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A SIMPLE STABILITY CRITERION FOR CONTROL SYSTEMS WITH VARYING DELAYS

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Abstract: Stability in presence of (possibly varying) time-delays in a control loop consisting of a continuous-time plant and a discrete-time controller is studied. A simple stability criterion is developed, consisting of a graphical test in a Bode plot of the closed loop system. The graphical test makes it very easy to design the controller for time-delay robustness.

Keywords: Jitter, Delay, Stability, Digital control, H-infinity

1. INTRODUCTION

This paper considers the problem of time-varying delays in a linear control system with a continuous-time stable plant and a discrete-time controller.

Time-delay robustness is a large research topic, and many sub-problems have been extensively explored. One such problem is "stability independent of delay", where the system stability is tested for *any* delay, see (Chen and Latchman (1995); Huang and Zhou (2000); Verriest *et al.* (1993)). Another problem is "delay-dependent stability" (implying restrictions on the delays), which has been explored in (Yan (2001); Verriest (1994); Gu and Han (2000); Li and de Souza (1997)) among others. All these references have used continuous-time systems only. In this paper, the focus is on the interconnection of continuous-time and discrete-time systems, as this is the common case in real-world systems (with continuous-time plant and discrete-time controller). This kind of problem has been studied before, see e.g. (Nilsson *et al.* (1998)), but often the probability densities of the delays are assumed to be known.

In this paper, delays are time-varying, stochastically or worst-case, and bounded. The main advantage of the method is that the stability criterion is a simple graphical check in a closed loop Bode plot, which makes it easy to design for robustness.

All sampled-data theory in this paper is known, and the papers main contribution is to apply it to the highly relevant control problem of delay stability. The result, thanks to its simplicity, is very usable in practice.

2. PROBLEM FORMULATION

Consider the control system in Figure 1. The plant $P(s)$ is a continuous-time stable strictly proper linear system, and the controller $C(z)$ is a discrete-time linear system with sample period h. Discretizing the plant $P(s)$ with a zero-order-hold input gives $\tilde{P}(z)$. The time delay $\tau(n)$ for the system for sample n may be stochastic, but must be bounded:

$$0 \le \tau(n) < Nh, \qquad (1)$$

where N is a positive integer. A simple put powerful criterion for stability of the closed loop system for *all* delays fulfilling (1) (including stochastic or worst-case time-varying) will be shown.

3. STABILITY FOR SHORT DELAYS

For clarity of the presentation, the case where $N = 1$ will be shown first, and then extended to

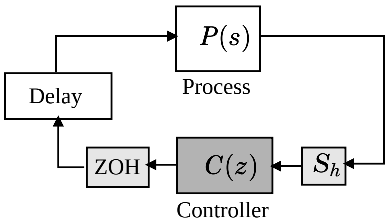

Fig. 1 The control system with delay after the controller.

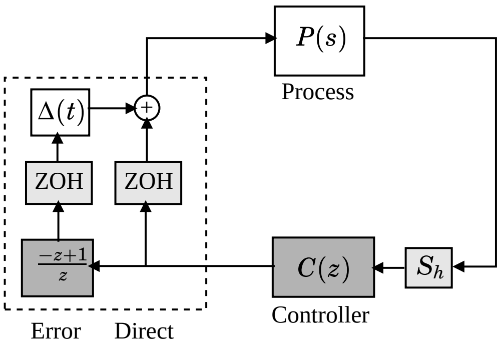

Fig. 2 An equivalent description of the control loop with delay.

longer delays.

The output of the controller is denoted $u(n)$, where n is the time step. The output from the delay is (for $N = 1$)

$$u_{\text{del}}(t) = \begin{cases} u(n-1) & t_n \le t < t_n + \tau(n) \\ u(n) & t_n + \tau(n) \le t < t_{n+1}. \end{cases} \tag{2}$$

Therefore, it can be rewritten as

$$\begin{aligned} u_{\text{del}}(t) = u(n) + u_{\text{err}}(t) = \\ u(n) + \big(-u(n) + u(n-1)\big)\Delta(t) \end{aligned} \tag{3}$$

where

$$\Delta(t) = \begin{cases} 1 & t_n \le t < t_n + \tau(n) \\ 0 & t_n + \tau(n) \le t < t_{n+1}. \end{cases} \tag{4}$$

The operator Δ is defined by $\Delta f = \Delta(t) \cdot f(t)$. Note that the L_2-induced gain of Δ, $\gamma(\Delta) \le 1$. The system in Figure 1 can now be transformed via Figure 2 to the standard feedback in Figure 3.

