15th Triennial World Congress, Barcelona, Spain

ON NONLINEAR RLC NETWORKS: PORT-CONTROLLED HAMILTONIAN SYSTEMS DUALIZE THE BRAYTON-MOSER EQUATIONS

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"Black fish blue fish old fish new fish." *(Dr. Seuss, 1960)*

Abstract: In this paper it is shown that the recently proposed port-controlled Hamiltonian systems with dissipation precisely dualize the classical Brayton-Moser equations. As a consequence, useful and important properties of the one framework can be translated to the other. For both frameworks a novel method is proposed to deal with networks containing capacitor-only loops or inductor-only cutsets using the Lagrange multiplier. This leads to the notion of implicit Brayton-Moser equations. Furthermore, the form and existence of the mixed-potential function is rederived from an external port point of view.

Keywords: Physical models, Hamiltonian systems, Brayton-Moser equations, passive elements, electrical networks.

1. INTRODUCTION

From the early sixties until the early eighties many researchers have concentrated on the development of systematic tools for the formulation of the dynamic behavior of nonlinear electrical circuits. Most of these works have in common that the methods are based on the use of the energy and the topological properties of the system. Pioneering results where reported by e.g. Brayton and Moser (Brayton and Moser, 1964) and MacFarlane (MacFarlane, 1970). Their method is mainly based on the definition of some mixed-potential function. Another approach was considered by Chua and McPherson (Chua and McPherson, 1974). Their method used the classical Lagrangian framework, but the choice of coordinates departed radically from conventional thinking. Almost a decade later, in (Kwatny *et al.*, 1982) a generalized Lagrangian framework is proposed in which some severe limitations of the previous methods are relaxed. After this period the area became relatively quiet, until recently with the introduction of port-controlled Hamiltonian (PCH) systems (Schaft, 2000) and Lagrangian modeling of power electronic systems in (Ortega *et al.*, 1998) and (Scherpen *et al.*, 2000). In the context of switched-mode systems it is shown that the dynamics, correspond to systems derivable from a Lagrangian or port-controlled Hamiltonian point of view. This brings the advantage that control techniques, like passivity-based control, can be successfully applied to such circuits.

In this paper we will concentrate on two specific formulations: the Brayton-Moser equations and PCH systems. In view of its practical applications related to controller design, we want to establish a connection between the two formalisms and discuss their advantages and disadvantages. The most trivial duality between the two frameworks is that PCH systems assume the circuit elements to be flux and charge controlled only, while the Brayton-Moser equation impose the restriction that the elements are current or voltage controlled. If the frameworks are used to design feedback controllers, the controller will consequently rely on some output or state measurements, i.e., measurements of fluxes and charges or currents and voltages. In a practical situation the off-the-shelf available sensors give as output the measurements in terms of current or voltage quantities only. In the linear case the relation between flux and current or charge and voltage is a static one, but if a system contains highly nonlinear elements complicated state transformations have to be included or quality degrading approximations have to be made. Since in general the elements may not have bijective relations, even more serious problems may arise.

One reason to work with PCH systems is that the dynamic equations are formulated in physical or 'natural' variables. In case of autonomous LC circuits this can be considered a reasonable argument, but, on the other hand, the inclusion of converter elements, like sources and resistors seems not so natural in the PCH framework. In principal, the constitutive relations of controlled voltage sources, current sources and dissipative elements are rather considered in terms of currents or voltages, instead of fluxes or charges, see e.g. (MacFarlane, 1970). It seems then to be more natural to choose for the Brayton-Moser formalism. Therefore, it is of interest to study if there exists some fundamental relation, in a mathematical sense, between both frameworks. Indeed, as will be shown throughout the paper, under some reasonable assumptions such a relation exists. As a consequence, essential and important properties of one framework can be translated to the other.

The paper is organized as follows. In Section 2, we briefly recall the definition of the Brayton-Moser equations and PCH systems. In Section 3, the connections between both frameworks are first established for autonomous LC circuits. Section 4 deals with the concept of implicit PCH systems. This concept is then to be translated to the Brayton-Moser equations. As a result, we provide a novel procedure to obtain minimal state space representations of circuits containing inductor-only cutsets and/or capacitor-only loops. The section ends with an illustrative example. Finally, we derive the dissipation and power-supply part of the mixed-potential function for the Brayton-Moser equations from an PCH external port point of view. Both frameworks are compared in the presence of power sources and resistive elements.

