15th Triennial World Congress, Barcelona, Spain

ANALYSIS AND CRONE CONTROL OF TIME VARYING SYSTEMS WITH ASYMPTOTICALLY CONSTANT COEFFICIENTS

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Abstract: Continuous time varying systems with asymptotically constant coefficients are analysed in the frequency domain through their representation using time varying frequency responses. A stability theorem for feedback systems including time varying systems with asymptotically constant coefficients is proposed. Finally, Crone control (robust control method based on fractional differentiation) is extended to robust control of time varying systems with asymptotically constant coefficients.

Keywords: CRONE control, robust control, time varying systems, Time varying frequency responses

1. INTRODUCTION

Transfer functions and associated frequency responses are powerful tools for the analysis and synthesis of stationary systems. Thus, several authors have extended these tools to treat time varying systems. In particular, Zadeh defined the system function notion (Zadeh, 1950) to which the time varying frequency response (TVFR) can be associated.

Many aspects of the definition of TVFRs correspond to the definition of stationary equivalents. However there has been little interest in TVFRs since Zadeh. The major reason is the difficulty to calculate the TVFR representing a general time varying system.

However, for some classes of systems, TVFR computation procedures have been developed (Sabatier et al., 1998; Sabatier and Garcia, 2000; Rudnitskii, 1960). In this study, TVFRs are used for the analysis and robust control of a particular class of time varying systems : time varying systems with asymptotically constant coefficients (Bellman, 1953; Kaplan, 1962).

In this paper, section 2 defines considered systems. Section 3 gives an algebraic transformation which leads to a state space representation of the considered systems with a constant state matrix (Lyapunov transformation). Using this transformation, section 4 explains how to compute a TVFR representation for time varying systems with asymptotically constant coefficients. Also, the representation of feedback systems using TVFRs is studied. Section 5 gives some rules for the truncation of TVFR infinite series when numerical implementations are required. In section 6, a frequency stability criterion is proposed for feedback systems including time varying systems with asymptotically constant coefficients.
Third generation Crone control is extended in section 7 to the control of time varying systems with asymptotically constant coefficients.
This extension leads to the synthesis of controllers which ensure :

- a near stationary behaviour of the closed loop system ;
- performances set by the designer such as rapidity and the resonance ratio in tracking of the closed loop system.

Whenever the plant is reparametrated, these controllers also ensure :

- robust closed loop stability and performance ;

- satisfactory immunity of the closed loop to the time varying character of the plant.

In section 8, the extension of Crone control is applied to an example.

2. SYSTEMS STUDIED

The systems studied in this paper admit the following state space description :

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases}, \qquad (1)$$

with $x(t) \in \mathbb{R}^{nxn}$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$.

Matrices $A(t)$, B(t) and C(t) are supposed continuous and bounded on $\mathbb{R}^+$. They respect the following relations :

$$\lim_{t\to\infty} A(t) = A_c\,, \quad \lim_{t\to\infty} B(t) = B_c\,, \quad \lim_{t\to\infty} C(t) = C_c\,, \quad (2)$$

where A_C, B_C and C_C are time invariant matrices.

It is supposed that matrices $A(t)$, $B(t)$ and $C(t)$ admit the series expansions :

$$A(t) = \sum_{k\in\mathbb{N}} A_k e^{-k\alpha t}\,, \quad B(t) = \sum_{k\in\mathbb{N}} B_k e^{-k\alpha t}\,, \quad C(t) = \sum_{k\in\mathbb{N}} C_k e^{-k\alpha t} \qquad (3)$$

where $A_k \in \mathbb{R}^{nxn}$, $B_k \in \mathbb{R}^{nx1}$, $C_k \in \mathbb{R}^{1xn}$, $\forall k \in \mathbb{N}$ and where $\alpha \in \mathbb{R}_+^*$. Indeed, relation (3) are Taylor expansions of matrices $A(\zeta)$, $B(\zeta)$ and $C(\zeta)$ if $\zeta = e^{-t}$ with $\zeta \in [0..1]$ (Sansone, 1959). Matrices $A(\zeta)$, $B(\zeta)$ and $C(\zeta)$ are thus analytic on [0..1].

