15th Triennial World Congress, Barcelona, Spain

MOTION PLANNING FOR A LINEARIZED KORTEWEG-DE VRIES EQUATION WITH BOUNDARY CONTROL

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Abstract: Explicit motion planning of a linearized Korteweg-de Vries equation with boundary control is achieved. The control is obtained through a "parametrization" of the trajectories of the system which is a generalization of the Brunovsky decomposition for finite-dimensional linear systems.*Copyright © 2002 IFAC*

Keywords: boundary control, motion planning, Korteweg-de Vries equation, Brunovsky form, flatness, controllability

INTRODUCTION

In this paper we illustrate a method for explicit approximate motion planning on the linearized Korteweg-de Vries equation with boundary control

$$\begin{cases} X_t + X_{xxx} + X_x = 0 \\ X(0,t) = X(1,t) = 0 \\ \qquad X_x(1,t) = u(t). \end{cases} \tag{1}$$

More precisely, given any final time T, any initial state $\Xi_0 \in L^2(0,1)$ and any final state $\Xi_T \in L^2(0,1)$, we compute an open-loop control $t \in [0,T] \mapsto u(t)$ steering the system from Ξ_0 to an arbitrarily small neighborhood of Ξ_T. In other words, we prove approximate controllability for every time T. Though a much stronger mathematical result can be found in (Rosier, 1997) (exact controllability of (1) and of the original (nonlinear) KdV equation), the interest of our approach is an explicit construction based on a "parametrization" of the trajectories of (1). The idea is reminiscent of flatness (Martin *et al.*, 1997) and can be seen as a generalization of the Brunovsky decomposition for finite-dimensional linear systems. The present point of view, introduced in (Laroche and Martin, 2000), is related to (Laroche *et al.*, 2000), but the approach is rather different; in particular, it is not restricted to all the boundary conditions except the control being on the same side. It seems applicable to many linear boundary control problems (Laroche, 2000).

1. PARAMETRIZATION OF TRAJECTORIES AND MOTION PLANNING

Definition 1. A linear boundary control problem on $[0,1] \times [0,T]$ is *parametrizable* if there exists a family $(\alpha_i)_{i\in\mathbb{N}}$ of functions in $L^2(0,1)$, a family $(a_i)_{i\in\mathbb{N}}$ of real numbers and a subspace $\mathcal{G}$ of $\mathcal{C}^\infty(0,T)$ such that:

- $\forall y \in \mathcal{G}$, the series

$$X(x,t) = \sum_{i=0}^{\infty} \alpha_i(x) y^{(i)}(t) \tag{2}$$

$$u(t) = \sum_{i=0}^{\infty} a_i y^{(i)}(t) \tag{3}$$

are normally convergent and define a (classical) solution of the problem on $[0,1]\times[0,T]$

- $\mathcal{G}$ contains all the polynomials on $[0,T]$
- $\overline{\operatorname{span}\{\alpha_i,\ i\in\mathbb{N}\}} = L^2(0,1)$.

If the subspace $\mathcal{G}$ is big enough, namely contains Gevrey functions of order > 1, such a parametrization directly yields an explicit solution to approximate motion planning (provided the boundary control problem is well-posed), see (Laroche *et al.*, 2000; Laroche, 2000) for more details.

Recall that $y \in \mathcal{C}^\infty(0,T)$ is *Gevrey of order* σ if

$$\exists M, R > 0, \forall i \in \mathbb{N}, \quad \sup_{t\in[0,T]} \left|y^{(i)}(t)\right| \le M\frac{(i!)^\sigma}{R^i}.$$

If $\gamma \le 1$, y is analytic (entire if $\gamma < 1$), hence has a convergent Taylor expansion around any point. But if $\gamma > 1$, y has around at least one point a divergent Taylor expansion; the larger σ, the "more divergent" the expansion. A fundamental example for motion planning is the "smooth unit step" defined by $\Phi_\gamma(0) := 0$, $\Phi_\gamma(1) := 1$ and

$$\Phi_\gamma(t) := \frac{\displaystyle\int_0^t \exp\left(\frac{-1}{\left((1-t)t\right)^\gamma}\right) d\tau}{\displaystyle\int_0^1 \exp\left(\frac{-1}{\left((1-t)t\right)^\gamma}\right) d\tau}, \quad t \in]0,1[.$$

Whatever $\gamma > 0$, Φ_γ is in $\mathcal{C}^\infty(0,1)$ and is Gevrey of order $1 + 1/\gamma$; its Taylor expansion around 0 and 1 is divergent.

