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IDENTIFICATION OF THE STABILITY OF FEEDBACK SYSTEMS IN THE PRESENCE OF NONLINEAR DISTORTIONS

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Abstract: A criterion to verify experimentally the stability of a nonlinear system, captured in a feedback loop, is proposed. The basic idea is to split the output power in coherent (linearly related to the input) and noncoherent power (the remaining power). A nonlinear power gain, measuring the sensitivity of the noncoherent power to input variations, is introduced. Using the small gain theorem, it is possible to check the local stability of the feedback for the actual class of excitation signals. *Copyright ' 2002 IFAC* .

Keywords**:** identification, nonlinear distortion, stability tests

1. INTRODUCTION.

Consider a nonlinear system captured in a linear feedback loop (see Fig. 1.) that is designed using linear techniques. Using the methods presented in

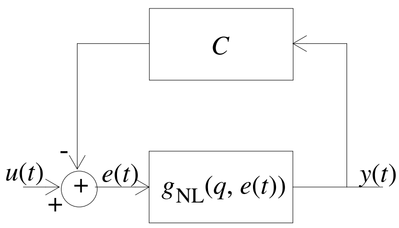

Fig. 1.: A nonlinear system captured in a linear feedback loop.

Pintelon and Schoukens (2001) and Schoukens *et al.* (2001), the level of the nonlinearities at the output of the closed loop system is verified for a given input level, and the results are shown in Fig. 2. From this experiment it turns out that the inband nonlinearity is more than 20 to 40 dB below the linear output contributions. The Nyquist plot of the linearized

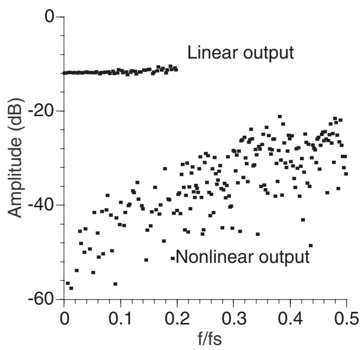

Fig. 2.: Measurement of the level of nonlinearity at the output of the closed loop system.

system (Fig. 3) shows a stable loop, so that the user is inclined to believe that she/he is on the safe side, no unstable operation is to be expected. However, it turns out that even very small disturbances create a jump in the output, so that the system is locally unstable. This observation brings us to the major question that we want to answer in this paper. Is it possible to verify the stability of a nonlinear system captured in a linear feedback loop, on the basis of a simple experiment? The alternative would be to consider the full nonlinear nature of the system and apply specific analysis and design methods, like Lyapunov based methods. This requires an (expensive) identification of a full blown,

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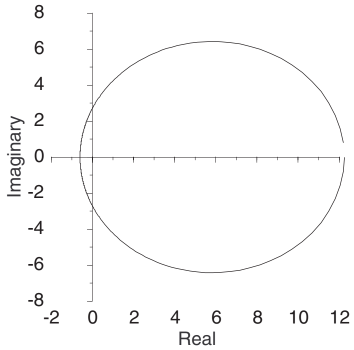


Fig. 3.: Nyquist plot of the linearized system.

validated nonlinear model. The linearized approach not only avoids this step, it also allows to use the intuitively appealing linear control theory. Of course the question raises if this simplification of reality is allowed, and as observed in the previous example, 'small' nonlinear distortions can push the system towards an unstable behaviour, even if the linear analysis predicts the system to be stable.