In the present study, matrix $A(t)$ is supposed to respect the following hypothesis.

Hypothesis 1

Let $\Lambda_k = \{\mu_i = \lambda_i - k\alpha, i \in [1,...,n]\}$ be the eigenvalue set of matrix $A_0+\alpha k I_n$, $\forall k \in \mathbb{N}$. Sets Λ_k respect the following condition : $\Lambda_{k_1} \cap \Lambda_{k_2} = \varnothing$ if $k_1 \neq k_2$.

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3. ALGEBRAIC TRANSFORMATION

Using $\zeta = e^{-\alpha t}$ and thus $d\zeta = -\alpha\zeta dt$, state space description (1) becomes :

$$\begin{cases} -\alpha\zeta \dfrac{dx(\zeta)}{d\zeta} = \left(\sum_{k\in\mathbb{N}} A_k \zeta^k\right) x(\zeta) + \left(\sum_{k\in\mathbb{N}} B_k \zeta^k\right) u(\zeta) \\ y(\zeta) = \left(\sum_{k\in\mathbb{N}} C_k \zeta^k\right) x(\zeta) \end{cases} \qquad (4)$$

If hypothesis 1 is met, variable change :

$$x(\zeta) = P(\zeta)z(\zeta) \quad \text{with} \quad P(\zeta) = \sum_{k\in\mathbb{N}} P_k \zeta^k\,, \quad P(0) = I\,, \qquad (5)$$

reduces representation (4) to (Wascow, 1976) :

$$\begin{cases} -\alpha\zeta \dfrac{dz(\zeta)}{d\zeta} = A_0 z(\zeta) + \tilde{B}(\zeta)u(\zeta) \\ y(\zeta) = \tilde{C}(\zeta)x(\zeta) \end{cases}. \qquad (6)$$

$$\tilde{B}(\zeta) = \sum_{k\in\mathbb{N}} \tilde{B}_k \zeta^k \qquad \tilde{B}(\zeta) = P^{-1}(\zeta)B(\zeta)$$

$$\tilde{C}(\zeta) = \sum_{k\in\mathbb{N}} \tilde{C}_k \zeta^k \qquad \tilde{C}(\zeta) = C(\zeta)P(\zeta)\,.$$

Matrix $P(\zeta)$ is the solution of equation (Wascow, 1976)

$$\alpha\zeta \frac{dP(\zeta)}{d\zeta} = -A(\zeta)P(\zeta) + P(\zeta)A_0\,. \qquad (7)$$

4. TIME VARYING FREQUENCY RESPONSE REPRESENTATION

4.1. Definitions

In the 1950s, Zadeh (Zadeh, 1950) demonstrated that linear time varying systems can be described by system functions $H(s, t)$. These system functions are linked to the impulse response of the system, $h(t, \xi)$, a function of the time variable t and of the moment of application of the impulse ξ, by :

$$H(s,t) = e^{-st}\int_{-\infty}^{\infty} h(t,\xi)e^{s\xi}d\xi\;, \qquad (8)$$

if $h(t, \xi)$ is piecewise constant $\forall\ t > \xi$, and if $\exists\ \sigma_c \in \mathbb{R}$ such that $\lim_{t-\tau\to\infty} e^{\sigma_c(t-\tau)}h(t,\xi) = 0$

If $\exists\ t > t_0 > \xi\ /\ |h(t,\xi)| < e^{-\sigma t}$ $\sigma \in \mathbb{R}$, $\forall\ \xi \in \mathbb{R}$, $t_0 \in \mathbb{R}$, system function notion previously introduced lead to the TVFR notion, denoted $H(j\omega, t)$ and obtained using $s = j\omega$. If $u(t) = e^{j\omega t}$, the TVFR of system (1) can also be defined by (Zadeh, 1950) :

$$H(j\omega,t) = \frac{\text{Response of the system to } e^{j\omega t}}{e^{j\omega t}}\;. \qquad (9)$$