Assume now the system is to be steered from the initial state

$$\forall x \in [0,1], \quad X(x,0) = \Xi_0(x), \quad \Xi_0 \in L^2(0,1),$$

to a final state "arbitrarily close" to

$$\forall x \in [0,1], \quad X(x,T) = \Xi_T(x), \quad \Xi_T \in L^2(0,1).$$

Thanks to the density of the α_i's, there exists given any $\varepsilon > 0$ polynomials

$$\Pi_0(x) = \sum_{i=0}^n p_i\alpha_i, \qquad p_i \in \mathbb{R}$$
$$\Pi_T(x) = \sum_{i=0}^n q_i\alpha_i, \qquad q_i \in \mathbb{R}$$

such that $\|\Xi_0 - \Pi_0\| \le \varepsilon$ and $\|\Xi_T - \Pi_T\| \le \varepsilon$ ($\|.\|$ denotes the usual norm on $L^2(0,1)$). On the other hand the function

$$Y(t) := \sum_{i=0}^n p_i\frac{t^i}{i!}\left(1 - \Phi_\gamma\left(\frac{t}{T}\right)\right) + q_i\frac{(t-T)^i}{i!}\Phi_\gamma\left(\frac{t}{T}\right)$$

is Gevrey of order $1 + 1/\gamma$ and satisfies

$$\begin{aligned} &Y^{(i)}(0) = p_i, \quad Y^{(i)}(T) = q_i, \qquad i = 0,\dots,n \\ &Y^{(i)}(0) = 0, \quad Y^{(i)}(T) = 0, \qquad i > n. \end{aligned}$$

It can then be shown that the open-loop control

$$U(t) := \sum_{i=0}^{+\infty} a_i Y^{(i)}(t), \quad t \in [0,T],$$

exactly steers the system from Π_0 to Π_T. Moreover, any C^2 approximate control such that

$$\sup_{t\in[0,T]} \left|\bar U(t) - U(t)\right| + \left|\dot{\bar U}(t) - \dot U(t)\right| \le \varepsilon,$$

in particular the truncated series

$$\bar U(t) := \sum_{i=0}^N a_i Y^{(i)}(t)$$

for N large enough, approximately steers the system from Ξ_0 to Ξ_T.

2. THE PROBLEM IS WELL-POSED

Considering controls $t \in [0,T] \mapsto u(t)$ in $C^2(0,T)$, (1) is a well-posed boundary control problem (see for instance (Curtain and Zwart, 1995, section 3.3)). Indeed, setting

$$Z(x,t) := X(x,t) + G(x)u(t),$$

where

$$G(x) := \sin x - \frac{\sin 1}{1-\cos 1}(1 - \cos x)$$

is the solution of

$$\begin{aligned} G''' + G' &= 0 \\ G(0) = G(1) &= 0 \\ G'(1) &= -1, \end{aligned}$$

we get the abstract evolution equation

$$\dot Z = FZ + G\dot u,$$

where the $L^2(0,1)$ valued operator F is defined on its domain

$$\mathcal{D}(F) = \{f \in H^3(0,1),\ f(0) = f(1) = f'(1) = 0\}$$

by $F(f) = -f''' - f'$. A simple but tedious computation (Laroche, 2000, section 6.3) shows that F is invertible, maximal and dissipative, hence that F is the infinitesimal generator of a strongly continuous semigroup of contractions (see for instance (Pazy, 1983)).

3. "DIRECT" APPROACH

To determine the a_i's and the α_i's, we plug (2)-(3) into the system equations (1) and sort along the derivatives of y. Since by definition $\mathcal{G}$ contain all the polynomials, we get the sequence of ordinary differential equations

$$\forall i \in \mathbb{N}, \quad \left\{\begin{aligned} \alpha_i''' + \alpha_i' + \alpha_{i-1} &= 0 \\ \alpha_i(0) = \alpha_i(1) &= 0 \\ \alpha_i'(1) &= a_i, \end{aligned}\right.$$

where we have set $\alpha_{-1} := 0$. By definition of F and G this means

$$\forall i \in \mathbb{N}, \quad \begin{cases} \alpha_i + a_i G \in \mathcal{D}(F) \\ F(\alpha_i + a_i G) = \alpha_{i-1}, \end{cases}$$

hence

$$\begin{aligned} \alpha_0 &= -a_0 G \\ \alpha_i &= F^{-1}\alpha_{i-1} - a_i G, \quad i \geq 1. \end{aligned}$$