2. LINEAR REPRESENTATION OF THE NONLINEAR SYSTEM

Consider the single-input, single-output discrete time nonlinear dynamic system

$$y(t) = g_{\mathrm{NL}}(q, u(t)), \tag{1}$$

with, q^{-1} the backwards shift operator, and $g_{\mathrm{NL}}(q, u(t))$ a nonlinear operator acting on $u(t)$. For simplicity we assume that no disturbing noise is present (no process noise nor observation noise), so that we can focus completely on the nonlinear effects. In Section 6 we show how the proposed procedure can be generalized to include also process noise. The feedback C is assumed to be known. For simplicity we restrict it here, without any loss of generality, to be a constant. To simplify the simulations, we also assume that the direct term of $g_{\mathrm{NL}}(q, u(t))$ equals zero, so that $y(t)$ depends only upon the past values $u(t-k)$, $k \geq 1$. The system in (1) can always be replaced by a linear model plus an error term that takes care for all unmodelled effects.

$$y(t) = G_y(q)u(t) + v_y(t) \tag{2}$$

Using the knowledge of the controller, eq. (2) can be turned into an explicit feedback representation, as given in Fig. 4 where $G(q)$ and $v(t)$ are defined from $G_y(q)$ and $v_y(t)$ in (3):

$$G_y(q) = \frac{G(q)}{1 + CG(q)}, \text{ and } v_y(t) = \frac{v(t)}{1 + CG(q)}. \tag{3}$$

Note that $v(t)$ is a function of $e(t)$, the error signal in the loop, where $e(t) = u(t) - Cy(t)$, is independent of the particular choice of $G(q)$. On the other hand, $v(t)$ strongly depends on this choice.

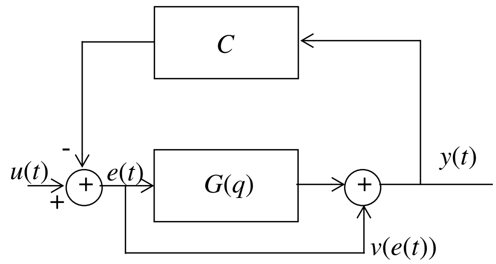


Fig. 4.: Feedback representation of the nonlinear loop for a given linear model.

3. STUDY OF THE BIBO STABILITY

Using classical transformation rules, the system in Fig. 4 becomes:

$$\begin{aligned} e(t) &= \frac{u(t)}{1 + CG(q)} - \frac{Cv(e(t))}{1 + CG(q)} \\ &= u_e(t) - Cv_y(e(t)) \end{aligned} \tag{4}$$

leading to the system in Fig. 5, with $v_y(e(t))$ the nonlinear function that describes the relation between the error signal $e(t)$ and the unmodelled output contributions. Assuming that $1/(1 + CG(q))$ is stable, the overall system will be stable if also the nonlinear feedback system in Fig. 5 is stable. This can be checked with the small gain theorem (Khalil, 1996).

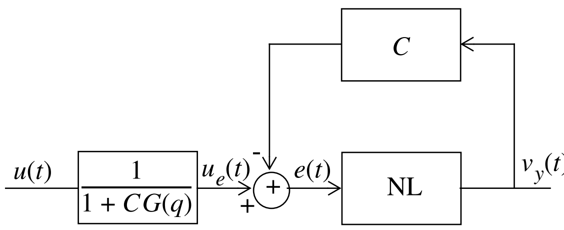


Fig. 5.: Feedback representation of the nonlinear loop for a given linear model $G(q)$.

Definition 1: G_{NL} is the nonlinear gain

$$G_{\mathrm{NL}} = \max_{u \in S_u} \frac{\|v_y(t)\|_2}{\|e(t)\|_2} \tag{5}$$

where S_u is the considered class of excitation signals.

For example S_u is the set of normally distributed excitations, clipped at the 3σ level, with a specified power spectrum.

If $CG_{NL} < 1$, the small gain theorem guarantees that

$$\max_{u \in S_u} \frac{\|v_y(t)\|_2}{\|u_e(t)\|_2} \leq \frac{G_{NL}}{1 - CG_{NL}} \qquad (6)$$

As a consequence, the overall system remains stable and the following theorem follows immediately.

