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ANALYSIS OF TIME VARYING SYSTEMS USING TIME VARYING S-TRANSFORMS AND TIME VARYING Z-TRANSFORMS

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Abstract: This paper deals with the analysis of continuous and discrete time varying systems using two unusual representations. These representations, based on time varying s-transform and time-varying *z*-transform notions, prove to be highly efficient in the field of automatic control. They also permit the extension of many well known theorems to time varying systems such as the initial and final value theorems.

Keywords: Time varying systems, time varying s-transform, time varying z-transform, final and initial value theorems

1. INTRODUCTION

Transfer functions and associated frequency responses are powerful tools for the analysis and synthesis of stationary systems. Thus, several authors have extended them to time varying systems. For example, Zadeh defined the system function notion (Zadeh, 1961), also called in this paper time varying s-transform notion (TVST), to which the time varying frequency response (TVFR) can be associated, and Jury defined the time varying z-transform notion (TVZT) (Jury, 1964).
Many aspects of the definition of TVSTs and TVZTs correspond to the definition of stationary equivalents. However there has been little interest in TVSTs and TVZTs since Zadeh and Jury, for several reasons.
First of all, the computation of the TVST or of the TVZT representing a time varying system proves to be very difficult in the general case.
Also, the stability of a time varying system can not be directly deduced from the poles of its TVST or its TVZT (Gibson, 1963).
Finally, relations used to determine the transfer function of the connection of several stationary systems, have no equivalent for time varying systems (Gibson, 1963).

In this paper, TVSTs and TVZTs are used for the analysis of time varying systems. These two tools are indeed used to extend many well known theorems to time varying systems, particularly the initial and final value theorems.

The paper is organized as follows. Section 2 deals with the representation of continuous time varying systems using TVSTs and three classes of systems are considered : time varying systems with periodic coefficients, time varying systems with asymptotically constant coefficients and time varying systems with polynomial coefficients. Section 3 deals with the representation of discrete time varying systems with periodic coefficients. Section 4 gives extensions of several well known theorems to time varying system.

2. TIME VARYING FREQUENCY RESPONSES

In the 1950s, Zadeh (Zadeh, 1961) demonstrated that linear time varying systems can be described by TVSTs (also called systems functions) $H(s, t)$. TVSTs are linked to the impulse response of the system, $h(t, \xi)$, which is both a function of the time variable t and of the point in time ξ when the impulse is applied, by the relation :

$$H(s,t) = \mathcal{S}[h(t,\xi)] = e^{-st} \int_{-\infty}^{\infty} h(t,\xi) e^{s\xi} d\xi \quad . \tag{1}$$

This representation of time varying systems is particularly attractive in the area of automatic control because it allows the computation of the steady and transitory states of the system. Indeed, if $y(t)$ is the output of a system described by the TVST $H(s, t)$, then

$$y(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} H(s,t) U(s) e^{st} ds \quad , \tag{2}$$

where c denotes the convergence abscissa of $H(s, t)$ and where $U(s)$ denotes the Laplace transform of the system input. The computation of the TVST of a time varying system is not easy in the general case. However, for some classes of systems several authors have provided solutions.

2.1. Time varying systems with periodic coefficients

We consider continuous time periodic systems characterized by the state space description :

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases}, \tag{3}$$

where $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $x(t) \in \mathbb{R}^{qx1}$ and where coefficients $A(t)$, $B(t)$ and $C(t)$ are real-valued matrices of appropriate dimensions. Matrices $A(t)$, $B(t)$ and $C(t)$ are periodic functions of time variable t, namely :

$$A(t) = A(t+T), \quad B(t) = B(t+T), \quad C(t) = C(t+T), \tag{4}$$

where period T represents the smallest value satisfying relation (4). Matrices $A(t)$, $B(t)$ and $C(t)$ are also supposed continuous on $[0, T]$, respectively elements of $L_2^{qxq}[0,T]$, $L_2^{qx1}[0,T]$ and $L_2^{1xq}[0,T]$ and their derivatives are supposed piecewise constant on $[0, T]$. Matrix $A(t)$, thus admits the following Fourier series expansion :

$$A(t) = \sum_{k \in \mathbb{Z}} A_k e^{jk\omega_0 t} \quad \text{with} \quad \omega_0 = \frac{2\pi}{T}. \tag{5}$$