4.2. TVFR of a time varying system with asymptotically constant coefficients

For time varying systems with asymptotically constant coefficients, if $u(t) = e^{j\omega t}$, then, output $y(t)$ is given, using relation (6) and if initial conditions are supposed equal to 0, by :

$$y(t) = \tilde{C}(t)x(t) = \tilde{C}(t)\int_0^t e^{A_0(t-\tau)}\tilde{B}(t)e^{j\omega\tau}d\tau\;. \qquad (10)$$

Relation (10) demonstrates that the state $x(t)$ of the system is :

$$x(t) = X(j\omega,t)e^{j\omega t} = \sum_{k\in\mathbb{N}} X_k(j\omega)e^{-k\alpha t}e^{j\omega t}\;, \qquad (11)$$

and that the TVFR of the system admits the following series expansion :

$$H(j\omega,t) = \sum_{k\in\mathbb{N}} H_k(j\omega)e^{-k\alpha t}\;. \qquad (12)$$

Using relations (11) and (12) in equation (1), then regrouping terms in $e^{-k\alpha t}$ (which form an orthogonal basis), it is possible to express frequency responses $H_k(j\omega)$ which appear in function $H(j\omega, t)$ (relation (12)).

Theorem 1

Frequency responses $H_k(j\omega)$ of relation (12) are given by :

$$H_k(j\omega) = \sum_{l=0}^{k} C_{k-l} X_l(j\omega), \tag{13}$$

with

$$X_l(j\omega) = [(j\omega - l\alpha)I_n - A_0]^{-1}\left(B_l - \sum_{i=0}^{l-1} C_{l-i} X_i(j\omega)\right), \tag{14}$$

and

$$X_0(j\omega) = [j\omega I_n - A_0]^{-1} B_0 , \tag{15}$$

if and only if $\det[(j\omega - l\alpha)I_n - A_0] \neq 0 \quad \forall l \in \mathbb{N}$ and $\forall \omega \in \mathbb{R}^*$.

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Proof : See comments of section 4.2

4.3. Feedback system case

The standard control scheme of Fig. 1 is considered, in which time varying systems with asymptotically constant coefficients $\mathcal{C}$ and $\mathcal{P}$ can be described by TVFRs $C(j\omega, t)$ and $P(j\omega, t)$ with :

$$C(j\omega,t) = \sum_{k\in\mathbb{N}} C_k(j\omega)e^{-k\alpha t} \text{ and}$$

$$P(j\omega,t) = \sum_{k\in\mathbb{N}} P_k(j\omega)e^{-k\alpha t} . \tag{16}$$

Let $\beta(j\omega, t)$ be the TVFR of the system resulting from the cascade connection of systems $\mathcal{C}$ and $\mathcal{P}$, with :

$$\beta(j\omega,t) = \sum_{k\in\mathbb{N}} \beta_k(j\omega)e^{-k\alpha t} . \tag{17}$$

Theorem 2

Frequency responses $\beta_k(j\omega)$ of relation (17) are given by :

$$\beta_k(j\omega) = \sum_{i=0}^{k} C_i(j\omega)P_{k-i}(j\omega - i\alpha) . \tag{18}$$

❑

Proof : Compute the output of $\mathcal{C}$ and $\mathcal{P}$ *using relation (9) if the input of* $\mathcal{C}$ *is* $\varepsilon(t)=e^{j\omega t}$ *. Then by identification, find frequency responses* $\beta_k(j\omega)$.