As pointed out in (Laroche and Martin, 2000; Laroche, 2000), this construction is very similar to the Brunovsky decomposition in finite dimension. Indeed, consider the finite dimension system

$$\dot{X} = FX + FGu, \quad X \in \mathbb{R}^n,$$

with F an invertible $n \times n$ matrix. This system is to be thought of as (1) since it is transformed by

$$Z(t) := X(t) + Gu(t)$$

into

$$\dot{Z} = FZ + G\dot{u}.$$

If we want a parametrization of the form,

$$\begin{aligned} X(t) &= \sum_{i=0}^{n-1} \alpha_i y^{(i)}(t) \\ u(t) &= \sum_{i=0}^{n} a_i y^{(i)}(t), \end{aligned}$$

with $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}^n$ we readily find

$$\begin{aligned} \alpha_0 &= -a_0 G \\ \alpha_i &= F^{-1}\alpha_{i-1} - a_i G, \quad i = 1, \ldots, n-1. \end{aligned}$$

Moreover, the a_i's must be (up to a constant factor) the coefficients of the characteristic polynomial of F. If $\alpha_1, \ldots, \alpha_n$ form a basis (i.e., if the system is controllable), the system written on this basis is in controller canonical form.

The intuitive idea in infinite dimension is thus to take for the a_i's the coefficients of the *Fredholm determinant* of the operator F, i.e., the entire function Δ_F whose zeros are the eigenvalues of F (such an entire function, which obviously generalizes the characteristic polynomial, exists because F is the inverse of an integral operator). Let $f_i(x, \lambda)$, $i = 1, 2, 3$ be the fundamental solutions at $x = 1$ of

$$f''' + f' + \lambda f = 0, \tag{4}$$

i.e., the solutions which satisfy the initial conditions

$$f_i^{(j-1)}(1, \lambda) = \delta_{ij}, \quad i, j = 1, 2, 3. \tag{5}$$

Now, λ is an eigenvalue of F if and only if (4) has a nonzero solution $f \in \mathcal{D}(F)$. Since $f = c_1 f_1 + c_2 f_2 + c_3 f_3$ for some real numbers c_1, c_2, c_3, this is possible if and only if

$$\Delta_F(\lambda) := \begin{vmatrix} f_1(0,\lambda) & f_2(0,\lambda) & f_3(0,\lambda) \\ f_1(1,\lambda) & f_2(1,\lambda) & f_3(1,\lambda) \\ f_1'(1,\lambda) & f_2'(1,\lambda) & f_3'(1,\lambda) \end{vmatrix} = 0.$$

Using (5), the Fredholm determinant of F is thus

$$\Delta_F(\lambda) = f_3(0, \lambda).$$

We now have compute the coefficients in the expansion of Δ_F. In this example, we could do this directly by first computing $f_3(0, \lambda)$ in closed form. However, this is very tedious and we use instead a more general alternative approach. Indeed, it can be seen that

$$\Delta_F(\lambda) = \sum_{i=0}^{\infty} \big(W^i \tilde{f}_3\big)(0)\lambda^i,$$

where

$$\tilde{f}_3(x) := f_3(x, 0) = 1 - \cos(1 - x)$$

and W is the operator defined on $L^2(0,1)$ by

$$f = Wg \quad \Leftrightarrow \quad \begin{cases} -f''' - f' = g \\ f(1) = f'(1) = f''(1) = 0, \end{cases}$$

in other words

$$\big(Wg\big)(x) = \int_x^1 \big(\cos(x - \xi) - 1\big) g(\xi) d\xi.$$

Moreover, taking $a_i := \big(W^i \tilde{f}_3\big)(0)$ ensures the series (3) is normally convergent when y is Gevrey of order at most 3. The proof is exactly similar to that of proposition 2 in the following section. We are now in position to compute the α_i's. Unfortunately, we have not yet been able to directly prove the convergence of (2) and the density of the α_i's, because our present proof requires F to be a Riesz-spectral operator, a fact we have not been able to establish. In the following section we prove by an "indirect" approach that (1) is parametrizable. We conjecture (and have experimentally checked the first terms using Maple) that the α_i's obtained by the "direct" and the "indirect" approach are identical.