Similar series expansions are also possible for matrices $B(t)$ and $C(t)$, but using matrices B_k and C_k instead of matrix A_k.
As demonstrated by Zadeh, system (3) can be represented by a TVST, $H(s, t)$, of the form :

$$H(s,t) = \sum_{k \in \mathbb{Z}} H_k(s) e^{jk\omega_0 t}. \tag{6}$$

Theorem 1 (Sabatier *et al*, 1998)

Transmittances $H_k(s)$ of relation (6) are given by

$$\mathcal{H} = \mathcal{C}(\mathcal{N} - \mathcal{A})^{-1}\mathcal{B}, \text{ if } (\mathcal{N} - \mathcal{A})^{-1} \text{exist}, \tag{7}$$

in which vectors $\mathcal{H}$ and $\mathcal{B}$, and matrix $\mathcal{A}$ are respectively given by :

$$\mathcal{H}^T = [\ \dots\ H_{-1}(s)\, H_0(s)\, H_1(s)\ \dots\ ], \tag{8}$$

$$\mathcal{B}^T = \lfloor \dots\ B_{-2}{}^T\ \ B_{-1}{}^T\ \ B_0{}^T\ \ B_1{}^T\ \ B_2{}^T\ \dots \rfloor, \tag{9}$$

$$\mathcal{A} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \dots & A_0 & A_{-1} & A_{-2} & \dots \\ \dots & A_1 & A_0 & A_{-1} & \dots \\ \dots & A_2 & A_1 & A_0 & \dots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{10}$$

Matrix $\mathcal{C}$ is defined as matrix $\mathcal{A}$ but using C_k, and $\mathcal{N} = blkdiag((s + k\omega_0)I_q)$ where I_q denotes the identity matrix of dimension q.

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2.2. Time varying systems with asymptotically constant coefficients

We consider continuous systems characterized by a state space description (3) where matrices $A(t)$, $B(t)$ and $C(t)$ are supposed continuous, bounded and analytic on $\mathbb{R}^+$. They also met the following relations :

$$\lim_{t\to\infty} A(t) = A_c, \quad \lim_{t\to\infty} B(t) = B_c, \quad \lim_{t\to\infty} C(t) = C_c, \tag{11}$$

A_C, B_C and C_C being constant matrices.
Without introducing restrictions, it is supposed that matrix $A(t)$ respects the following series expansion :

$$A(t) = \sum_{k \in \mathbb{N}} A_k e^{-k\alpha t}, \quad \alpha \in \mathbb{R}_+^*, \tag{12}$$

and similarly for matrices $B(t)$ and $C(t)$, but using matrices B_k and C_k, where $A_k \in \mathbb{R}^{qx1}$, $B_k \in \mathbb{R}^{qx1}$, $C_k \in \mathbb{R}^{qx1}$, $\forall k \in \mathbb{N}$

Such a system can be represented by a TVST, $H(s, t)$, of the form (Garcia, 2001) :

$$H(s,t) = \sum_{k \in \mathbb{Z}} H_k(s) e^{-k\alpha t}. \tag{13}$$

Theorem 2 (Garcia, 2001)

Transmittances $H_k(s)$ of relation (13) are given by :

$$\mathcal{H} = \mathcal{C}(\mathcal{N} - \mathcal{A})^{-1}\mathcal{B} \quad \text{if} \quad (\mathcal{N} - \mathcal{A})^{-1} \text{exist}, \tag{14}$$

in which vectors $\mathcal{H}$ and $\mathcal{B}$, and matrix $\mathcal{A}$ are respectively given by :

$$\mathcal{H}^T = [H_0(s)\, H_1(s)\, H_2(s)\ \dots\ ], \tag{15}$$

$$\mathcal{B}^T = \lfloor B_0{}^T\ \ B_1{}^T\ \ B_2{}^T\ \dots \rfloor, \tag{16}$$

$$\mathcal{A} = \begin{bmatrix} A_0 & & & \\ A_1 & A_0 & (0) & \\ A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{17}$$

Matrix $\mathcal{C}$ is defined as matrix $\mathcal{A}$ using C_k, and $\mathcal{N} = blkdiag((s + jk\omega_0)I_q)$ where I_q denotes the identity matrix of dimension q.