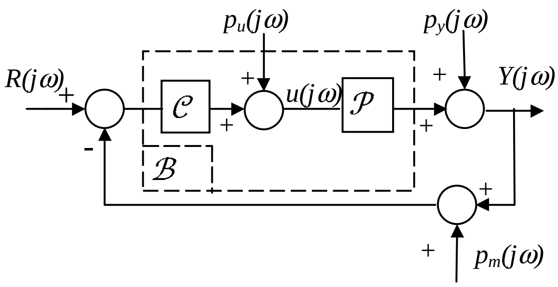

Fig.1. *Feedback system*

Let $T(j\omega, t)$, $S(j\omega, t)$, $CS(j\omega, t)$ and $PS(j\omega, t)$ denote respectively TVFRs connecting (see Fig. 1) :

- reference input $r(t)$ to output $y(t)$;
- output disturbance $p_y(t)$ to output $y(t)$;
- reference input $r(t)$ to input $u(t)$;
- plant input disturbance $p_u(t)$ to output $y(t)$.

As the feedback system of Fig. 1 being a time varying systems with asymptotically to constant coefficients, TVFR $T(j\omega, t)$ is of the form :

$$T(j\omega,t) = \sum_{k\in\mathbb{N}} T_k(j\omega)e^{-k\alpha t} , \tag{19}$$

where frequency responses $T_k(j\omega)$ are given by the theorem 3.

Theorem 3

Frequency responses $T_k(j\omega)$ of relation (19) are given by :

$$T_k(j\omega) = \frac{\beta_k(j\omega) - \sum_{i=0}^{k-1} T_i(j\omega)\beta_{k-i}(j\omega - i\alpha)}{1 + \beta_0(j\omega - k\alpha)} \tag{20}$$

and

$$T_0(j\omega) = \beta_0(j\omega)/(1 + \beta_0(j\omega)) \tag{21}$$

if and only if $1 + \beta_0(j\omega - k\alpha) \neq 0 \quad \forall k \in \mathbb{N}$ and $\forall \omega \in \mathbb{R}^*$.

❑

Proof : Compute the output of $\mathcal{B}$ *using relation (9) if the input of* $\mathcal{T}$ *is* $r(t)=e^{j\omega t}$ *. Then by identification, find frequency responses* $T_k(j\omega)$.

Similar theorems for the other TVFRs ($S(j\omega, t)$, $PS(j\omega, t)$ and $CS(j\omega, t)$) can be established in the same way (Garcia, 2001).

5. NUMERICAL TREATMENTS

TVFRs of time varying systems with asymptotically constant coefficients admit infinite series expansions that must be truncated at an order N for a numerical application. Given the recursive form of relations relations (13), (14), (18), (20) which defines frequency responses $H_k(j\omega)$, $\beta_k(j\omega)$, $T_k(j\omega)$, (but also $CS_k(j\omega)$, $S_k(j\omega)$, $PS_k(j\omega)$ and $C_k(j\omega)$), and given the convergence of these relations, (for proof see (Garcia, 2001), and (Wascow, 1976) for an example of proof) the truncation problem can be easily solved. Indeed, relation (13), (14), (18) and (20) can be computed until the modulus of frequency responses $H_k(j\omega)$, $\beta_k(j\omega)$, $T_k(j\omega)$ (but also $CS_k(j\omega)$, $S_k(j\omega)$, $PS_k(j\omega)$ and $C_k(j\omega)$), $k > N$, is less than a fixed real number ε.

6. FREQUENCY STABILITY CONDITION

This paragraph gives a criterion which permit to check the stability of the feedback system of Fig. 1. To establish this criterion, the following property must first be given.

Property 1

If $T(j\omega, t)$ denotes the TVFR of the feedback system of Fig. 1, and if $\{p_1, \cdots, p_m\}$ denotes the set of poles of transmittance $T_0(s)$, then poles of transmittance $T_k(s)$ are elements of the union of sets $\{p_1 + i\alpha, \cdots, p_m + i\alpha\}$ with $i \in [0, \cdots k]$.

