{cc}\mathrm{sgn}(\tilde{y}), \end{aligned} \tag{25}$$

where we have used the fact that $C(x, y)z = C(x, z)y$ (Spong and Vidyasagar, 1989).

Let $x = (\tilde{q}^T, \dot{y}^T, \tilde{y}^T, \dot{\tilde{y}}^T)^T$ be the state of the overall system (plant + observer). Then this system is described by

$$\begin{aligned} \dot{\tilde{q}} &= \dot{y}, \\ \ddot{y} &= M^{-1}(y)\left[-C(y,\dot{y})\dot{y} - g(y) - F(\dot{y}) + u\right], \\ \dot{\tilde{y}} &= \dot{\tilde{y}}, \\ \ddot{\tilde{y}} &= M^{-1}(y)[-C(y,\dot{y})\dot{\tilde{y}} - C(y,\dot{\tilde{y}})\dot{\tilde{y}} - F_v\dot{\tilde{y}} \\ &\quad -H_P\tilde{y} - h_D M(y)\dot{\tilde{y}} - F_c\mathrm{sgn}(\dot{y}) \\ &\quad +F_c\mathrm{sgn}(\dot{\hat{y}}) - K_{cc}\mathrm{sgn}(\tilde{y})]. \end{aligned} \tag{26}$$

Let us consider the control law

$$u = g(q) - K_c\mathrm{sgn}(\tilde{q}) - K_p\tilde{q} - K_d\dot{\hat{y}}, \tag{27}$$

where $K_c = \mathrm{diag}\{k_{c_i}\}_{i=1}^m > 0$ is a diagonal matrix such that $K_c > F_c$, and $K_p = \mathrm{diag}\{k_{p_i}\}_{i=1}^m \geq 0$ is also a diagonal matrix. Let us also suppose that the friction coefficients F_v and F_c are not exactly known, such that in the observer we use an approximation F_{va} and F_{ca}. Then the overall controlled system is given by

$$\begin{aligned} \dot{\tilde{q}} &= \dot{y}, \\ \ddot{y} &= M^{-1}(y)[-C(y,\dot{y})\dot{y} - F(\dot{y}) - K_c\mathrm{sgn}(\tilde{q}) \\ &\quad -K_p\tilde{q} - K_d(\dot{y} - \dot{\tilde{y}})], \\ \dot{\tilde{y}} &= \dot{\tilde{y}}, \\ \ddot{\tilde{y}} &= M^{-1}(y)[-C(y,\dot{y})\dot{\tilde{y}} - C(y,\dot{\tilde{y}})\dot{\tilde{y}} - H_P\tilde{y} \\ &\quad -h_D M(y)\dot{\tilde{y}} - F(\dot{y}) + F_a(\dot{\hat{y}}) - K_{cc}\mathrm{sgn}(\tilde{y})], \end{aligned} \tag{28}$$

where $F_a(\dot{\hat{y}}) = F_{va}\dot{\hat{y}} + F_{ca}\mathrm{sgn}(\dot{\hat{y}})$, $F_{ja} = \mathrm{diag}\{f_{ja_i}\}_{i=1}^m$, $j \in \{\text{“}v\text{”}, \text{“}c\text{”}\}$, are diagonal matrices such that $F_{ja} > F_j$. A candidate Lyapunov function for this system is

$$V(x) = \tilde{q}^T K_c\mathrm{sgn}(\tilde{q}) + \tilde{y}^T K_{cc}\mathrm{sgn}(\tilde{y}) + \frac{1}{2}x^T B(q)x, \tag{29}$$

where $B(q) = \mathrm{diag}\{K_p, M(y), H_P, M(y)\} \geq 0$ is a block-diagonal matrix. It is not difficult to show that the time-derivative of V along the trajectories of the overall system satisfies

$$\begin{aligned} \dot{V} \leq -\dot{q}^T &\left[F(\dot{q}) + K_d(\dot{y} - \dot{\tilde{y}})\right] - \dot{\tilde{y}}^T\left[F(\dot{q}) - F_a(\dot{\hat{y}})\right] \\ &-h_D\dot{\tilde{y}}^T M(q)\dot{\tilde{y}} + k_q(\|\dot{q}\|_2 + \|\dot{\tilde{y}}\|)\left\|\dot{\tilde{y}}\right\|_2^2, \end{aligned} \tag{30}$$

where $k_q = \max_{q\in\mathbb{R}^m}\sum_{k=1}^m \|C_k(q)\|_2/2$, the elements of matrices C_k being given by

$$C_{k,ij}(q) = \frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}$$

for $i, j, k = 1, \ldots, m$. M_{ij} are the components of matrix M (Spong and Vidyasagar, 1989). We also used the fact that $\dot{\tilde{y}}^T F_c\left[\mathrm{sgn}(\dot{\hat{y}}) - \mathrm{sgn}(\dot{q})\right] \leq 0$.