2. THE BRAYTON-MOSER AND THE PCH EQUATIONS

In this section we briefly recall both the concept of PCH systems and the Brayton-Moser equations. Consider a, possibly nonlinear, electrical network Σ consisting of ρ capacitors and σ inductors. We start by restricting the discussion to networks without elements in excess, i.e., we do not admit inductor-only cutsets and capacitor-only loops. This condition will be relaxed in Section 4. The order of the network equals $n = \rho + \sigma$. Under the assumption that the inductors are current controlled and that the capacitors are voltage controlled, Brayton and Moser (Brayton and Moser, 1964) have shown the dynamical behavior of such circuit is governed by the following system of differential equations

$$\mathcal{M}^*(x)\dot{x} = \Upsilon \frac{\partial P(x)}{\partial x}, \qquad (1)$$

where $x = [v_{C_1}, \ldots, v_{C_\rho}, i_{L_1}, \ldots, i_{L_\sigma}]^T$ denote the inductor currents and capacitor voltages, respectively, $\Upsilon = \text{diag}\{-I_{\rho\times\rho}, I_{\sigma\times\sigma}\}$, and $\mathcal{M}^*(x) = \text{diag}\{\mathcal{C}(v_C), \mathcal{L}(i_L)\}$ contains the capacitance and inductance matrices. The scalar function $P(x)$ is called the mixed-potential function to be specified later. As stated in (Kwatny *et al.*, 1982), equations (1) does not establish a Lagrangian system in the classical sense, but it can be viewed as some degenerate Lagrangian form. On the other hand, in (Weiss and Mathis, 1997) it is shown that (1) corresponds to a classical Lagrangian or Hamiltonian system, but their generalized coordinates do not correspond to physical intuition, since they include inductor charges and capacitor fluxes. However, if we now assume that the inductors, resistors and voltage sources are flux controlled, and the capacitors, conductivities and current sources are charge controlled, the port-controlled Hamiltonian system with Dissipation (PCHD) is represented by the following equation (Schaft, 2000)

$$\dot{y} = [\mathcal{J}(y) - \mathcal{D}(y)]\frac{\partial H(y)}{\partial y} + \mathcal{F}(y) \qquad (2)$$

Here $y = [q_{C_1}, \ldots, q_{C_\rho}, \varphi_{L_1}, \ldots, \varphi_{L_\sigma}]^T \in \mathbb{R}^n$, denotes the capacitor charges and the inductor fluxes, respectively, the scalar function $H(y)$ is the total stored energy in the circuit, called the *Hamiltonian*, and the $n \times n$ structure matrix $\mathcal{J}(y)$ is a Dirac structure associated with the circuit topology. Since $\mathcal{J}(y)$ is a power-preserving internal interconnection structure, it is easily checked that $\mathcal{J}$ satisfies the important property

$$\mathcal{J}(y) = -\mathcal{J}^T(y), \text{ (skew-symmetry)}. \qquad (3)$$

For electrical circuits without switches $\mathcal{J}(y)$ is usually a constant matrix. For that, we set $\mathcal{J}(y) = \mathcal{J}$. Finally, the vector $\mathcal{F}(y)$ represents the external sources, and the matrix $\mathcal{D}(y)$ is a positive semi-definite symmetric matrix containing the values of the resistive and conductive elements.

In order to be able to relate the Brayton-Moser equations and the PCH framework, we impose the assumption that all capacitors can be both voltage or charge controlled, and that all inductors can be both flux or current controlled, i.e.,

*Assumption 1.*Throughout the document it is assumed that all dynamic elements have bijective relations, i.e.,

$$q_{C_k} = \mathfrak{C}_k^*(v_C) \leftrightarrow v_{C_k} = \mathfrak{C}_k(q_C),\ k = 1, \ldots, \rho$$
$$\varphi_{L_j} = \mathfrak{L}_j^*(i_L) \leftrightarrow i_{L_j} = \mathfrak{L}_j(\varphi_L),\ j = 1, \ldots, \sigma$$

where $\mathfrak{C}_k^*$ and $\mathfrak{L}_j^*$ are smooth functions $\mathfrak{C}_k^* : \mathbb{R} \to \mathbb{R}_k$ and $\mathfrak{L}_j^* : \mathbb{R} \to \mathbb{R}_j$, respectively.

Both frameworks impose similar inherent limitations, i.e., the PCH framework assumes the inductors to be only flux-controlled and the capacitors to be only charge-controlled, while the Brayton-Moser equations are restricted to current and voltage controlled elements. In the following section we will show that the frameworks bear an interesting similarity in structure. The analysis is first carried out for nonlinear autonomous LC circuits.

3. AUTONOMOUS LC CIRCUITS

In the present study, our perspective is to view the capacitor charges, q_{C_k}, and the inductor fluxes, φ_{L_j}, as what we call, the *energy* variables, and the capacitor voltages, v_{C_k}, and the inductor currents, i_{L_j}, as the *co-energy* (or power) variables. The relation between the energy and co-energy variables is given by

$$\frac{dq_C}{dt} = \psi^T i_L, \quad \frac{d\varphi_L}{dt} = -\psi v_C \qquad (4)$$

where $q_C = [q_{C_1}, \dots, q_{C_\rho}]^T$, $\varphi_L = [\varphi_{L_1}, \dots, \varphi_{L_\sigma}]^T$, etc., and ψ is a constant matrix of appropriate dimensions. Note that (4) constitutes Kirchhoff's current and voltage laws, where we used the properties $\dot{q}_C = i_C$ and $\dot{\varphi}_L = v_L$. For autonomous LC circuits, the Brayton-Moser equations (1) can be written as