Theorem 1: *If* $u \in S_u$, $1/(1 + CG(q))$ *is stable, and* $CG_{NL} < 1$, *then the output* $y(t)$ *is bounded by*

$$\|y(t)\|_2 < M\|u\|_2, \qquad (7)$$

with M *a positive, finite constant.*

The previous result can be generalized without any problem for small perturbations around a given set of excitations. Consider the nonlinear closed loop, excited with an excitation $u \in S_u$. Will the system remain stable for small distortions $\delta u \in S_{\delta u}$ around this signal? Using a similar approach as in the first part of this section, the following result is found immediately.

Consider the closed loop nonlinear system in Fig. 1 with a stable operation for $u \in S_u$, excited with $u(t) + \delta u(t)$, resulting in an output $y(t) + \delta y(t)$ and an output error $\delta v_y(t)$. Consider also the same linear model $G_y(q)$ to describe the system for $u(t)$, and $u(t) + \delta u(t)$. Define

$$\delta G_{NL} = \max_{\substack{\delta u \in S_{\delta u} \\ u \in S_u}} \frac{\|\delta v_y(t)\|_2}{\|\delta e(t)\|_2}, \qquad (8)$$

Theorem 2: If $u \in S_u$, $\delta u \in S_{\delta u}$, $1/(1 + CG(q))$ *is stable,* $CG_{NL} < 1$ *and* $C\delta G_{NL} < 1$, *the output* $\delta y(t)$ *is bounded by*

$$\|\delta y(t)\|_2 < M_\delta \|\delta u\|_2, \qquad (9)$$

with M_δ *a positive, finite constant.*

While Theorem 1 guarantees that a bounded input results in a bounded output, it does not bound the variation of the output for small input variations. Theorem 2 should be met in order to get that result.

4. EXPERIMENTAL STABILITY VERIFICATION

4.1 Method

Using the previous results it is possible to verify experimentally the stability of the closed loop system. In a first experiment, a linear approximation $\hat{G}(q, \theta)$ starting from the input $u_1(t)$ and the output $y_1(t)$ using one of the identification procedures (Ljung, 1999; Pintelon and Schoukens, 2001). Next a second experiment is made, applying the excitation $u_2(t) = u_1(t) + \delta u(t)$, and $v_{y2}(t) = y_2(t) - \hat{G}(q, \theta)u_2(t)$ is obtained. Eventually, the nonlinear gain is estimated by:

$$\delta\hat{G}_{NL} = \frac{\|v_{y2}(t) - v_{y1}(t)\|_2}{\|e_2(t) - e_1(t)\|_2}. \qquad (10)$$

with $e_i(t) = u_i(t) - Cy_i(t)$, $i = 1, 2$.

Form the first experiment, also $\hat{G}_{NL}$ can be estimated

$$\hat{G}_{NL} = \frac{\|v_{y1}(t)\|_2}{\|e_1(t)\|_2}. \qquad (11)$$

4.2 Discussion

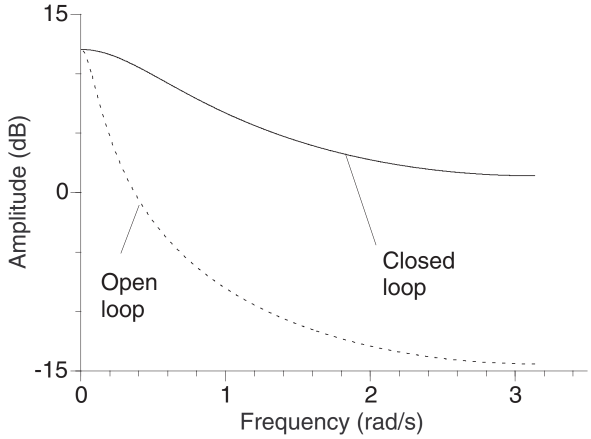
1) Remark that the identification step in this procedure is not critical. Firstly, it can be noted that this is an open loop identification. Under these conditions it is much easier to get a good estimate than under closed loop conditions. Secondly, the reader should be aware that even in the case of a 'non-optimal' linear approximation, the method still works. The only price that will be paid is that a smaller fraction of the output power will be explained by the linear model, and more power will be shifted to the unmodelled output $v_y(t)$. Since the stability of this fraction of the output is checked using the small gain criterion that can be too conservative, the conclusions may be also too conservative.