The right side of (30) is non-positive if

$$f_{c_i} > \frac{f_{ca_i}}{2},$$

$$(F_v + K_d)_m > \frac{(F_{vs} + K_d)_M^2}{4F_{va_m}},$$

$$\frac{h_D M_m}{k_q} > \|\dot{q}\|_2 + \|\dot{\tilde{y}}\|_2,$$

where $A_{m(M)}$ is the minimum (maximum) eigenvalue of matrix A. An analysis similar to those presented in the previous sections leads to conclude that the point $(\tilde{q}, \dot{y}, \tilde{y}, \dot{\tilde{y}}) = 0$ is a global and asymptotically stable equilibrium point of this system. Figure 3 shows a typical response obtained for a 2-DOF robot manipulator controlled with (27) and using the observed proposed above with a mismatch of about $+20\%$ of the nominal values between the model and the observer parameters (dashed lines). This figure also shows the results when the exact velocity is used in the controller (solid lines). Note that there is no significant difference between the two responses. Finally, figure 4 shows the response of the same system and the same control law with the observer included ($K_d > 0$, solid lines) and without the estimation of velocity ($K_d = 0$, dashed lines). Observe that the closed-loop system displays a better response with the observer. However, in both cases the origin is asymptotically stabilized.

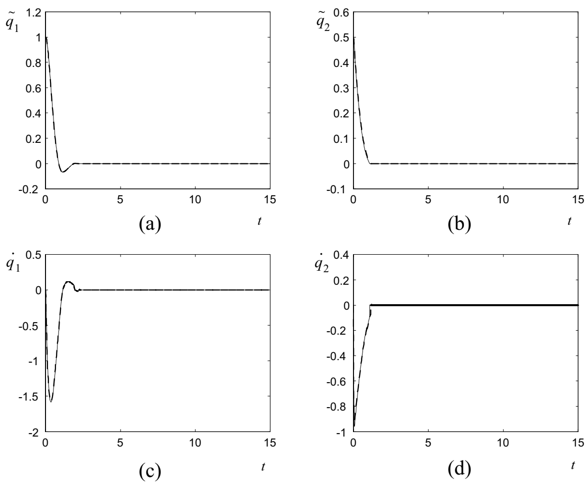

Fig. 3. Response with the exact velocity (solid lines) and with the observed velocity (dashed lines) of a 2-DOF robot with Coulomb friction fedback with the control $u = g(q) - K_c \mathrm{sgn}(\tilde{q}) - K_p \tilde{q} - K_d \dot{\hat{y}}$.

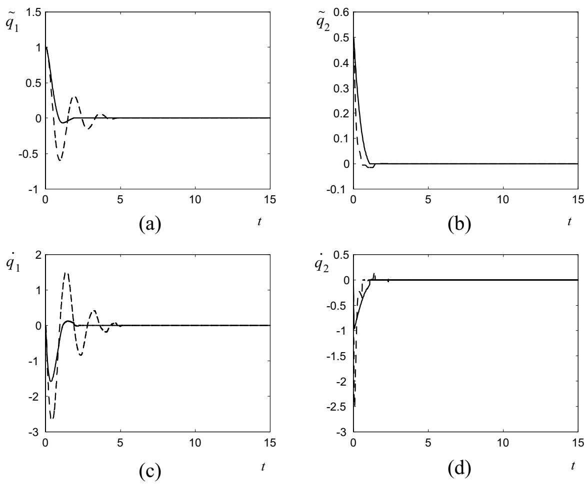

Fig. 4. Response of a 2-DOF robot with Coulomb friction, fedback with the control law $u = g(q) - K_c \mathrm{sgn}(\tilde{q}) - K_p \tilde{q} - K_d \dot{\hat{y}}$, with the observer included ($K_d > 0$, solid lines) and without the observer ($K_d = 0$, dashed lines).

5. CONCLUSIONS

Several discontinuous controllers for the regulation of robotic manipulators with Coulomb friction have been proposed. The design is based on the theory of discontinuous differential equations developed by Filippov and an invariance principle recently established. The proposed controllers make use only of position measurements, and to improve the performance of the closed-loop system a discontinuous observer has also been developed. The stability of the overall structure has been shown, as well as its robustness with respect to parameter mismatches. Like all the discontinuous controllers, those presented here exhibit chattering in the steady state. This behavior can be removed by approximating the sign functions with some continuous functions at the price of having a steady position error. This smoothing action can be made on the control signal, but it is not needed in the observer if this model is implemented in a digital computer.

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