As the upper part of the loop in Figure 3 consists of both discrete-time and continuous-time linear elements, the resulting transfer function Ψ is linear and time-periodic. The H_∞ gain of Ψ will now be calculated. For the intuition, a frequency domain calculation is preferable. Such a result is presented in (Yamamoto and Araki (1994)). It is shown that (in notation from (Lindgärde (1999)))

$$\|\Psi\|_\infty = \sup_{\omega \in (-\omega_h/2, \omega_h/2]} \|\Psi(\omega)\|, \tag{5}$$

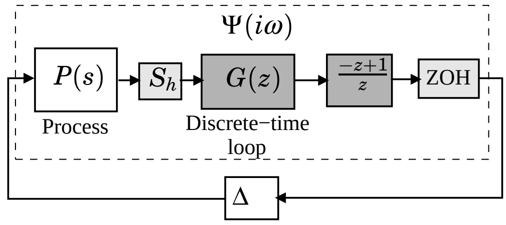

Fig. 3 The control loop and Δ function rewritten on standard feedback form. $G(z) = \frac{C(z)}{1-\tilde{P}(z)C(z)}$.

where

$$\|\Psi(\omega)\| = \sup_{\|\bar{u}\|=1} \frac{\|y\|_P}{\|u\|_P}, \tag{6}$$

in which

$$u = \sum_{k=-\infty}^{\infty} u_k e^{i(\omega + k\omega_h)t} \tag{7}$$

$$\bar{u} = \begin{pmatrix} \dots & u_{-1} & u_0 & u_1 & \dots \end{pmatrix} \tag{8}$$

$$\|\cdot\|_P = \lim_{\tau\to\infty} \sqrt{\frac{1}{2\tau}\int_{-\tau}^{\tau} \|\cdot(t)\|^2 dt} \tag{9}$$

A general formula for $\|\Psi(\omega)\|$ is presented in (Yamamoto and Araki (1994); Lindgärde (1999)), but in this case, it simplifies significantly to

$$\|\Psi(\omega)\| = \underbrace{\|-e^{i\omega h}+1\|}_{B_1(\omega)} \cdot \|G(e^{i\omega h})\| \cdot \underbrace{\sqrt{\sum_{k=-\infty}^{\infty} \|P(i(\omega + k\omega_h))\|^2}}_{P_{\text{alias}}(\omega)}, \tag{10}$$

where $G(z) = \frac{C(z)}{1-\tilde{P}(z)C(z)}$. See Appendix for details. The aliasing sum of P converges since P is strictly proper. Note that $P_{\text{alias}}(\omega)$ is close to $\tilde{P}(e^{i\omega h})$ if the sample rate h is chosen well.

It should be noted that the above infinite sums are easily numerically approximated, but there is also an algebraic calculation presented in (Braslavsky *et al.* (1998)).

3.1 *A Stability Criterion*

To prove closed-loop stability of the control system in Figure 3 using the Small Gain Theorem,

$$\gamma(\Delta)\gamma(\Psi) < 1 \tag{11}$$

is required, which is true if

$$\|\Psi(\omega)\| < 1, \ \forall\omega \tag{12}$$

due to (5) and that $\gamma(\Delta) \leq 1$. Thus

$$\|\Psi(\omega)\| = \|B_1(\omega)G(e^{i\omega h})P_{\text{alias}}(\omega)\| < 1 \Leftrightarrow$$
$$\| \underbrace{G(e^{i\omega h})P_{\text{alias}}(\omega)}_{T^*(\omega)} \| < \frac{1}{B_1(\omega)} \qquad (13)$$

is a sufficient condition for stability. This can easily be checked in a Bode diagram, where also a "stability margin" can be obtained from the minimum distance from $\|T^*(\omega)\|$ to $\frac{1}{B_1(\omega)}$. See Figure 4 for an example.