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2.3. Time varying systems with polynomial coefficients

Let a system characterized by differential equation :

$$\sum_{k=0}^{n} a_k(t) \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k(t) \frac{d^k u(t)}{dt^k}, \tag{18}$$

where $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and where coefficients $a_k(t)$ and $b_k(t)$ are real-valued functions :

$$a_k(t) = \sum_{l=0}^{q} a_{k,l} t^l \quad \text{and} \quad b_k(t) = \sum_{l=0}^{r} b_{k,l} t^l. \tag{19}$$

An expression of the TVST $H(s, t)$ of system (18) is given in (Rudnitskii, 1960).

3. TIME VARYING Z-TRANSFORM

If $h(n, k)$ denotes the response at time nT_e (T_e being the sampling period) of a discrete time-varying system whose input is a Delta Kronecker function δ_{nk}, ($\delta_{nk} = 1$ if $n = k$, $\delta_{nk} = 0$ if $n \neq k$) then by analogy to the stationary case, the TVZT of this system can be defined by (Jury, 1964) :

$$H(n,z) = \mathfrak{Z}[h(n,k)] = \sum_{r=0}^{\infty} h(n,n-r)z^{-r} \qquad z \in \mathbb{C}, \quad (20)$$

or, using $k = n - r$ (assuming no input before time $kT_e = 0$) :

$$H(n,z) = \sum_{k=-\infty}^{n} h(n,k)z^{-n+k} = z^{-n}\sum_{k=0}^{n} h(n,k)z^{k}\ . \quad (21)$$

Using this representation, the output of the system at time nT_e, $y(n)$, is related to its input by (Jury, 1964) :

$$y(n) = \frac{1}{2\pi j}\int_{\Gamma} H(n,z)U(z)z^{n-1}\mathrm{d}z\,, \qquad (22)$$

where $U(z)$ denotes the z-transform of the input, and Γ is a closed path in the z-plane which encircles the singularities of integral (22) counterclockwise.

3.1. Time varying systems with periodic coefficients

We consider a discrete periodic system characterized by the state space description :

$$\begin{cases} x(n+1) = A'(n)x(n) + B'(n)u(n) \\ y(n) = C'(n)x(n) \end{cases}, \qquad (23)$$

where $u(n) \in \mathbb{R}$, $y(n) \in \mathbb{R}$, $x(n) \in \mathbb{R}^{qx1}$ and where coefficients $A'(n)$, $B'(n)$ and $C'(n)$ are real-valued matrices of appropriate dimensions.

Matrices $A'(n)$, $B'(n)$ and $C'(n)$ are also periodic functions of variable n, namely :

$$A'(n) = A'(n+T'), \qquad B'(n) = B'(n+T'), \qquad C'(n) = C'(n+T'), \qquad (24)$$

where period $T' = MT_e$, $M \in \mathbb{N}$, represents the smallest value satisfying relation (24) and where T_e denotes the sampling period. These matrices are respectively elements of $l_2^{nxn}[0,T']$, $l_2^{nx1}[0,T']$ and $l_2^{1xn}[0,T']$. Matrix $A'(n)$ thus admits the following Fourier series expansion (Garcia, 2001):

$$A'(n) = \sum_{k=0}^{M-1} A'_k\, e^{jk\frac{2\pi}{M}n}\ . \qquad (25)$$

Similar series expansions are also possible for matrices $B'(n)$ and $C'(n)$, but using matrices B'_k and C'_k.