2) A second critical issue is the fact that the 'max' operator in (5) and (8) is replaced by a measured value. In pathological cases, this can lead to completely wrong conditions. However, in general, it will be very unlikely that the system is potentially unstable if the gain factor $C\delta\hat{G}_{NL}$ is much smaller than 1, compared to the uncertainty on the estimate. Moreover, a 'full nonlinear model based' approach suffers from exactly the same problem. In order to make a firm statement, it is necessary to have 'hard error bounds' on the model, which is impossible to obtain from experiments only. And even if the bounds are available, it will be very hard to maximize (8) over all possible excitations.

3) It should be noticed that (10) is a stochastic variable. In the absence of process noise and measurement noise, its variability is completely due to the variability of the reference signal $u(t)$.

5. SIMULATION

We illustrate the method on two examples. In the first



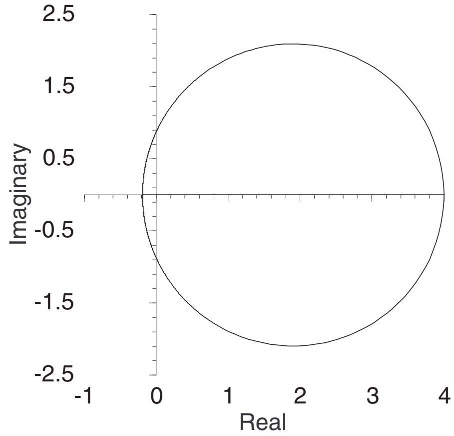


Fig. 6.: Open and closed loop transfer function (Left); Nyquist plot (Right) of the linear part of the system.

example, a stable and unstable nonlinear feedback system is considered. In the second example, the method is applied to the system used in the introduction of this paper.

5.1 Example 1

In the simulation, a Wiener system is used because it allows to verify the stability of the closed loop system using the circle criterion (Zames, 1966; Khalil, 1996). A Wiener system consists of the cascade of a linear dynamic part ($x(t) = w(q)u(t)$) and a static nonlinear part ($y = f(x)$):

The linear system is given by:

$$w(q) = 2q^{-1}/(1.1 - q^{-1}) \ , C = 0.2 . \qquad (12)$$

The nonlinear system is defined as:

$$\text{if } |x| < x_{\max}\text{: } y = x + 0.1x^3 ; \qquad (13)$$

$$\text{if } |x| \geq x_{\max}\text{: } y = x + \alpha x , \qquad (14)$$

with α s.t. the characteristic is continuous in $|x| = x_{\max}$ ($\alpha = 0.1x_{\max}^2$);

$$\text{if } |y| \geq 1000\text{: } y = 1000\,\mathrm{sign}(y) . \qquad (15)$$

The sector gain of $f(x)$ is set by the value of $x_{\max}$. If it is small enough, the circle criterion guarantees stability (bounded input, bounded output). If it is too large, the system can become unstable. Both situations are considered in the simulation. The characteristics of the linear system are given in Fig. 6., the nonlinear characteristic is given in Fig. 7..