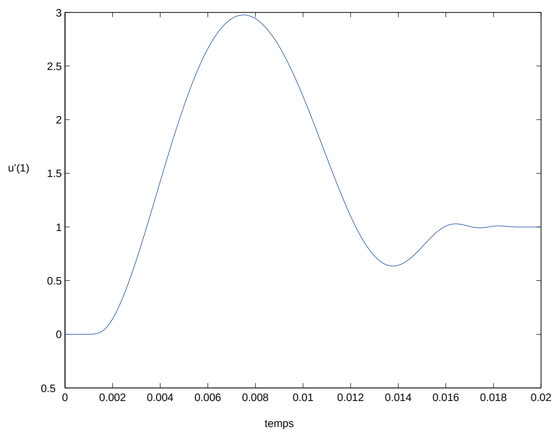

Fig. 1. Open-loop control $U(t)$.

Before going further, we illustrate the relevance of this motion planning method with a numerical simulation. The goal is to steer the system from the initial steady state $X(x, 0) = 0$ to (a neighborhood of) the final steady state $X(x,T) = \frac{\sin 1}{1-\cos 1}(1-\cos x) - \sin x$, with $T := 0.02$. Notice that every steady state is a multiple of α_0, hence of G; in other words, we do not need to compute the α_i, $i>0$ to steer the system from rest to rest. We have thus taken (with the notations of

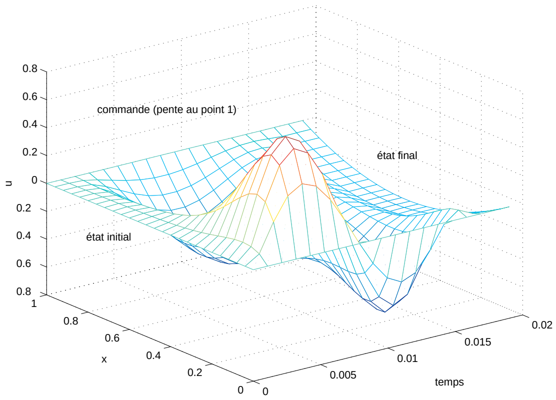

Fig. 2. Motion of the system.

section 1)

$$Y(t) := \Phi_\gamma\left(\frac{t}{T}\right), \qquad 1 < 1+1/\gamma \leq 3.$$

and as the approximate control the truncated series

$$\bar{U}(t) := \sum_{i=0}^{5} a_i Y^{(i)}(t).$$

The results (with $\gamma := 1$) are displayed on figures 1 and 2.

4. "INDIRECT" APPROACH

We now tackle the same problem but first perform the boundary feedback

$$u(t) := X_x(0,t) + v(t),$$

which symmetrizes the boundary conditions and transforms (1) into

$$\begin{cases} X_t + X_{xxx} + X_x = 0 \\ X(0,t) = X(1,t) = 0 \\ X_x(1,t) - X_x(0,t) = v(t). \end{cases} \tag{6}$$

The motivation for this feedback is to give rise to an operator with antisymmetric hence normal algebraic inverse, easily shown to be Riesz-spectral. This feedback is just a technical intermediary and need not be actually performed. As mentioned earlier, we conjecture it does not change the α_i's.

The problem (6) is also well-posed: setting

$$Z(x,t) := X(x,t) + B(x)u(t),$$

where

$$B(x) := \frac{1}{2}\left(\sin x - \frac{\sin 1}{1-\cos 1}(1-\cos x)\right) = \frac{1}{2}G(x)$$

is the solution of

$$\begin{aligned} B''' + B' &= 0 \\ B(0) = B(1) &= 0 \\ B'(1) - B'(0) &= -1, \end{aligned}$$

we get the abstract evolution equation

$$\dot{Z} = AZ + B\dot{v}$$

where the operator A is $L^2(0,1)$ valued, and defined on its domain

$$\mathcal{D}(A) = \{f \in H^3(0,1), f(0) = f(1) = f'(1) - f'(0) = 0\}$$

by $A(f) = -f''' - f'$. Its algebraic inverse is the integral operator K on $L^2[0,1]$ defined by

$$(Kg)(x) := \int_0^1 k(x,\xi)g(\xi)d\xi,$$

with the antisymmetric kernel

$$k(x,\xi) := \begin{cases} \dfrac{2}{\sin\frac{1}{2}} \sin\dfrac{\xi}{2} \sin\dfrac{x-\xi}{2} \sin\dfrac{1-x}{2} & \text{if } 0 \leq \xi \leq x, \\ \dfrac{2}{\sin\frac{1}{2}} \sin\dfrac{x}{2} \sin\dfrac{x-\xi}{2} \sin\dfrac{1-\xi}{2} & \text{if } x \leq \xi \leq 1. \end{cases}$$