As demonstrated by Jury, system (23) can be represented by a TVZT, $H(n, z)$, of the form (Jury, 1964) :

$$H(n,z) = \sum_{k=0}^{M-1} H'_k(z)e^{jk\frac{2\pi}{M}n}\ . \qquad (26)$$

Theorem 3 (Sabatier and Garcia, 2000)

Transmittances $H'_k(z)$ of relation (26) are given by :

$$\mathcal{H}' = \mathcal{C}'(\mathcal{N}' - \mathcal{A})^{-1}\mathcal{B}' \quad \text{if} \quad (\mathcal{N}' - \mathcal{A})^{-1} \text{ exist,} \quad (27)$$

in which vectors $\mathcal{H}'$ and $\mathcal{B}'$, and matrices $\mathcal{A}'$ are respectively given by :

$$\mathcal{H}'^{T} = [\ H'_0(z) \quad H_1(z) \quad \ldots \quad H'_{M-1}(z)]\ , \qquad (28)$$

$$\mathcal{B}'^{T} = \left[\, B_0'^{\,T} \quad B_1'^{\,T} \quad \ldots \quad B'^{\,T}_{M-1} \,\right], \qquad (29)$$

$$\mathcal{A}' = \begin{bmatrix} A'_0 & A'_{M-1} & A'_{M-2} & \cdots & A'_2 & A'_1 \\ A'_1 & A'_0 & A'_{M-1} & \cdots & A'_3 & A'_2 \\ A'_2 & A'_1 & A'_0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & A'_{M-1} & A'_{M-2} \\ A'_{M-2} & A'_{M-3} & A'_{M-4} & \cdots & A'_0 & A'_{M-1} \\ A'_{M-1} & A'_{M-2} & A'_{M-3} & \cdots & A'_1 & A'_0 \end{bmatrix} . \quad (30)$$

Matrix $\mathcal{C}'$ is defined as matrix $\mathcal{A}'$ using C'_k, and $\mathcal{N}' = blkdiag\left(e^{jk2\pi/M} I_q\right)$, $k \in [0,\cdots,M-1]$, where I_q denotes the identity matrix of dimension q.

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4. ANALYSIS OF TIME VARYING SYSTEMS USING TVST AND TVZT

4.1. Continuous time systems

Using the definitions given by relations (1) and (2), all the following properties can be demonstrated (Garcia, 2001). They are extensions to time varying systems of well known properties for stationary systems.

Frequency displacement

$$\mathfrak{S}\left[e^{a(t-\xi)}h(t,\xi)\right] = H(s-a,t) \qquad (31)$$

Time displacement : displacement in relation to observation instant t

$$\mathfrak{S}[h(t-a,\xi)] = e^{-sa}\ H(s,t-a) \qquad (32)$$

Time displacement : displacement in relation to impulse instant ξ

$$\mathfrak{S}[h(t,\xi+a)] = e^{-sa}\ H(s,t) \qquad (33)$$

Scaling

$$\mathfrak{S}\left[h\left(\frac{t}{a},\frac{\xi}{a}\right)\right] = a\,H\left(sa,\frac{t}{a}\right) \qquad (34)$$

Complex differentiation

$$\mathfrak{S}\left[(-1)^k(t-\xi)^k h(t,\xi)\right] = \frac{d^k H(s,t)}{(ds)^k} \qquad (35)$$

Complex integration

$$\mathcal{S}[h(t,\xi)/(t-\xi)] = \int_{s}^{+\infty} H(s,t)ds \tag{36}$$

Real differentiation in relation to observation instant t

$$\mathcal{S}\left[\frac{\mathrm{d}h(t,\xi)}{\mathrm{d}t}\right] = sH(s,t) + \frac{\mathrm{d}\left[H(s,t)\right]}{\mathrm{d}t} \tag{37}$$