The simulation consists of two parts. In the first one, the maximum gain of the nonlinearity is set such that the circle criterion guarantees stability: $x_{\max} = 5.94$. In the second one, the maximum gain of the nonlinearity is set such that the circle criterion is violated: $x_{\max} = 7.94$, and the system can become unstable. For both situations the nonlinear power gain was measured. The system is excited with a filtered white noise sequence (2nd order Butterworth filter, cut off frequency of 0.05 Hz., $f_s = 1$ Hz). In each experiment a record length of 8096 points is used. The excitation signal is clipped at the 3 sigma level, to avoid extremely large spikes. The closed loop linear approximation is obtained using an output error method. The distortion signal $\delta u(t)$ is generated as filtered white noise (2nd order Butterworth filter, cut off frequency of 0.25 Hz) with a standard deviation $\sigma_{\delta u} = 0.01\sigma_{u_1}$. The effective value of the input (σ_{u_1}) is varied, and for each value the corresponding squared nonlinear gain $\delta\hat{G}_{\mathrm{NL}}^2$ is measured. This process is repeated 1000 times. The results for both simulations are given in Fig. 8. and Fig. 10

Instead of plotting the 1000 measurements of the nonlinear gain for each input amplitude, the 1%, 5%, 50%, 95% and 99% percentiles are drawn. For the higher amplitudes, the system became unstable and no nonlinear gain could be determined. This is shown on the figure by plotting the probability of such event. This figure shows nicely that the high gain system can become locally unstable once the nonlinear power gain comes close to 1. On the other hand, if a low nonlinear gain is measured, the user can conclude that

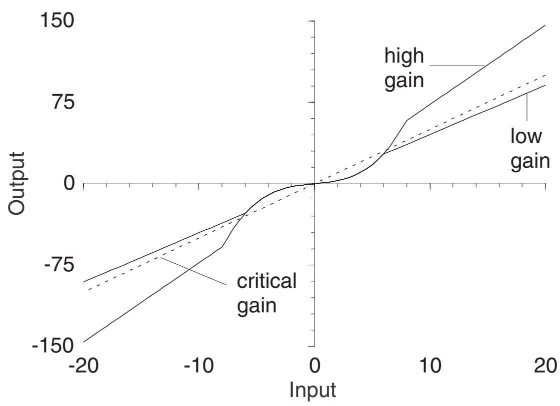


Fig. 7.: The characteristic of the static nonlinear part of the system, for two different gains. The broken line gives the critical gain (5.25) as it is obtained from the circle criterium.

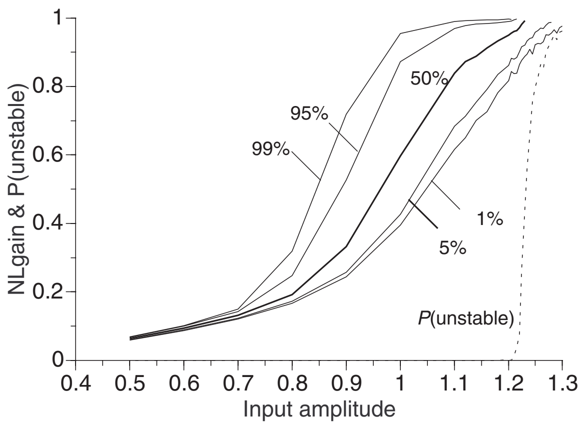


Fig. 8.: The percentiles of the squared nonlinear power gain $\delta\hat{G}^2_{\mathrm{NL}}$ and the probability to get an unstable realization as a function of the input amplitude for the high gain system.

there is no risk that the system will become unstable. The Nyquist plot for the best linear approximation of the high gain system close to the instability bound is given in Fig. 9 It is clear that this system is still far

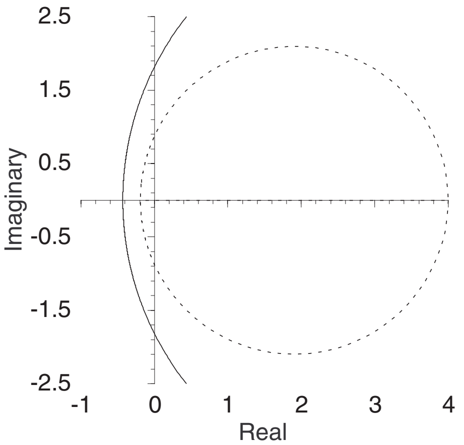


Fig. 9.: Nyquist plot of the original linear system (broken line), and the linearized system (full line) of the high gain nonlinearity.