$$\frac{d}{dt}\left[\frac{\partial H^*(x)}{\partial x}\right] = \Upsilon \frac{\partial P_T(x)}{\partial x}, \tag{5}$$

where $H^*(x)$ is the sum of the total magnetic and electric co-energy, i.e., $H^*(x) = H^*(v_C, i_L) = V^*(v_C) + T^*(i_L)$, with $V^*(v_C)$ and $T^*(i_L)$ smooth functions $V^*(v_C) : \mathbb{R}^\rho \to \mathbb{R}$ and $T^*(i_L) : \mathbb{R}^\sigma \to \mathbb{R}$ defined as

$$V^*(v_C) := \sum_{k=1}^{\rho} \int_0^{v_{C_k}} \mathfrak{C}_k^*(v'_{C_k}) dv'_{C_k}$$

$$T^*(i_L) := \sum_{j=1}^{\sigma} \int_0^{i_{L_j}} \mathfrak{L}_j^*(i'_{L_j}) di'_{L_j},$$

respectively. At this point it is interesting to remark that in theoretical mechanics $H^*(x)$ is often referred to as the *co-Hamiltonian*. The scalar function $P_T(x)$ defines the potential forces, which denote the rate of power in the circuit. A very interesting property is given in the following lemma

Lemma 1. Given the Brayton-Moser equations (5), then the gradient of the total rate of power stored in the capacitors and the inductors, $P_C(x)$ and $P_L(x)$, satisfies the following relation

$$\Upsilon \frac{\partial^2 P_T(x)}{\partial x^2} = \begin{bmatrix} 0_{\rho\times\rho} & \psi^T \\ -\psi & 0_{\sigma\times\sigma} \end{bmatrix}, \tag{6}$$

with ψ a constant $\sigma \times \rho$ matrix as defined in (4) and $P_T(x) = P_L(x) = -P_C(x)$. Hence, (6) defines a power-preserving relation between the energy variables and the power variables called a Dirac structure.

Proof: The proof is mainly based on a straight forward application of Tellegen's Theorem. Consider the rate of power conserved in the inductors defined as $P_L(v_C, i_L) = \sum_{j=1}^{\sigma} i_{L_j} v_{L_j} = i_L^T v_L$. From (4) we know that $v_L = -\psi v_C$ and therefore that $i_L^T v_L = -i_L^T \psi v_C$. In a similar fashion, the rate of power conserved in the capacitors, defined as $P_C(v_C, i_L) = \sum_{k=1}^{\rho} v_{C_k} i_{C_k} = v_C^T i_C$, can be written as $v_C^T i_C = v_C^T \psi^T i_L$, from which we conclude that $P_T(v_C, i_L) = P_C(v_C, i_L) = -P_L(v_C, i_L)$. Calculation of the second partial derivatives yields the result. ■

In these notations, the dynamics of the network can be represented as a set of first order equations given by

$$\mathcal{M}^*(x)\dot{x} + \mathcal{J}^T x = 0, \tag{7}$$

where $\mathcal{J}$ denotes the interconnection (Dirac) structure defined in (6), i.e.,

$$\mathcal{J} := \Upsilon \frac{\partial^2 P_T(x)}{\partial x^2}, \tag{8}$$

and $\mathcal{M}^*(x) := \frac{\partial^2 H^*(x)}{\partial x^2}$ is a positive definite $n \times n$ matrix referred to as the co-energy matrix. In order to establish the connection between both frameworks we have to rewrite (5) in terms of the 'natural' energy variables q_C and φ_L. Before we continue, let us first study the structure of the co-energy matrix $\mathcal{M}^*(x)$. It is easily checked that if the circuit contains no magnetically coupled-inductors, no inductor-only cutsets, and no capacitor-only loops, $\mathcal{M}^*(x)$ is a diagonal matrix, and thus symmetric. If, on the other hand, there exists a coupling between one or more inductors, additional paths for the energy transfer are introduced which do not contribute extra current coordinates to the formulation. Under the assumption that the coupling coefficients satisfy the reciprocity condition, i.e,

$$\frac{\partial \mathfrak{L}_j^*(i_L)}{\partial i_{L_k}} = \frac{\partial \mathfrak{L}_k^*(i_L)}{\partial i_{L_j}},\ j,k = 1,\dots,\sigma,\ j \neq k,$$

we may conclude that $\mathcal{M}^*(x)$ remains symmetric. For a detailed discussion on the inclusion of coupled-magnetics, see e.g. (Scherpen *et al.*, 2000). We thus have the following property

$$\mathcal{M}^*(x) = \mathrm{diag}\left\{\frac{\partial \mathfrak{C}^*(v_C)}{\partial v_C}, \frac{\partial \mathfrak{L}^*(i_L)}{\partial i_L}\right\}.$$