K is compact (Curtain and Zwart, 1995, Theorem $A.3.24$) and normal. Therefore (Curtain and Zwart, 1995, Theorem $A.4.25$), A is closed, its eigenvalues are isolated and its eigenvectors form an orthogonal basis for $L^2(0,1)$. Since moreover (Laroche, 2000, section 6.3) its eigenvalues are simple, A is a Riesz-spectral operator. Since K is antisymmetric, all the eigenvalues of A lie on the imaginary axis and are pairwise conjugate; therefore (Curtain and Zwart, 1995, Theorem 2.3.5), A is the infinitesimal generator of a strongly continuous semigroup of contractions.

Exactly as in the previous section, we get the sequence of ordinary differential equations

$$\forall i \in \mathbb{N}, \quad \begin{cases} \alpha_i''' + \alpha_i' + \alpha_{i-1} = 0 \\ \alpha_i(0) = \alpha_i(1) = 0 \\ \alpha_i'(1) - \alpha_i'(0) = a_i, \end{cases} \tag{7}$$

where we have set $\alpha_{-1} := 0$. In other words

$$\begin{aligned} \alpha_0 &= -a_0 B \\ \alpha_i &= K\alpha_{i-1} - a_i B, \quad i \geq 1, \end{aligned}$$

The Fredholm determinant of A is given by

$$\begin{aligned} \Delta_A(\lambda) &:= \begin{vmatrix} e_1(0,\lambda) & e_2(0,\lambda) & e_3(0,\lambda) \\ e_1(1,\lambda) & e_2(1,\lambda) & e_3(1,\lambda) \\ e_1'(1,\lambda) & e_2'(1,\lambda) - e_2'(0,\lambda) & e_3'(1,\lambda) \end{vmatrix} \\ &= e_2(1,\lambda)e_3'(1,\lambda) - e_3(1,\lambda)e_2'(1,\lambda) + e_3(1,\lambda), \end{aligned}$$

where $e_i(x,\lambda)$, $i = 1,2,3$ are the fundamental solutions at $x=0$ of (4), i.e., the solutions which satisfy the initial conditions

$$e_i^{(j-1)}(1,\lambda) = \delta_{ij}, \quad i,j = 1,2,3.$$

We can simplify $\Delta_A(\lambda)$ by using the natural symmetries of (4). Clearly,

$$f_i(x,\lambda) = e_i(1-x,-\lambda), \quad i = 1,2,3.$$

On the other hand $f_3 = c_1e_1+c_2e_2+c_3e_3$ for some real numbers c_1, c_2, c_3; elementary computations show that

$$c_1 = e_2(1,\lambda)e_3'(1,\lambda) - e_3(1,\lambda)e_2'(1,\lambda) = f_3(0,\lambda).$$

This implies

$$\Delta_A(\lambda) = e_3(1,\lambda) + e_3(1,-\lambda).$$

As in the previous section, we compute the coefficients in the expansion of Δ_A in term of iterates of

$$\tilde{e}_3(x) := e_3(x,0) = 1-\cos x$$

by the operator V defined on $L^2(0,1)$ by

$$f = Vg \quad \Leftrightarrow \quad \begin{cases} -f''' - f' = g \\ f(0) = f'(0) = f''(0) = 0, \end{cases}$$

in other words

$$\big(Vg\big)(x) = \int_0^x \big(\cos(x-\xi)-1\big)g(\xi)d\xi.$$

Proposition 2. Δ_A is an entire function,

$$\Delta_A(\lambda) = \sum_{i=0}^{\infty} a_i\lambda^i,$$

where the coefficients a_i are defined by

$$\forall i \in \mathbb{N}, \quad \begin{cases} a_{2i} = 2\big(V^{2i}\tilde{e}_3\big)(1) \\ a_{2i+1} = 0. \end{cases}$$

Moreover the a_i's satisfy the estimates

$$\exists M, R > 0, \forall i \in \mathbb{N}, \quad a_i \le M\frac{R^i}{(3i)!},$$

hence Δ_A is of Weierstrass order (at most) $\frac{1}{3}$.