3.2 *Some Approximate Stability Criteria*

If the $P(s)$ is sampled with a high enough sampling rate,

$$T^*(i\omega) \approx \frac{\tilde{P}(e^{i\omega h})C(e^{i\omega h})}{1 - \tilde{P}(e^{i\omega h})C(e^{i\omega h})} = T(e^{i\omega h}),$$

where T is the complementary sensitivity function. Thus, (13) can be approximately checked in the standard closed loop Bode diagram.

To simplify matters further, (13) can also be tested approximately by studying only using one (well chosen) frequency point. First, the closed-loop bandwidth of the control system is defined as the largest frequency $\omega_b \leq \frac{\pi}{h}$ which satisfies

$$\|T(e^{i\omega_b h})\| = \frac{1}{\sqrt{2}}. \qquad (14)$$

If T has a roll-off of 1 and does not have any resonance peaks, the comparison (13) can be done in the point $\omega = \omega_b$.

$$\|T(e^{i\omega_b h})\| = \frac{1}{\sqrt{2}} < \|\frac{1}{-e^{i\omega_b h} + 1}\| \Leftrightarrow \omega_b h < \frac{\pi}{2} \qquad (15)$$

By designing the control system so that (15) holds with some margin, no delay less than one sample-period (stochastic or constant) can destabilize the system. This stability criterion is rather weak, due to the roll-off one and no resonance assumptions.

A better approximate criterion is the following: Define ω_c as the frequency point (if unique) where $\|\tilde{P}(e^{i\omega_c h})C(e^{i\omega_c h})\| = 1$, and note that this point usually gives the approximate resonance peak of T. Then

$$\|T(e^{i\omega_c h})\| = \frac{1}{\| -1 - e^{i(\pi - \varphi_m)}\|} = \frac{1}{\|1 - e^{i\varphi_m}\|}, \qquad (16)$$

where φ_m is the phase-margin of the system.

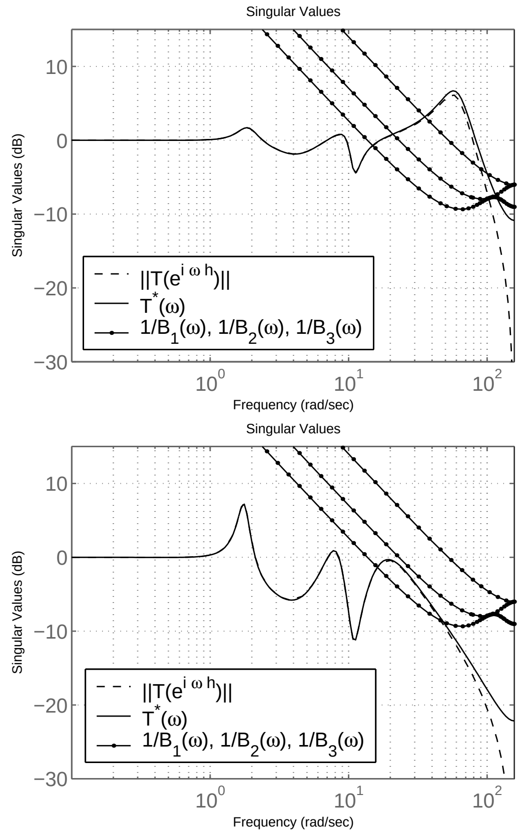

Fig. 4 A DVD focus loop with two different PID controllers. The upper loop can be destabilized by delays within a sample period, whereas the lower is guaranteed to be stable for delays within two sample periods. Note the similarity of T and T^*, as the aliasing does not change much.

The comparison in point $\omega = \omega_c$ becomes

$$\|T(e^{i\omega_c h})\| < \frac{1}{B_1(\omega_c)} \Leftrightarrow$$
$$\frac{1}{\|1 - e^{i\varphi_m}\|} < \frac{1}{\|1 - e^{i\omega_c h}\|} \Leftrightarrow$$
$$\varphi_m > \omega_c h \qquad (17)$$

Note that this approximate criterion is the same as the exact stability criterion for a stable system with a fixed delay h (which is an allowed delay in this criterion!). From this some conclusions can be drawn:

a) (13) is not very conservative.

b) For stability issues, stochastic delays are not much worse than fixed. This does of course not account for delay compensation, etcetera.