❑

Proof : Suppose that $\{p_1, \cdots, p_m\}$ *denotes the set of poles of transmittance* $T_0(s)$*, then use relation (20).*

If it is taken that transmittance $T_0(s)$ has m_r real poles and m_c pair of complex conjugate poles (where ρ_i and σ_i are respectively the real part and the imaginary part of pole p_i), then, using property 1, $T_k(s)$ is :

$$T_k(s) = \sum_{i=1}^{m_r}\sum_{l=0}^{k}\frac{A_{i,l}}{s - l\alpha - p_i} + \sum_{i=1}^{m_c}\sum_{l=0}^{k}\frac{B_{i,l}s + C_{i,l}}{\frac{s^2}{(\rho_i + l\alpha)^2 + \sigma_i^2} - \frac{2s(\rho_i + l\alpha)}{(\rho_i + l\alpha)^2 + \sigma_i^2} + 1}. \quad (22)$$

Given that the output of the system characterised by TVFR $T(j\omega, t)$ is (Zadeh, 1950):

$$y(t) = \sum_{k\in\mathbb{N}}\frac{1}{2\pi}e^{-k\alpha t}\int_{-\infty}^{+\infty}T_k(j\omega)R(j\omega)e^{-j\omega t}d\omega\,, \quad (23)$$

the feedback system of Fig. 1 (system $\mathcal{T}$: $r(t) \rightarrow y(t)$) is thus exponentially stable if the real part of poles of $T_k(s)$ is less than $k\alpha$, or given property 1, if the $T_0(s)$ has only negative real part poles. Given that analogous conditions for TVFRs $S(j\omega, t)$, $PS(j\omega, t)$ and $CS(j\omega, t)$ can be established, the following theorem can be given.

Theorem 4

Feedback system of Fig. 1 is exponentially stable if and only if transmittance $T_0(s)$ has only negative real part poles or equivalently given relation (21), if transmittance $\beta_0(s)$ respect the Nyquist criterion.

❑

Proof : see comments above

7. CRONE CONTROL OF TIME VARYING SYSTEMS WITH ASYMPTOTICALLY CONSTANT COEFFICIENTS

7.1. *Objectives*

To extend Crone control to the control of time varying plants with asymptotically constant coefficients, an open loop behaviour for the nominal parametric state of the plant is required which :

- ensures a near stationary behaviour of the closed loop system (not perfectly stationary due to the truncation of TVFR of the controller) ;
- ensures performances set by the designer such as rapidity and the resonance ratio in tracking of the closed loop system ;
- takes into account the behaviour of the plant at the low and the high frequencies to ensure satisfactory accuracy of steady state, and immunity of the plant input to measurement noise.

Whenever the plant is reparametrated, namely if the plant $\mathcal{P}$ is element of the description family $\mathbb{P}$, this open loop must also ensure :

- robust closed loop stability and performance ;
- satisfactory immunity of the closed loop to the time varying character of the plant.

As for the stationary case, the behaviour thus defined can be described for stable plants, by a transmittance based on frequency limited complex fractional integration (Oustaloup *et al*., 2000) :

$$\beta(p) = K\left(1+\frac{\omega'_b}{p}\right)^{n_b}\left(\frac{1+\frac{p}{\omega_b}}{1+\frac{p}{\omega'_b}}\right)\left(\frac{1+\frac{p}{\omega_h}}{1+\frac{p}{\omega_b}}\right)^{a}$$

$$\left(Re\left[\left(C_0\frac{1+\frac{p}{\omega_h}}{1+\frac{p}{\omega_b}}\right)^{ib}\right]\right)^{-sign(b)}\left(\frac{1+\frac{p}{\omega'_h}}{1+\frac{p}{\omega_h}}\right)^{2}\frac{1}{\left(1+\frac{p}{\omega'_h}\right)^{n_h}} \quad (25)$$

where

$$C_0 = \left[\left(1+\frac{\omega_r^2}{\omega_b^2}\right)\Big/\left(1+\frac{\omega_r^2}{\omega_h^2}\right)\right]^{1/2}. \quad (26)$$

K ensures the open loop unit gain frequency ω_u set by the designer. ω'_b, ω_b, ω_h and ω'_h are transitional frequencies. $n_b \in \mathbb{N}$ and $n_h \in \mathbb{N}$ are respectively the asymptotic behaviour orders in open loop at the low ($\omega < \omega'_b$) and the high ($\omega > \omega'_h$) frequencies. $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ are the real and imaginary orders of integration. ω_r is the resonance frequency close to ω_u.