PROOF. We first prove

$$\forall i \ge 0, \forall x \in [0,1], \quad \big|\big(V_0^i\tilde{e}_3\big)(x)\big| \le M\frac{x^{3i}}{(3i)!}$$

where $M := \sup_{x\in[0,1]}|\tilde{e}_3(x)|$. Indeed, assume this holds true. Using

$$|\cos x - 1| \le \frac{x^2}{2}$$

and integrating by part twice, this implies for $x \in [0,1]$

$$\begin{aligned} \big|\big(V_0^{i+1}f\big)(x)\big| &= \left|\int_0^x \big(1-\cos(x-\xi)\big)\big(V_0^if\big)(\xi)d\xi\right| \\ &\le M\int_0^x \frac{(x-\xi)^2}{2}\frac{\xi^{3i}}{(3i)!}d\xi \\ &= M\frac{\xi^{3i+3}}{(3i+3)!}. \end{aligned}$$

Using the same technique, we get similar estimates for $\big(V_0^i\tilde{e}_3\big)$, $\big(V_0^i\tilde{e}_3\big)'$, $\big(V_0^i\tilde{e}_3\big)''$ and $\big(V_0^i\tilde{e}_3\big)'''$.

We then claim

$$e_3(x,\lambda) = \sum_{i=0}^{\infty}\big(V_0^i\tilde{e}_3\big)(x)\lambda^i.$$

Indeed, the previous estimates ensure this series is normally convergent, as well as its first, second and third derivatives. It thus defines a function in $\mathcal{C}^3$, which is easily seen to satisfy (4) with the required initial conditions.

The proposition is now obvious, the Weierstrass order resulting from (Saks and Zygmund, 1965, chapter 7). $\square$

Proposition 3. $\forall y \in \mathcal{C}^\infty(0,T)$ with Gevrey order (at most) 3, the series

$$\begin{aligned} X(x,t) &= \sum_{i=0}^{\infty}\alpha_i(x)y^{(i)}(t) \\ u(t) &= \sum_{i=0}^{\infty}\alpha_i'(1)y^{(i)}(t) \end{aligned}$$

are normally convergent and define a classical solution of (1) on $[0,1]\times[0,T]$.

PROOF. The key point is to establish the L^2 estimate

$$\|\alpha_i\| \le M\frac{R^i}{(3i)!}.$$

Since A is antisymmetric, its eigenvalues are purely imaginary and pairwise conjugate. Moreover 0 is not an eigenvalue. Denoting the eigenvalues by λ_i, we can therefore index them on $\mathbb{Z}^\star$ so that $\lambda_{-i} = \lambda_i$ and

$$\frac{\lambda_i}{\sqrt{-1}} < \frac{\lambda_{i+1}}{\sqrt{-1}}$$

(remember all the eigenvalues are simple). Let $(u_i)_{i\in\mathbb{Z}^\star}$ be the associated Hilbert basis of eigenfunctions of (this basis exists since A is Riesz-spectral).

By proposition 2, Δ_A is entire of Weierstrass order $\le$ 1, hence (Saks and Zygmund, 1965, chapter 7) can be factored into the normally convergent infinite product

$$\Delta_A(\lambda) = a_0\prod_{i\in\mathbb{Z}^\star}\left(1-\frac{\lambda}{\lambda_i}\right) = a_0\prod_{i\in\mathbb{N}^\star}\left(1+\frac{\lambda^2}{|\lambda_i|^2}\right)$$

Identifying this product with the expansion of Δ_A, we readily find for $l \in \mathbb{N}$

$$\begin{aligned} a_{2l} &= a_0\sum_{\substack{i_1,\dots i_l\neq 0\\ i_1<\dots<i_l}}\frac{1}{|\lambda_{i_1}|^2\dots|\lambda_{i_l}|^2} \\ a_{2l+1} &= 0. \end{aligned}$$