3.3 *Comparisons to Other Criteria*

One other way to treat the delay is to split it up in direct feed-through and error (just like in this case), but write it as

$$U_{\text{del}}(s) = e^{-sD}U = U + \underbrace{(e^{-sD} - 1)U}_{\text{Delay error}} \qquad (18)$$

where D is the delay which changes once per sample. The maximum gain of $(e^{-sD} - 1)$ is 2. Therefore, it is required require that

$$G(e^{i\omega h})P_{\text{alias}}(\omega) < \frac{1}{2} \quad \forall\omega, \qquad (19)$$

meaning that the (approximate) complementary sensitivity function has to be low for *all* frequencies. The difference between (19) and (13) is the high pass filter leading to the $\|\frac{1}{-e^{i\omega h}+1}\|$ bound. This filter does also have a H_∞ gain of 2, but it is only achieved for high frequencies. Using (13) the complementary sensitivity may be high at lower frequencies.

4. STABILITY FOR LONGER DELAYS

This section presents a generalization of the stability criterion for any N. No other structure to the delays is required than (1), which means that the output of the delay is

$$u_{\text{del}}(t) = \delta(t)^T \begin{pmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(n-N) \end{pmatrix}, \qquad (20)$$

where $\delta(t)$ is a "selector function" – for each time t, one element of $\delta(t)$ is 1 and the others are zero. Since it models a delay, it is piecewise constant with at most N switches per sample period. The delay can then, again, be rewritten as feed-through plus error:

$$u_{\text{del}}(t) = u(n) + \Delta(t)^T \begin{pmatrix} u(n) - u(n-1) \\ \vdots \\ u(n) - u(n-N) \end{pmatrix}, \qquad (21)$$

where $\Delta(t)$ is the selector function which may be all zero for some time. Naturally, the L_2-induced gain $\gamma(\Delta) \leq 1$. The same reasoning as for $N = 1$ can now be used, with the difference that $\Psi(\omega)$ is SIMO and Δ is MISO. The gain of Ψ now becomes

$$\|\Psi(\omega)\| = \underbrace{\left\| \begin{pmatrix} -e^{i\omega h} + 1 \\ \vdots \\ -e^{Ni\omega h} + 1 \end{pmatrix} \right\|}_{B_N(\omega)} \underbrace{\|G(e^{i\omega h})\| P_{\text{alias}}(\omega)}_{T^*(\omega)} \qquad (22)$$

and the stability criterion $(\gamma(\Psi) < 1)$

$$T^*(\omega) < \frac{1}{B_N(\omega)} \quad \forall\omega \qquad (23)$$

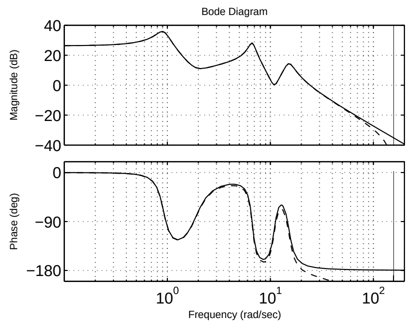

Fig. 5 The Bode plot for the DVD focus process (sampled and continuous). Note that the time scale is changed by a factor of 1000.

4.1 *Approximate criteria, again*

The approximate criterion (17) extends to

$$\varphi_m > \omega_c N h, \qquad (24)$$

which, again, is the same as for a constant delay of Nh.

5. EXAMPLE

In this section, a real-world process is presented, for which the criterion can be used to check robustness to delays. The process is the focus lens system of a DVD player. As a DVD rotates in the player, the laser spot has to be kept in focus on the DVD surface. Therefore a tight control loop detecting the focus error and moving the lens has to be closed. A Bode plot of the transfer function from control signal to focus error is shown in Figure 5.