7.2. *Optimisation of the open loop behaviour*

The optimisation of the open loop behaviour consists in determining the seven optimal parameters of the nominal open loop transmittance $\beta(s)$:

- optimal real integration order a_{opt}, and optimal gain K_{opt} ;
- optimal imaginary integration order b_{opt} ;
- optimal transitional frequencies ω'_{bopt}, ω_{bopt}, ω_{hopt} and ω'_{hopt}.

The unit gain frequency and the tangency to an iso-overshoot contour of magnitude Q are chosen by the designer, so only five independent parameters need to be considered.

The open loop behaviour which satisfies the objectives defined in section 7.1 can be computed by solving a constrained optimisation problem.
The performance criterion and constraints which provide optimal open loop behaviour respecting the objectives of section 7.1 thus comprise terms which guarantee :

- robustness of the stability degree of the control
- immunity of the control to the time varying character of the plant
- the performance objectives set by the designer.

To define the constrained optimisation problem analytically, the standard control scheme of Fig. 1 is considered.

Robustness of stability, and immunity to the time varying character of the plant

In the time domain, the stability degree can be estimated by the first overshoot of the step response in tracking. In the frequency domain, if the closed loop is near-stationary, this first overshoot can be estimated from the resonance ratio in tracking, Q, deduced from the frequency response of $T_0(j\omega)$. This extension of third generation Crone control thus ensures the robustness of the stability degree through :

- the minimisation of the resonance ratio variations of the stationary part of $T(j\omega, t)$, $T_0(j\omega)$, (extreme variations being characterised by $Q_{\min}$ and $Q_{\max}$)
- the minimisation of the time varying part of $T(j\omega, t)$ at reparametration of the plant.

This lead to the minimisation of the criterion :

$$J = (Q_{\max} - Q)^2 + (Q_{\min} - Q)^2 + \delta \sup_{\substack{P\in\mathbb{P},\omega\in\mathbb{R}^+ \\ k\in\mathbb{N}}} |T_k(j\omega)| . \quad (27)$$

With a judicious choice of weighting coefficient δ, the minimisation of criterion (27) ensures the minimisation of the resonance ratio variations of $T_0(j\omega)$ and the minimisation of the time varying part of the TVFR $T(j\omega, t)$.

Performance objectives : shaping of the control loop TVFRs

Minimisation of criterion (27) sometimes produce undesirable closed loop behaviours for one or more plant $\mathcal{P}$ of the description family $\mathbb{P}$. We thus define how each function $CS(j\omega, t)$, $S(j\omega, t)$ and $T(j\omega, t)$ should be shaped to eliminate these behaviours.

The solicitation level of the plant input is taken into account through limitation of the TVFR $CS(j\omega, t)$, namely :

$$\sup_{P\in\mathbb{P},t\in\mathbb{R}^+} |CS(j\omega,t)| < CS_{adm}(\omega), \qquad \forall\omega\in\mathbb{R}^+, \quad (28)$$

where $CS_{adm}(\omega)$ is the maximum admissible value of the modulus of $CS(j\omega, t)$.

Also, in order to ensure a satisfactory rejection of plant output disturbances, function $S(j\omega, t)$ is bounded by the constraint :

$$\sup_{P\in\mathbb{P},t\in\mathbb{R}^+} |S(j\omega,t)| < S_{adm}(\omega), \qquad \forall\omega\in\mathbb{R}^+, \quad (29)$$

where $S_{adm}(\omega)$ is the maximum admissible value of the modulus of $S(j\omega, t)$.