Recall that $q_{C_k} = \mathfrak{C}_k^*(v_C)$ and $\varphi_{L_j} = \mathfrak{L}_j^*(i_L)$. Hence, by multiplying (5) with the interconnection structure (8) and by using Lemma 1, the Brayton-Moser equations can be written as

$$\psi^T \frac{di_L}{dt} = -\psi^T \left[\frac{\partial \mathfrak{L}^*(i_L)}{\partial i_L}\right]^{-1} \psi\, v_C \tag{9}$$

$$\psi \frac{dv_C}{dt} = \psi \left[\frac{\partial \mathfrak{C}^*(v_C)}{\partial v_C}\right]^{-1} \psi^T i_L \tag{10}$$

Since we assume the (q_C, v_C)- and (φ_L, i_L)-curves to be bijective,

$$\frac{\partial \mathfrak{C}(q_C)}{\partial q_C} := \left[\frac{\partial \mathfrak{C}^*(v_C)}{\partial v_C}\right]^{-1}, \text{ and}$$

$$\frac{\partial \mathfrak{L}(\varphi_L)}{\partial \varphi_L} := \left[\frac{\partial \mathfrak{L}^*(i_L)}{\partial i_L}\right]^{-1}$$

exist. Finally, by using (4), we obtain a system of second order equations in terms of the inductor fluxes and capacitor charges given by

$$\ddot{q}_C = \psi^T \frac{\partial \mathfrak{L}(\varphi_L)}{\partial \varphi_L} \dot{\varphi}_L \tag{11}$$

$$\ddot{\varphi}_L = -\psi \frac{\partial \mathfrak{C}(q_C)}{\partial q_C} \dot{q}_C. \tag{12}$$

The PCH equations (2) for autonomous LC circuits can be written in partitioned form as

$$\dot{q}_C = \psi^T \frac{\partial H(q_C, \varphi_L)}{\partial \varphi_L} \tag{13}$$

$$\dot{\varphi}_L = -\psi \frac{\partial H(q_C, \varphi_L)}{\partial q_C}. \tag{14}$$

Comparing (11) and (12) with the PCH equations (13) and (14), it is easily recognized that (9) and (10), in terms of the fluxes and charges, are precisely describing the time-derivative or 'lifted' version of the PCH equations. Thus, the port-controlled Hamiltonian equations can be obtained from the Brayton-Moser equations by a multiplication with a constant ψ and taking the integral with respect to time. Of course, the most direct relation is that both formalisms describe the Kirchhoff laws. Furthermore, the Hamiltonian, $H(q_C, \varphi)$ is defined as the sum of the total magnetic and electric energy in terms of q_C and φ_L, i.e., $H(q_C, \varphi_L) = V(q_C) + T(\varphi_L)$. As a direct consequence of Assumption 1, the relation between the co-Hamiltonian and the Hamiltonian is defined through a full Legendre transformation defined as follows. Let $y = [q_{C_1}, \ldots, q_{C_\rho}, \varphi_{L_1}, \ldots, \varphi_{L_\sigma}]^T$, with

$$y_j := \frac{\partial H^*(x)}{\partial x_j},\ j = 1, \ldots, n \tag{15}$$

then one defines the Hamiltonian function $H(y)$ as the Legendre transform, $x = \mathfrak{F}(y)$, of $H^*(x)$, i.e.,

$$\tilde{H}(y, x) = \sum_{j=1}^{\rho+\sigma} y_j x_j - H^*(x). \tag{16}$$

The Legendre transformation is then completed if we define the Hamiltonian as $H(y) := \tilde{H}(y, \mathfrak{F}(y))$, with $y = \mathcal{M}^*(x)x$. Since $\dot{y} = \mathcal{J}x$ and $H(y) = \frac{1}{2}y^T[\mathcal{M}^*(\mathfrak{F}(y))]^{-1}y$, we obtain the PCH dynamics in a similar fashion as in (7)

$$\dot{y} + \mathcal{J}^T\mathcal{M}(y)y = 0, \tag{17}$$

where $\mathcal{M}(y) := [\mathcal{M}^*(\mathfrak{F}(y))]^{-1}$ is a diagonal and positive definite symmetric $n \times n$ matrix referred to as the energy matrix. Hence, we have shown

Proposition 1. The port-controlled Hamiltonian equations (13) and (14), with the Hamiltonian being the total energy, dualize the Brayton-Moser equations given by (5), with the co-Hamiltonian expressing the total co-energy.

Consequently, equations (5) establish a port-controlled *co-Hamiltonian* (PCH*) framework with a Dirac structure given by the interconnection matrix $\mathcal{J}$.

4. EXCESS ELEMENTS: IMPLICIT SYSTEMS

In this section we extend the formulation to LC networks which contain capacitor-only loops and/or inductor-only cutsets. In (Kwatny *et al.*, 1982) it is stated that the excess elements do not contribute a generalized coordinate or velocity to the formulation, but they do contribute extra co-energy terms to their Lagrangian. Consequently, the order of the network is not simply $n = \rho + \sigma$. Although their method seems the most direct and simple one when dealing with Lagrangian dynamics, it becomes much more involved for Hamiltonian systems, especially when the constitutive relations are nonlinear. Here we propose an alternative method which is based on the introduction of implicit systems using Lagrange multipliers.