The process is sampled at 50 kHz, and it is controlled by a digital PID controller. For the example two parameter settings for the PID have been used: Controller C_1 has a high gain and gives the control loop a high bandwidth. Controller C_2 is C_1 but with a lower gain. The complementary sensitivity functions for the system with the two different controllers can be seen in Figure 4.

Apparently, using C_1 (upper part of the figure) stability of the system cannot be guaranteed even for delays $\tau(n) \leq h$. With some process and sample noise, the variance of the system can indeed be shown to go to infinity for delays close to h.

Using C_2 (lower part), the system is guaranteed to be stable for delays $\tau < 2h$. Again, for slightly longer delays (with $\tau < 3h$), and some process noise, the variance goes to infinity.

Note, that even if the system can be destabilized when the criterion is not fulfilled in this case, the result is not strict.

6. CONCLUSIONS

A simple stability criterion in a digital control system with delays has been shown. The criterion can be checked in a Bode plot. An approximate version of the criterion effectively puts a bound on the phase margin of the system.

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A. CALCULATION OF Ψ GAIN

Consider a linear system Ψ consisting of a strictly proper continuous-time part $P(s)$, a sampler, a discrete-time part SIMO $G_N(z)$, and a (diagonal) zero order hold $Z(s)$. The gain $\|\Psi(\omega)\|$ is calculated as (from (Yamamoto and Araki (1994)))

$$\|\Psi(\omega)\| = \lim_{k\to\infty} \bar{\sigma}(\psi^k(i\omega)), \tag{25}$$

where

$$\psi^k(i\omega) = \underbrace{\begin{pmatrix} \frac{1}{h}Z(i\omega + ki\omega_h) \\ \frac{1}{h}Z(i\omega + (k-1)i\omega_h) \\ \vdots \\ \frac{1}{h}Z(i\omega - ki\omega_h) \end{pmatrix}}_{Z^k(\omega)} G_N(e^{i\omega h}) \cdot$$

$$\underbrace{\begin{pmatrix} P(i\omega + ki\omega_h) & \dots & P(i\omega - ki\omega_h) \end{pmatrix}}_{P^k(\omega)}. \tag{26}$$

As $\psi^k(i\omega)$ has at most rank 1, the maximum singular value is simply

$$\bar{\sigma}(\psi^k(i\omega)) = \|Z^k(\omega)G_N(e^{i\omega h})\| \cdot \|P^k(\omega)\| \tag{27}$$

$Z(s)$ is a zero-order-hold, such that $Z(s) = \text{diag}\left\{\frac{1-e^{-sh}}{s}, \frac{1-e^{-sh}}{s}, \dots\right\}$, and thus it holds that

$$\lim_{k\to\infty} \|Z^k(\omega)G_N(e^{i\omega h})\| =$$

$$\lim_{k\to\infty} \left\| \begin{pmatrix} \frac{1-e^{-i(\omega+k\omega_h)h}}{i(\omega+k\omega_h)h} G_N(e^{i\omega h}) \\ \frac{1-e^{-i(\omega+(k-1)\omega_h)h}}{i(\omega+(k-1)\omega_h)h} G_N(e^{i\omega h}) \\ \vdots \\ \frac{1-e^{-i(\omega-k\omega_h)h}}{i(\omega-k\omega_h)h} G_N(e^{i\omega h}) \end{pmatrix} \right\| =$$

$$\lim_{k\to\infty} \sqrt{\sum_{n=-k}^{k} \left| \frac{1-e^{-i(\omega+n\omega_h)h}}{i(\omega+n\omega_h)h} \right|^2} \cdot \|G_N(e^{i\omega h})\| =$$

$$\|G_N(e^{i\omega h})\| \tag{28}$$

and thus (as the sums converge)

$$\|\Psi(\omega)\| = \|G_N(e^{i\omega h})\| \lim_{k\to\infty} \|P^k(\omega)\|. \tag{29}$$

In this case, $G_N(z)$ is the closed loop $G(z)$ and N difference filters

$$G_N(z) = \begin{pmatrix} \frac{-z^1+1}{z^1} \\ \vdots \\ \frac{-z^N+1}{z^N} \end{pmatrix} G(z). \tag{30}$$

This gives (10) and (22).