Finally, in order to cancel the effects of hauling on the step response in relation to its value in steady state and to ensure a satisfactory rejection of measurement noise, the three following constraints are introduced :

$$\sup_{P\in\mathbb{P},0<\omega<\omega_1} |T_0(j\omega)| < h_1, \qquad \inf_{P\in\mathbb{P},0<\omega<\omega_2} |T_0(j\omega)| > h_2,$$

and (30)

$$\sup_{P\in\mathbb{P},\omega_3<\omega} |T_0(j\omega)| < h_3 \quad h_1\in\mathbb{R}, \quad h_2\in\mathbb{R}, \quad h_3\in\mathbb{R}.$$

Optimisation of the open loop behaviour

Given the previous comments, the optimisation of the open loop behaviour consists in determining the five optimal parameters of the nominal open loop transmittance $\beta(s)$ which minimise criterion (27) and satisfy the constraints (28), (29) and (30). The optimisation algorithm is based on the non linear simplex (Oustaloup and Mathieu, 1999).

7.3 - *Optimal controller*

The optimal controller is computed by pseudo-inversion so that the cascade connection of the controller and the nominal plant can be described by transmittance $\beta(j\omega)$. The resulting controller can be described by a TVFR $C(j\omega, t)$ of the form (Garcia, 2001) :

$$C(j\omega,t) = C_0(j\omega) + \sum_{k=1}^{\infty} C_k(j\omega) e^{-k\alpha t} . \quad (31)$$

The synthesis of the controller thus consists in the approximation of transmittances $C_k(s)$ by transmittances of the form :

$$C_k(j\omega) = \sum_{i=0}^{n_k} b_{k,i}(j\omega)^i \Bigg/ \sum_{i=0}^{d_k} a_{k,i}(j\omega)^i , \quad (32)$$

where degrees n_k and d_k are set by the designer. Two techniques can be used to determine coefficients $a_{k,i}$ and $b_{k,i}$ of relation (32). The first is a non iterative synthesis method based on the elementary symmetrical functions of Vietes roots (Oustaloup and Mathieu, 1999) and the second is based on the resolution of a linear programming problem (Oustaloup and Mathieu, 1999).
In general, to allow the implantation of the $C(j\omega, t)$ controller, relation (23) must be truncated.

8. APPLICATION

8.1. Description of the plant

Third generation Crone control is applied to the synthesis of a robust controller $C(j\omega, t)$ which ensures stability and a near stationary behaviour to the feedback system of Fig. 2.

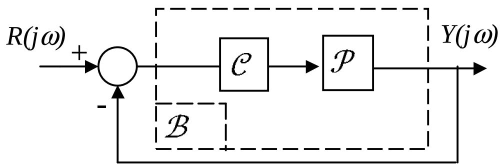

Fig. 2. *Feedback system*

Plant $\mathcal{P}$ is described for the nominal parametric state of the plant by the state space description :

$$\begin{bmatrix}\dot{x}(t)\\ \ddot{x}(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & -A_0 - A_1 e^{-\alpha t}\end{bmatrix}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} u(t), \qquad y(t)=x(t), \quad (33)$$

with $A_0 = 5$, $A_1 = 8$ and $\alpha = 2$. Plant $\mathcal{P}$ is submitted to the parametric variations characterised by :

$$3 \le A_0 \le 7 , \qquad \text{and} \qquad 5 \le A_1 \le 10 . \quad (34)$$

Using theorem 1, time varying frequency of plant $\mathcal{P}$ is given by :

$$P(j\omega,t) = \sum_{k\in\mathbb{N}} P_k(j\omega)\mathrm{e}^{-k\alpha t} \quad \text{with} \quad P_0(s) = \frac{1}{s^2 + A_0 s + 1}, \tag{35}$$

and where frequency responses $P_k(j\omega)$ are defined, if $k > 0$, by :

$$P_k(j\omega)\left((j\omega - k\alpha)^2 + A_0(j\omega - k\alpha) + 1\right) + P_{k-1}(j\omega)A_1(j\omega - (k-1)\alpha) = 0 . \tag{36}$$

The objective of this example is to control output $y(t)$.