In the context of mechanical systems it is well-known (Schaft *et al.*, 1996) that the kinematic constraints can be expressed as $\mathcal{A}^T(q)\dot{q} = 0$, with $\dot{q} \in \mathbb{R}^m$ the vector of generalized velocities and $\mathcal{A}^T(q)$ some $m \times k$ matrix of rank k. The corresponding constraint forces are of the form $\mathcal{A}^T(q)\lambda$, where the Lagrange multipliers $\lambda \in \mathbb{R}^m$ are determined by the requirement that the constraints $\mathcal{A}^T(q)\dot{q} = 0$ need to be satisfied at all time. If we transform the latter properties to the electrical domain, the corresponding PCH equations become

$$\dot{\tilde{y}} = \tilde{\mathcal{J}}\frac{\partial H(\tilde{y})}{\partial \tilde{y}} + \mathcal{A}(\tilde{y})\lambda \tag{18}$$

$$0 = \mathcal{A}^T(\tilde{y})\frac{\partial H(\tilde{y})}{\partial \tilde{y}}, \tag{19}$$

where we denote $\tilde{y} \in \mathbb{R}^{\tilde{n}}$, $\tilde{\mathcal{J}}$, and $H(\tilde{y})$ as the *augmented*- energy variables, interconnection structure, and Hamiltonian, respectively. In other words, the constraint PCH equations are possibly non-minimal in the sense that certain energy variables have to be eliminated first to obtain a minimal representation of order $n \le \tilde{n} := \rho + \sigma$. In the electrical domain $\mathcal{A}$ is constant, i.e., $\mathcal{A}(\tilde{y}) = \mathcal{A} \in \mathbb{R}^{(\rho+\sigma)\times k}$, with k the number of independent constraints. Equation (18) together with (19) is often called an *implicit generalized* PCH system (Schaft *et al.*, 1996). In order to accommodate the use of Lagrange multipliers with the Brayton-Moser equations, (5) must be altered as follows

$$\frac{d}{dt}\left[\frac{\partial H^*(\tilde{x})}{\partial \tilde{x}}\right] = \Upsilon\frac{\partial \tilde{P}(\tilde{x})}{\partial \tilde{x}}, \tag{20}$$

where $\tilde{x} = [\tilde{v}_C^T, \tilde{i}_L^T,]^T \in \mathbb{R}^{\tilde{n}}$, $\tilde{P}(\tilde{x}) := -\tilde{v}_C^T\tilde{\psi}^T\tilde{i}_L - \tilde{v}_C^T\mathcal{A}_C\lambda_C + \tilde{i}_L^T\mathcal{A}_L\lambda_L$, $\lambda = [\lambda_C^T, \lambda_L^T]^T$ and

$$\mathcal{A}^T\tilde{x} := \begin{bmatrix} \mathcal{A}_C^T & 0 \\ 0 & \mathcal{A}_L^T \end{bmatrix}\begin{bmatrix} \tilde{v}_C \\ \tilde{i}_L \end{bmatrix}, \tag{21}$$

with $\mathcal{A}_C^T\tilde{v}_C = 0$ and $\mathcal{A}_L^T\tilde{i}_L = 0$. Following (Schaft *et al.*, 1996), we call (20) together with (21) an *implicit generalized* PCH* system. In case the network contains capacitor-only loops or inductor-only cutsets, the coordinates of the resulting elements in excess can be viewed as *intermediate* help-variables. These help-variables are finally removed using the constraint equation. Let us next demonstrate the procedure to obtain a minimal set of equations by studying a simple example.

Example: Consider the simple LC network as depicted in Fig. 1. The network is obtained from (Chua and

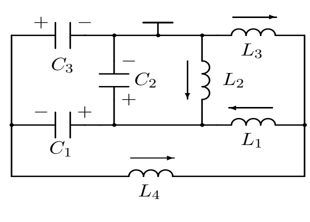

Fig. 1. LC circuit with elements in excess.

McPherson, 1974), but we have interchanged the inductor L_1 with capacitor C_3 and assume for simplicity that the constitutive relations are linear. It contains two excess elements; one arising from the inductor cutset formed by $\{L_1, L_3, L_4\}$, and one arising from the capacitor loop $\{C_1, C_2, C_3\}$. An appropriate choice is to take L_1 and C_3 as the excess elements. The corresponding co-Hamiltonian is defined as $H^*(v_C, i_L) = \frac{1}{2}\sum_{k=1}^{3} C_k v_{C_k}^2 + \frac{1}{2}\sum_{j=1}^{4} L_j i_{L_j}^2$. The algebraic constraints corresponding to the capacitor-only loop and the inductor-only cutset are given by $v_{C_1} - v_{C_2} + v_{C_3} = 0 \Rightarrow \mathcal{A}_C^T = [1\ -1\ 1]$ and $-i_{L_1} + i_{L_3} + i_{L_4} = 0 \Rightarrow \mathcal{A}_L^T = [-1\ 0\ 1\ 1]$. It follows that $\lambda \in \mathbb{R}^2$. Using Kirchhoff's current or voltage law, the matrix $\tilde{\psi}$ is readily found as