On the other hand for $i \in \mathbb{N}$

$$\alpha_i = \frac{1}{a_0}\sum_{k\in\mathbb{Z}^\star}u_kc_k\sum_{j=0}^{i}a_j\lambda_k^{j-i}, \tag{8}$$

where the c_k's are the coordinates of α_0, i.e.,

$$\alpha_0 = \sum_{k\in\mathbb{Z}^\star}c_ku_k.$$

Indeed, assuming (8) holds true,

$$\begin{aligned}\alpha_{i+1} &= K\alpha_i + \frac{a_{i+1}}{a_0}\alpha_0 \\ &= \frac{1}{a_0}\sum_{k\in\mathbb{Z}^\star} c_k(Ku_k)\sum_{j=0}^{i} a_j\lambda_k^{j-i} + \frac{a_{i+1}}{a_0}\sum_{k\in\mathbb{Z}^\star} c_k u_k \\ &= \frac{1}{a_0}\sum_{k\in\mathbb{Z}^\star} u_k c_k \sum_{j=0}^{i+1} a_j\lambda_k^{j-i-1},\end{aligned}$$

since $Ku_k = \frac{u_k}{\lambda_k}$ (K is the algebraic inverse of A). This implies

$$\|\alpha_i\|^2 = \frac{1}{|a_0|^2}\sum_{k\in\mathbb{Z}^\star} |c_k|^2 \left|\lambda_k^{-i}\right|^2 \left|\sum_{j=0}^{i} a_j\lambda_k^j\right|^2$$

We recognize in the last factor the expansion at order i of $\Delta_A(\lambda_k)$. Since $a_{2l+1} = 0$ for $l \in \mathbb{N}$,

$$\left|\sum_{j=0}^{2l} a_j\lambda_k^j\right| = \left|\sum_{j=0}^{2l+1} a_j\lambda_k^j\right|$$

and

$$\begin{aligned}\left|\sum_{j=0}^{2l} a_j\lambda_k^j\right| &= |a_0|\left|1 + \sum_{\substack{i_1,\dots i_l\neq 0\\ i_1<\dots<i_l}} \frac{\lambda_k^2}{|\lambda_{i_1}|^2\dots|\lambda_{i_l}|^2}\right| \\ &= |a_0|\left|1 - \sum_{\substack{i_1,\dots i_l\neq 0\\ i_1<\dots<i_l}} \frac{|\lambda_k|^2}{|\lambda_{i_1}|^2\dots|\lambda_{i_l}|^2}\right| \\ &= |a_0|\left|\sum_{\substack{i_1,\dots i_l\neq 0,k\\ i_1<\dots<i_l}} \frac{|\lambda_k|^{2l}}{|\lambda_{i_1}|^2\dots|\lambda_{i_l}|^2}\right| \\ &\leq |a_0|\,|a_{2l}|\left|\lambda_k^{2l}\right|.\end{aligned}$$

This implies

$$\|\alpha_{2l}\|, \|\alpha_{2l+1}\| \leq |a_{2l}|\sqrt{\sum_{k\in\mathbb{Z}^\star}|c_k|^2} = |a_{2l}|\,\|\alpha_0\|.$$

Thanks to property 2, we finally get (up to renaming M and R) the estimate

$$\|\alpha_i\| \leq M\frac{R^i}{(3i)!}.$$

We next find a similar estimate for the uniform norm. By the Cauchy-Schwartz inequality

$$\begin{aligned}\left\|\left(K\alpha_i\right)(x)\right\| &= \left|\int_0^1 k(x,\xi)\alpha_{i-1}(\xi)d\xi\right| \\ &\leq \|k(x,.)\|\,\|\alpha_{i-1}\| \leq \sup_{[0,1]^2}|k(.,.)|\,\|\alpha_{i-1}\|,\end{aligned}$$

which implies (up to renaming M and R)

$$\sup_{x\in[0,1]}|\alpha_i(x)| \leq M\frac{R^i}{(3i)!}.$$

Since the kernel k is twice differentiable with respect to x with a bounded second derivative, we readily obtain (up to renaming M and R) similar estimates for α_i' and α_i''. A similar estimates for α_i''' is then derived from (7).

These estimates clearly imply that the formal solution of (6)

$$\begin{aligned}X(x,t) &= \sum_{i=0}^{\infty}\alpha_i(x)y^{(i)}(t) \\ v(t) &= \sum_{i=0}^{\infty} a_i y^{(i)}(t)\end{aligned}$$

is normally convergent and define a classical solution for any y Gevrey of order at most 3, which ends the proof. □

Finally, we have:

Proposition 4. $\overline{\operatorname{span}\{\alpha_i,\ i\in\mathbb{N}\}} = L^2(0,1)$.

The proof relies only on the fact that A is Riesz-spectral and is similar to (Laroche and Martin, 2000, theorem 1) (see (Laroche, 2000, section 7.4) for more details).

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