away from an unstable behaviour. The results for the low gain system are given in Fig. 10. The gain comes close to 1 and drops then again for increasing

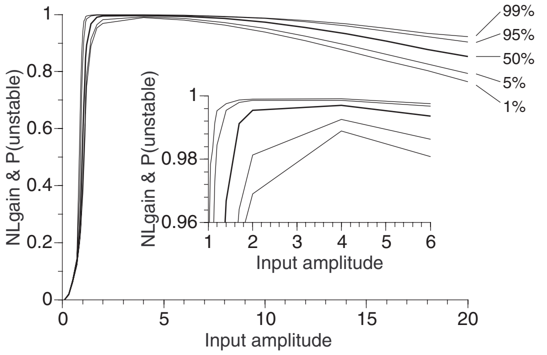


Fig. 10.: The squared nonlinear power gain $\delta\hat{G}^2_{\mathrm{NL}}$ as a function of the input amplitude. The results are plotted for a nonlinear system with a sub critical gain (low gain nonlinearity).

amplitudes. For these amplitudes, the system behaves again more linearly. Remark that in this case, using the nonlinear gain criterion, it would not be possible to guarantee the stability in the zone where the gain is very close to 1. Without prior knowledge, this zone would be classified as potential unstable if only one experiment is made.

5.2 Example 2

In this example we show the results for the example in the introduction. It is the same system as in example 1, but the feedback gain is changed to $C = 0.5$. Only the high gain case is analysed here The same

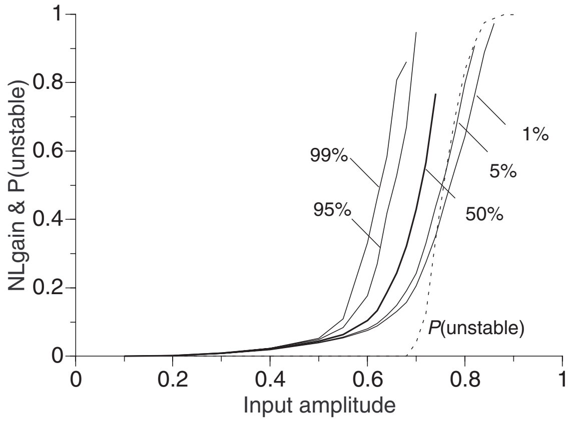


Fig. 11.: The squared nonlinear power gain $\delta\hat{G}^2_{\mathrm{NL}}$ as a function of the input amplitude. The results are plotted for a nonlinear system with a sub critical gain (Low gain nonlinearity) and a super critical gain (High gain nonlinearity).

conclusion can be made as before.

6. DEALING WITH PROCESS NOISE

In the presence of process noise, the previous method can not be directly applied since $\|\delta v_y(t)\|_2$ and $\|\delta v_e(t)\|_2$ will also include the impact of the process noise. So we need a procedure to separate the process noise from the error term $v_y(t)$. This can be easily done by making the input periodic, for example by repeating the random excitation periodically:

$$u_P(t) = u(\mathrm{rem}(t, N)), \tag{16}$$

with $\mathrm{rem}(t, N)$ the remainder of t/N, so that a signal with period N is created. Once the transients disappeared, and assuming that a periodic input results in a periodic output for the considered class of nonlinear systems (e.g. no bifurcations are allowed), the process noise variance on the signal $x = v_y$ or $x = e$ is estimated as:

$$\hat{\sigma}^2_{\mathrm{pn}}(x) = \frac{1}{MN-1}\sum_{t=1}^{MN}(x(t) - \hat{x}(\mathrm{rem}(t, N)))^2, \tag{17}$$

with M the number of measured periods, and

$$\hat{x}(t) = \frac{1}{M}\sum_{r=0}^{M-1} x(t+rN) \qquad (18)$$

the sample mean of the signal. For sufficiently large data samples, the uncertainty on the variance estimate is an $O(N^{-1/2})$, for example for white noise and $M = 2$, it is $\sigma_{\hat{\sigma}^2_{\mathrm{pn}}} = \hat{\sigma}_{\mathrm{pn}}\sqrt{2/N}$.