$$\tilde{\psi}^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Plugging the latter into (20) together with (21), and after eliminating λ_1, λ_2, we obtain the equations of motion in the form $\mathcal{M}^*\dot{x} + \mathcal{J}^T x = 0$ as

$$\begin{aligned} (C_1 + C_3)\frac{dv_{C_1}}{dt} - C_3\frac{dv_{C_3}}{dt} - i_{L_4} &= 0 \\ (C_2 + C_3)\frac{dv_{C_2}}{dt} - C_3\frac{dv_{C_1}}{dt} - i_{L_2} - i_{L_3} &= 0 \\ L_2\frac{di_{L_2}}{dt} + v_{C_2} &= 0 \\ (L_1 + L_3)\frac{di_{L_3}}{dt} + L_1\frac{di_{L_4}}{dt} + v_{C_2} &= 0 \\ (L_1 + L_4)\frac{di_{L_4}}{dt} + L_1\frac{di_{L_3}}{dt} + v_{C_1} &= 0 \end{aligned}$$

Next, we derive the dynamics of the LC circuit using the implicit port- controlled Hamiltonian description. The corresponding Hamiltonian is defined as $H(q_C, \varphi_L) = \frac{1}{2}\sum_{k=1}^{3} C_k^{-1} q_{C_k}^2 + \frac{1}{2}\sum_{j=1}^{4} L_j^{-1}\varphi_{L_j}^2$. The constraint matrices $\mathcal{A}_C$, $\mathcal{A}_L$, and the augmented interconnection structure $\tilde{\psi}$ are as before. According to (19), the constraints are now given by $\frac{q_{C_1}}{C_1} - \frac{q_{C_2}}{C_2} + \frac{q_{C_3}}{C_3} = 0$ and $-\frac{\varphi_{L_1}}{L_1} + \frac{\varphi_{L_3}}{L_3} + \frac{\varphi_{L_4}}{L_4} = 0$. Hence, after plugging the information in (18), using (19) and eliminating the Lagrange multipliers yields

$$\begin{aligned} \dot{q}_{C_1} &= \frac{C_1(C_3 + C_1)}{\Gamma}\frac{\varphi_{L_4}}{L_4} + \frac{C_1C_2}{\Gamma}\left(\frac{\varphi_{L_2}}{L_2} + \frac{\varphi_{L_3}}{L_3}\right) \\ \dot{q}_{C_2} &= \frac{C_1(C_3 + C_1)}{\Gamma}\left(\frac{\varphi_{L_2}}{L_2} + \frac{\varphi_{L_3}}{L_3}\right) + \frac{C_1C_2}{\Gamma}\frac{\varphi_{L_4}}{L_4} \\ \dot{\varphi}_{L_2} &= -\frac{q_{C_2}}{C_2} \\ \dot{\varphi}_{L_3} &= -\frac{L_3(L_4 + L_1)}{\Lambda}\frac{q_{C_2}}{C_2} + \frac{L_1L_4}{\Lambda}\frac{q_{C_1}}{C_1} \\ \dot{\varphi}_{L_4} &= -\frac{L_4(L_3 + L_1)}{\Lambda}\frac{q_{C_1}}{C_1} + \frac{L_1L_3}{\Lambda}\frac{q_{C_2}}{C_2}, \end{aligned}$$

with $\Lambda = L_1L_3 + L_3L_4 + L_4L_1$ and $\Gamma = C_1C_2 + C_2C_3 + C_3C_1$. Notice that the interconnection structure $\mathcal{J}$ is extracted from $\tilde{\mathcal{J}}$ by deleting the zero rows and columns of $\tilde{\psi}$. Notice also that the PCH equations are not the same as the ones which can be obtained by inverting the co-energy matrix $\mathcal{M}^*$. ■

5. DISSIPATIVE ELEMENTS AND SOURCES

In this section we extend the Brayton-Moser equations with dissipative elements and sources in an alternative way in comparison to e.g. (Brayton and Moser, 1964; Massimo *et al.*, 1980). For sake of simplicity, we will again restrict the developments to circuits that do not contain excess elements. This restriction can be easily relaxed using the developments of the previous section. Let us start by decomposing the mixed-potential function into

$$P(x) = [P_T + (P_E - P_R) - (P_J - P_G)](x). \quad (22)$$

Recall that $P_T(x)$ represents the rate of internal power in the capacitors or inductors, i.e., $P_T(x) = P_C(x) = -P_L(x)$. The powers $P_R(x)$, $P_G(x)$ represent half the dissipated power in the resistive and conductive elements, respectively, while the supplied power by the external voltage and current sources is defined by $P_E(x)$ and $P_J(x)$. In Section 3 we have defined $P_T(x)$ using the notion of a power preserving interconnection of the port variables. In the following we extend this idea to derive the quantities $P_R(x)$, $P_G(x)$, $P_E(x)$, and $P_J(x)$. According to Proposition 1 and in a similar fashion as for PCH systems in (Schaft, 2000), we may define (1) in the following alternative way