Real differentiation in relation to impulse instant ξ

$$\mathcal{S}\left[\frac{\mathrm{d}^k h(t,\xi)}{\mathrm{d}\xi^k}\right] = (-s)^k H(s,t) + \sum_{i=0}^{k-1} (-s)^{k-1-i} \left.\frac{\mathrm{d}^i h(t,\xi)}{\mathrm{d}\xi^i}\right|_{\xi=t} \tag{38}$$

Integration

$$\mathcal{S}\left[\int_{-\infty}^{\xi} h(t,\xi)\,\mathrm{d}\xi\right] = \frac{\left.\int h(t,\xi)\,d\xi\right|_{\xi=t}}{s} - \frac{1}{s}H(s,t) \tag{39}$$

Real integration

$$\mathcal{S}\left[\int_{\xi}^{t} h(t,\xi)\,\mathrm{d}\xi\right] = \frac{H(s,t)}{s} \tag{40}$$

Initial value theorem

Let $h(t, \xi)$ be the impulse response, and $H(s, t)$ the TVST of a time varying system $\mathcal{H}$. Also, let $g(t, \xi)$ be the impulse response of a system $\mathcal{G}$ such that :

$$g(t,\xi) = \frac{d\left(h(t,\xi)\right)}{d\xi}. \tag{41}$$

Using differentiation property (relation (38)), the TVST of system $\mathcal{G}$ is given by :

$$G(s,t) = sH(s,t) - h(t,t). \tag{42}$$

The limit of $G(s, t)$ as s tends towards infinity is by definition :

$$\lim_{s\to\infty} G(s,t) = \lim_{s\to\infty} \int_0^{\infty} g(t,t-\tau)e^{-s\tau}d\tau. \tag{43}$$

It can be demonstrated if $\exists$ M $\in$ $\mathbb{R}^+$ such that $\int_0^M |g(t,t-\tau)|d\tau < \infty$ and $|g(t,t-\tau)| < \infty$ (Garcia, 2001), that relation (43) is equal to zero, thus using relation (42),

$$\lim_{s\to\infty} G(s,t) = \lim_{s\to\infty} \left(sH(s,t)\right) - h(t,t). \tag{44}$$

Thus, from relation (43) and (44) :

$$\lim_{s\to\infty} \left(sH(s,t)\right) = h(t,t). \tag{45}$$

Now, let $y(t)$ be the response of system $\mathcal{H}$ to the input $u(t)$ applied at time $t = \xi$, then by definition (relation (2)) :

$$y(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} H(s,t)U(s)e^{st}ds\ , \tag{46}$$

or

$$y(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} H(s,t)V(s)e^{s(t-\xi)}ds\ , \tag{47}$$

where $U(s) = \mathrm{e}^{-s\xi}\,V(s)$. Given relation (47), $y(t)$ can be considered as the response of a time varying system of TVST $H(s, t)V(s)$ to impulse $\delta(t\text{-}\xi)$. Thus using relation (45), the limit of $y(t)$ as t tends towards ξ is given by :

$$\lim_{t\to\xi} y(t) = \lim_{s\to\infty} s\,H(s,\xi)\,V(s)\,, \tag{48}$$

or :

$$\lim_{t\to\xi} y(t) = \lim_{s\to\infty} s\,H(s,\xi)\,\mathrm{e}^{s\xi}\,U(s)\,. \tag{49}$$

Relation (49) thus leads to the following theorem.

Theorem 4 – Initial value theorem

Let $h(t, \xi)$ be the impulse response and $H(s, t)$ the TVST of a time varying system $\mathcal{H}$. Let $y(t)$ be the response of system $\mathcal{H}$ to an input $u(t)$ applied at instant ξ. Thus, if the TVSTs of both $h(t, \xi)$ and of $\mathrm{d}h(t,\xi)/\mathrm{d}\xi$ exist, and if $\lim_{s\to\infty} sH(s,\xi)e^{s\xi}\,U(s)$ exist, then the initial value of $y(t)$ is :

$$\lim_{t\to\xi} y(t) = \lim_{s\to\infty} sH(s,\xi)\mathrm{e}^{s\xi}\,U(s)\ , \tag{50}$$

$U(s)$ being the Lapalce transform of $u(t)$.