8.2. Synthesis of the controller

The unit gain frequency is fixed at ω_u = 30 rd/s for the nominal behaviour of the plant, and the asymptotic behaviour orders in open loop at low and at high frequencies are fixed at n_b = 1 and n_h = 3.
The optimal open loop behaviour which minimises the criterion

$$J = (Q_{\max} - 2.3)^2 + (Q_{\min} - 2.3)^2 + \delta \sup_{\substack{P\in\mathbb{P},\nu\in\mathbb{R}^+,\\ k\in\mathbb{N}}} \left|T_k(j\omega)\right| , \tag{37}$$

(weighting coefficient δ is chosen equal to 0.5 after a few trials) is determined under constraints

$$\sup_{P\in\mathbb{P},0<\omega<\omega_1} \left|T_0(j\omega)\right| < h_1, \qquad \inf_{P\in\mathbb{P},0<\omega<\omega_2} p \left|T_0(j\omega)\right| > h_2,$$
and (38)
$$\sup_{P\in\mathbb{P},\omega_3<\omega} \left|T_0(j\omega)\right| < h_3 ,$$

with h_1 = 1 dB, h_2 = -1 dB, h_3 = -5 dB, ω_1 = 8 rd/s, ω_2 = 5 rd/s and ω_3 = 100 rd/s.

This optimisation gives :

a = 1.258, b = 0.9394, K = 34.14,
ω'_b = 1.33 rd/s, ω_b = 2.66 rd/s,
ω_h = 66.4 rd/s, ω'_h = 132.8 rd/s,

and then permits the synthesis of an optimal controller which has the form :

$$C(j\omega,t) = C_0(j\omega) + C_1(j\omega)e^{-\alpha t} , \tag{39}$$

where frequency responses $C_0(j\omega)$ and $C_1(j\omega)$ are respectively defined by (given comments of section 7.3):

$$C_0(j\omega) = \beta(j\omega)\left((j\omega)^2 + (j\omega)A_0 + 1\right)$$
and (40)
$$C_1(j\omega) = (j\omega)A_1\beta(j\omega) .$$

Fig. 3 shows the responses of the closed loop with two extreme parametric states of the plant, to the step function $r(t)$=H(t-τ) with τ= 0, τ= 1, τ= 2 and τ= 3 (H(t) denotes the Heaviside function). It also demonstrates the efficiency of the synthesis method in spite of plant uncertainties and the time varying character of the plant.

9 – Conclusion

In this paper, an extension of Crone control to time varying systems with asymptotically constant coefficients is given. This robust control method is based on the computation of an optimal open loop behaviour which minimises the stability degree variations of the closed loop system as the plant is reparametrated. This extension is possible through the representation of considered systems using time varying frequency responses.

So, frequency methods prove to be very efficient for robust control of time varying systems. Thus, we aim now to extend the synthesis method presented in this paper to other classes of time varying systems using time varying frequency response representations.

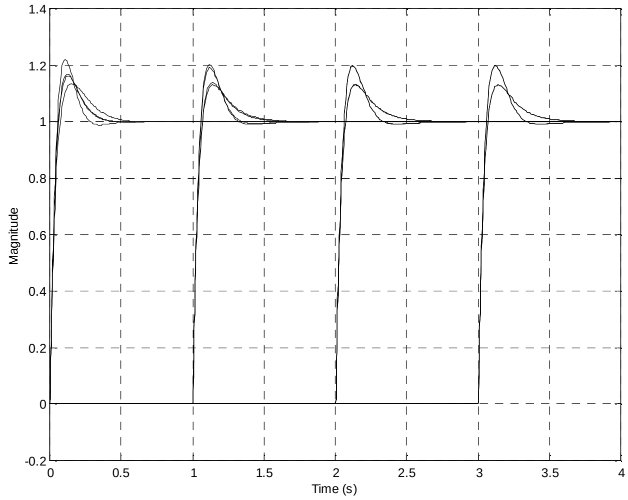

Fig. 3. Closed loop time responses corresponding to the extreme parametric states of the plant to step inputs $r(t)$=$H(t$-$\tau)$, with τ= 0, τ= 1, τ= 2 and τ= 3

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