$$\frac{\partial^2 H^*(x)}{\partial x^2}\frac{dx}{dt} - \Upsilon\frac{\partial P_T(x)}{\partial x} = \mathcal{K}\gamma \quad (23)$$

$$\xi = \mathcal{K}^T x, \quad (24)$$

where $\xi = \mathcal{K}^T x$, with $\mathcal{K}$ a constant $n \times m$ matrix, is the output equation of the circuit, and $\mathcal{K}\gamma$ are the generalized forces (control inputs) applied to the circuit. As in (Schaft, 2000), we like to view these inputs and outputs as external ports of the circuit. In case a circuit contains dissipative elements, some or all ports are terminated by the corresponding α (voltage-controlled) conductivities and β (current-controlled) resistances, $m = \alpha + \beta \leq n$. Termination of these ports can be considered as feedback laws describing the relation between the dynamic elements and the dissipative elements. Indeed, if we subdivide $\mathcal{K}\gamma$ in (23) and (24) as $\mathcal{K}\gamma := \mathcal{K}_G\gamma_G + \mathcal{K}_R\gamma_R$, then the corresponding outputs are given by ξ_G and ξ_R. The vector $\gamma_G, \xi_G \in \mathbb{R}^\alpha$, and the vector $\gamma_R, \xi_R \in \mathbb{R}^\beta$ denote the power variables at the ports which are terminated by conductive and resistive elements, respectively, i.e.,

$$\gamma_{G_k} = -\mathfrak{G}_k(\xi_G),\ k = 1, \ldots, \alpha \quad (25)$$

$$\gamma_{R_j} = -\mathfrak{R}_j(\xi_R),\ j = 1, \ldots, \beta, \quad (26)$$

where $\mathfrak{G}_k \geq 0$, $\mathfrak{G}_k(0) = 0$, and $\mathfrak{R}_j \geq 0$, $\mathfrak{R}_j(0) = 0$, are smooth functions $\mathfrak{G}_k : \mathbb{R} \to \mathbb{R}_k$ and $\mathfrak{R}_j : \mathbb{R} \to \mathbb{R}_j$ stemming from Ohm's law. The voltage and current potentials of the conductive and resistive elements are

$$P_G(\xi_G) = \sum_{j=1}^{\alpha}\int_0^{\xi_{G_j}} \mathfrak{G}_j(\xi'_G)d\xi'_{G_j}$$

$$P_R(\xi_R) = \sum_{k=1}^{\beta}\int_0^{\xi_{R_k}} \mathfrak{R}_k(\xi'_R)d\xi'_{R_k}.$$

Hence, we may rewrite (25) and (26) as

$$\gamma_G = -\frac{\partial P_G}{\partial \xi_G}(\xi_G) := -\mathcal{Q}_G(\xi_G)\xi_G$$

$$\gamma_R = -\frac{\partial P_R}{\partial \xi_R}(\xi_R) := -\mathcal{Q}_R(\xi_R)\xi_R,$$

with $\mathcal{Q}_G(\xi_G)$ and $\mathcal{Q}_R(\xi_R)$ some symmetric matrices. The dissipative elements are then included into the Brayton-Moser equations as follows. Let the scalar function $P_D(\xi) : \mathbb{R}^{\alpha+\beta} \to \mathbb{R}$ denote the difference between the dissipative voltage and current potentials, i.e., $P_D(\xi) := P_G(\xi_G) - P_R(\xi_R)$, then

$$\frac{\partial P_D(\mathcal{K}^T x)}{\partial x} = \mathcal{K}\tilde{\mathcal{Q}}(x)\mathcal{K}^T x, \quad (27)$$

with

$$\tilde{\mathcal{Q}}(x) := \operatorname{diag}\left\{\mathcal{Q}_G(\mathcal{K}_G^T x), -\mathcal{Q}_R(\mathcal{K}_R^T x)\right\}.$$

Equation (27) is precisely the definition of the dissipative current and voltage potentials (referred to as resistive content and co-content in (MacFarlane, 1970)) as derived in (Brayton and Moser, 1964). Finally, after substitution of the latter into $\mathcal{K}\gamma = \mathcal{K}_G\gamma_G + \mathcal{K}_R\gamma_R$ yields

$$\frac{\partial^2 H^*(x)}{\partial x^2}\frac{dx}{dt} = \Upsilon\frac{\partial P(x)}{\partial x}, \qquad (28)$$

with $P(x) = P_T(x) + P_D(x)$ the total mixed-potential

$$P(x) = -v_C^T\psi^T i_L + \int_0^x \mathcal{K}\tilde{\mathcal{Q}}(x')\mathcal{K}^T x' dx'. \qquad (29)$$