The gain estimate becomes

$$\delta\hat{G}^2_{\mathrm{NL}} = \max_{\substack{\delta u \in S_{\delta u} \\ u \in S_u}} \frac{\|\delta v_y(t)\|_2^2 - 2\hat{\sigma}^2_{\mathrm{pn}}(v_y)}{\|\delta e(t)\|_2^2 - 2\hat{\sigma}^2_{\mathrm{pn}}(e)} \qquad (19)$$

The factor 2 is due to the difference that is made to obtain $\delta v_y(t), \delta v_y(t)$. It is clear the $\sigma_{\hat{\sigma}^2_{\mathrm{pn}}}$ sets the minimum level of the nonlinear distortions that can be detected under the noise floor $\hat{\sigma}^2_{\mathrm{pn}}$. If the differences in (19) drop to far below $\sigma_{\hat{\sigma}^2_{\mathrm{pn}}}$, the estimate $\delta\hat{G}_{\mathrm{NL}}$ becomes unreliable. The appearance of such a situation can be detected by monitoring the levels during the measurements.

This method was applied to Example 5.1 where a white process noise with standard deviation 0.01 was added, $N = 8192$, $M = 2$, and $\sigma_{\delta u} = 0.01\sigma_{u_1}$. The results for 10 repeated runs are given in Fig. 12

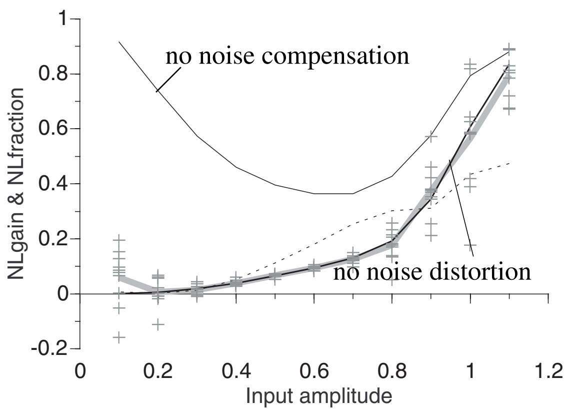


Fig. 12.: Gray line: mean value of the squared nonlinear power gain $\delta\hat{G}^2_{\mathrm{NL}}$; gray +: the squared nonlinear power gain $\delta\hat{G}^2_{\mathrm{NL}}$ in each realization; ---: the fraction of the nonlinear power in the measured power variation

From this figure it can be observed that

- It is not possible to measure the nonlinear gain in the presence of process noise without applying the correction.
- The corrected estimate coincides well with the gain obtained in the absence of process noise.
- The nonlinear errors can be detected even far below the process noise levels as can be observed from the low levels of the nonlinear fraction (ranging from a few% up to about 50% in this test).
- By increasing the experiment length (N) it is possible to decrease the nonlinear level that can still be detected.

7. CONCLUSIONS

In this paper we presented a simple experimental method to check if a nonlinear system is 'close' to its potential unstable operation. A nonlinear power gain is introduced, and a simple measurement procedure is setup to obtain this gain from 2 experiments. No nonlinear model of the system is needed. The nonlinear gain is used as an indication to the user how close the nonlinear feedback loop comes to a (local) unstable behaviour. The major advantage of the method is its simplicity. The major disadvantage is the stochastic behaviour of the gain factor, due to the stochastic nature of the input signal. This might lead to long experiments in order to reduce this variability in the amplitude range that comes close to the unstable behaviour. The method can also be applied in the presence of process noise.

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