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Final value Theorem

The limit of $G(s, t)$, TVST of $g(t, \xi)$ given by relation (41), as s tends towards 0 is :

$$\lim_{s\to 0} G(s,t) = \lim_{s\to 0} \int_0^{\infty} \frac{dh(t,t-\tau)}{d\tau}e^{-s\tau}d\tau\ , \tag{51}$$

or after computation of the right side of relation (51) if all the poles of $G(s, t)$ have a negative real part ,

$$\lim_{s\to 0} G(s,t) = \lim_{\tau\to\infty} h(t,t-\tau) - h(t,t-\tau)\big|_{\tau=0}\ , \tag{52}$$

or using relation (42),

$$\lim_{s\to 0} G(s,t) = \lim_{s\to 0} \left(sH(s,t)\right) - h(t,t). \tag{53}$$

Thus, from relation (52) and (53) :

$$\lim_{s\to 0} \left(sH(s,t)\right) = \lim_{\tau\to+\infty} h(t,t-\tau) = \lim_{t\to+\infty} h(t,\xi). \tag{54}$$

using t - $\tau = \xi$.
Now let $y(t)$ be the response of system $\mathcal{H}$ (defined in section 4.2) to input $u(t)$ applied at time $t = \xi$. Then by definition (relation (2)), $y(t)$ can be seen as the response of a time varying system of TVST $H(s, t)U(s)$ to impulse $\delta(t)$. Thus, using relation (54) the limit of $y(t)$, as t tends towards infinity, is :

$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty}\lim_{s\to 0} s\,H(s,t)U(s)\,. \tag{55}$$

Relation (55) leads to the following theorem.

Theorem 5 – Final value theorem

Let $h(t, \xi)$ be the impulse response and $H(s, t)$ the TVST of a time varying system $\mathcal{H}$. Let $y(t)$ be the response of system $\mathcal{H}$ to an input $u(t)$ applied at time ξ. Thus, if the TVST of both $h(t, \xi)$ and of $\mathrm{d}h(t,\xi)/\mathrm{d}\xi$ exist, and if $\lim_{t\to\infty}\lim_{s\to 0}(s)H(s,t)U(s)$ exist, then the initial value of $y(t)$ is:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\lim_{s\to 0} sH(s,t)U(s)\ , \qquad (56)$$

U(s) being the Laplace transform of $u(t)$.

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4.2. Discrete time systems

As in the continuous case, using the definitions given by relations (21) and (22) the following properties can be demonstrated (Garcia, 2001).

Complex displacement

$$\mathfrak{Z}\left[\mathrm{e}^{-a(n-k)T_e}\ h(n,k)\right] = H\left(n, z\,\mathrm{e}^{aT_e}\right) \qquad (57)$$

Time displacement : displacement in relation to observation instant n

$$\mathfrak{Z}[h(n-a,k)] = z^{-a}\ H(n-a,z) \qquad (58)$$

Time displacement : displacement in relation to impulse instant k

$$\mathfrak{Z}[h(n,k+a)] = z^{-a}\ H(n,z) \qquad (59)$$

Scaling

$$\mathfrak{Z}[h(an,ak)] = H\left(an, z^{1/a}\right) \qquad (60)$$

Complex differentiation

$$\mathfrak{Z}\left[\prod_{i=0}^{m-1}(k-n-i)h(n,k)\right] = z^{-m}\,\frac{\mathrm{d}^m H(n,z)}{\mathrm{d}z^m} \qquad (61)$$

Differentiation in relation to observation instant n

$$\mathfrak{Z}\left[\frac{\mathrm{d}}{\mathrm{d}n}h(n,k)\right] = \frac{\mathrm{d}}{\mathrm{d}n}H(n,z) + \ln z\, H(n,z) \qquad (62)$$

Differentiation in relation to impulse instant k

$$\mathfrak{Z}\left[\frac{\mathrm{d}}{\mathrm{d}k}h(n,k)\right] = -\ln z\, H(n,z) \qquad (63)$$