For circuits without sources we have thus re-derived the existence and the form of the dissipative parts of the mixed-potential function and we have a procedure to obtain such functions. A similar procedure can be followed to include the voltage and current sources. This will lead to an external potential function $P_F(x) := P_E(i_L) - P_J(v_C)$ (supplied content and co-content, respectively). The complete Brayton-Moser equations derived from an external port point of view are then defined by extending (29) as $P(x) = P_T(x) + P_D(x) + P_F(x)$. Summarizing, we may call the Brayton-Moser equations (28), a port-controlled co-Hamiltonian framework with dissipation (PCH*D). We may interpret (28) as the closed-loop system depicted in Fig. 2. From this Figure it is clearly seen that the dissipative elements can be viewed as feedback loops.

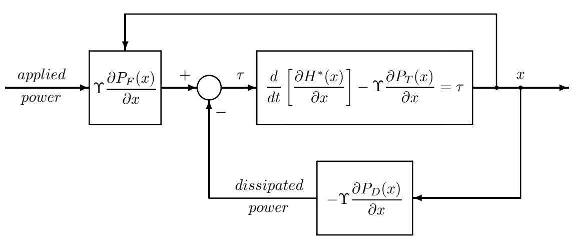

Fig. 2. Closed-loop interpretation of the Brayton-Moser equations.

6. SUMMARY AND CONCLUSION

In this paper we have established a direct connection between the classical Brayton-Moser equations (*old fish*) and the recently developed port-controlled Hamiltonian framework (*new fish*). A special Legendre transform is used to relate the energy and co-energy functions. It has been shown that if the Brayton-Moser equations are expressed in terms of the natural coordinates (inductor fluxes and capacitor charges) they lead to a time-differentiated version of the port-controlled Hamiltonian equations. As a result, many important properties can be exchanged between the two frameworks. We have developed a novel systematic procedure to deal with networks containing inductor-only cutsets and capacitor-only loops. This was inspired by the concept of implicit mechanical systems using Lagrange multipliers and kinematic constraints. Moreover, the mixed-potential function as defined by Brayton and Moser was shown, in partitioned form, to be derivable from an PCH external port point of view. For that reason, we may call the Brayton-Moser equations a port-controlled co-Hamiltonian system with dissipation (PCH*D).

During the developments in this paper we have seen that the frameworks exhibit dual inherent limitations. As a result, one sometimes has to make a choice between the one framework or the other. In case a given circuit contains non-bijective charge-controlled capacitors, one takes the port-controlled Hamiltonian equations, while in case of non-bijective current- or voltage controlled resistors, or a highly nonlinear model to be used for feedback control, one should rather vote for the Brayton-Moser equations. Although not shown here, we conclude by stating that the Brayton-Moser framework can also be easily accommodated for the inclusion of controlled switches. Results concerning this topic will be reported elsewhere.

7. REFERENCES

Brayton, R.K. and J.K. Moser (1964). A theory of nonlinear networks I and II. *Quart. Appl. Math.* **12**(1 and 2), 1–33.

Chua, L.O. and J.D. McPherson (1974). Explicit topological formulation of Lagrangian and Hamiltonian equations for nonlinear networks. *IEEE Trans. Circ. and Sys.* **21**(2), 277–285.

Kwatny, H.G., F.M. Massimo and L.Y. Bahar (1982). The generalized Lagrange formulation for nonlinear RLC networks. *IEEE Trans. Circ. and Sys.* **CAS-29**(4), 220–233.

MacFarlane, A.G.J. (1970). An integral invariant formulation of a canonical equation set for nonlinear elec. networks. *Int. J. Contr.* **CAS 11**(3), 449–467.

Massimo, F.M., H.G. Kwatny and L.Y. Bahar (1980). Derivation of the Brayton-Moser equations from a topological mixed-potential function. *J. Franklin Inst.* **310**(415), 259–269.

Ortega, R., A. Loría, P.J. Nicklasson and H. Sira-Ramírez (1998). *Passivity-Based Control of Euler-Lagrange Systems; Mechanical, Electrical and Electromechanical Applications*. Springer-Verlag.

Schaft, A.J. van der (2000). *$\mathcal{L}_2$-Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag.

Schaft, A.J. van der, M. Dalsmo and B.M. Maschke (1996). Mathematical structures in the network representation of energy-conserving physical systems. *Proc. 35th IEEE Conf. Dec. Contr., Kobe, Japan* pp. 201–206.

Scherpen, J.M.A., D. Jeltsema and J.B. Klaassens (2000). Lagrangian modeling and control of switching networks with integrated coupled magnetics. *in Proc. 39th IEEE Conf. Dec. Contr. Sydney, Australia* pp. 4054–4059.

Weiss, L. and W. Mathis (1997). A Hamiltonian formulation for complete nonlinear RLC-networks. *IEEE Trans. Circ. and Sys., Fund. Theory and Appl.* **44**(9), 843–846.