Initial value theorem

Let $h(n, k)$ be the response, at time nT_e of a time varying system $\mathcal{H}$, to the Delta Kroneker function δ_{nk}. TVZT $H(n, z)$ of $\mathcal{H}$ is by definition (relation (20)) :

$$H(n,z) = \sum_{r=0}^{+\infty} h(n,n-r)\,z^{-r} = h(n,n) + \sum_{r=1}^{+\infty} h(n,n-r)\,z^{-r}\ .(64)$$

The limit of $H(n, z)$, as z tends towards infinity, is thus :

$$\lim_{z\to\infty} H(n,z) = \lim_{z\to\infty}\left[h(n,n) + \sum_{r=1}^{+\infty} h(n,n-r)\,z^{-r}\right],\ (65)$$

or simply :

$$\lim_{z\to\infty} H(n,z) = h(n,n)\,. \qquad (66)$$

Thus, if the Delta Kroneker function is applied at time k_0T_e, $k_0 \in \mathbb{N}$, the initial value of system $\mathcal{H}$ is given by :

$$\lim_{n\to k_0} h(n,k_0) = h(k_0,k_0) = \lim_{z\to\infty} H(k_0,z)\,. \qquad (67)$$

Now let $y(n)$ be the response at time nT_e of system $\mathcal{H}$ to the input $u^*(t)$ defined by :

$$u^*(t) = \sum_{k=k_0}^{\infty} u(k)\,\delta(t-kT_e)\,,\quad k_0 \in \mathbb{N}. \qquad (68)$$

$y(n)$ is from definition (relation (22)) :

$$y(n) = \frac{1}{2\pi j}\int_{\Gamma} H(n,z)U(z)z^{n-1}\mathrm{d}z\,, \qquad (69)$$

or

$$y(n) = \frac{1}{2\pi j}\int_{\Gamma} H(n,z)V(z)z^{k_0}z^{n-1}\mathrm{d}z\,, \qquad (70)$$

where

$$V(z) = \sum_{k=0}^{\infty} u(k+k_0)\,z^{-k} = z^{k_0}\,U(z)\,, \qquad (71)$$

Given relation (71), $y(n)$ can be considered as the response of a time varying system of TVZT $H(n, z)V(z)$ to the Delta Kroneker function δ_{nk0}. Thus, using relation (45) the limit of $y(n)$, as n tends towards k_0, is:

$$\lim_{n\to k_0} y(n) = \lim_{z\to\infty} H(k_0,z)V(z) = \lim_{z\to\infty} H(k_0,z)\,z^{k_0}\,U(z)\,. \qquad (72)$$

Relation (72) thus leads to the following theorem.

Theorem 6 – Initial value theorem

Let $H(n, z)$ be the TVZT of a time varying system $\mathcal{H}$ whose response at time nT_e to the Delta Kroneker function δ_{nk} is $h(n, k)$. System $\mathcal{H}$ is supplied by the input signal whose z transform is $U(z)$ given by relation (68). If it exists, the initial value of the output of $\mathcal{H}$ is thus given by

$$\lim_{n\to k_0} y(n) = \lim_{z\to\infty} H(k_0,z)\,z^{k_0}\,U(z)\,. \qquad (73)$$

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Final value Theorem

Let $h(n, k)$ be the impulse response, and $H(n, z)$ the TVZT of a time varying system $\mathcal{H}$. Also, let

$$g(n, k) = h(n, k) - h(n, k+1) \qquad (74)$$

be the impulse response of a time varying system $\mathcal{G}$. By definition (relation (20)), the TVZT $G(n, z)$ of system $\mathcal{G}$ is given by :

$$G(n,z) = \sum_{k=-\infty}^{n} \left[h(n,k) - h(n,k+1)\right] z^{-n+k} . \qquad (75)$$

The limit of $G(n, z)$ as z tends towards 1, is thus :

$$\lim_{z\to 1} G(n,z) = \sum_{k=-\infty}^{n} \left[h(n,k) - h(n,k+1)\right] , \qquad (76)$$

or after simplification :

$$\lim_{z\to 1} G(n,z) = \lim_{i\to\infty} h(n,n-i) - h(n,n+1) . \qquad (77)$$

The impulse response $h(n, k)$ being equal to 0 if $n < k$, relation (77) becomes :

$$\lim_{z\to 1} G(n,z) = \lim_{i\to\infty} h(n,n-i) , \qquad (78)$$

Given relation (74) and using time displacement property (relation (58)), TVZT $G(n, z)$ is :

$$G(n,z) = H(n,z) - z^{-1} H(n,z) = \frac{z-1}{z} H(n,z) , \qquad (79)$$

Thus, using relations (78) and (79), the final value of the impulse response $h(n, k)$ is :

$$\lim_{n\to\infty} h(n,k) = \lim_{n\to\infty} \lim_{z\to 1} (z-1) H(n,z) . \qquad (80)$$

Now, if $y(n)$ denotes the response at time nT_e of system $\mathcal{H}$ to the input $u^*(t)$ given by relation (68), $y(n)$ is given by :

$$y(n) = \sum_{k=k_0}^{+\infty} h(n,k)\, u(k) , \qquad (81)$$

and the final value of $y(n)$ is :

$$\lim_{n\to+\infty} y(n) = \sum_{k=k_0}^{+\infty} \left[\lim_{n\to+\infty} h(n,k) \right] u(k) . \qquad (82)$$

Using relation (80), relation (82) becomes :

$$\lim_{n\to+\infty} y(n) = \sum_{k=k_0}^{+\infty} \left[\lim_{n\to+\infty} \lim_{z\to 1} (z-1) H(n,z) \right] u(k) , \qquad (83)$$

or :

$$\lim_{n\to+\infty} y(n) = \lim_{z\to 1} \left[\lim_{n\to+\infty} (z-1) H(n,z) \right] \left[\sum_{k=k_0}^{+\infty} u(k) z^{-k} \right] , \qquad (84)$$

or given the definition of $U(z)$ (relation (71)) :

$$\lim_{n\to+\infty} y(n) = \lim_{z\to 1} \lim_{n\to+\infty} (z-1) H(n,z) U(z) . \qquad (85)$$

Relation (85) thus leads to the following theorem.

Theorem 7 – Final value theorem

Let $H(n, z)$ be the TVZT of a time varying system $\mathcal{H}$ whose response at time nT_e to the Delta Kroneker function δ_{nk} is $h(n, k)$. System $\mathcal{H}$ is supplied by the input signal whose z transform is $U(z)$. The final value of the output of $\mathcal{H}$ is thus given by

$$\lim_{n\to+\infty} y(n) = \lim_{z\to 1} \lim_{n\to+\infty} (z-1) H(n,z) U(z) , \qquad (86)$$

if $\lim_{n\to+\infty} y(n)$ and $\lim_{n\to\infty} h(n,k)$ $\forall\, k \in \mathbb{N}$ exist.

❑

6. CONCLUSION

In this paper, time varying s-transform (or systems functions) and time varying z-transforms, respectively introduced by Zadeh (Zadeh, 1961) and Jury (Jury, 1964), have been used to extend to continuous and discrete time varying systems, several properties and theorems such as :

- time displacement properties,
- frequency displacement properties,
- scaling properties
- complex and real differentiation properties
- complex and real integration properties
- initial value theorems
- final value theorems.

Computation procedures for time varying s-transform and for time varying z-transforms have also been given for continuous time varying systems with :

- periodic coefficients,
- asymptotically constant coefficients,
- polynomial coefficients,

and for discrete time varying systems with periodic coefficients.

Thus, these results, along with previous results on robust control of periodic systems (Sabatier et al., 1998), (Sabatier and Garcia, 2000), show the efficiency of time varying s-transform, to which time varying frequency responses can be associated, and of time varying z-transforms for the analysis and control of time varying systems.

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