15th Triennial World Congress, Barcelona, Spain

CONTINUOUS-TIME AR PARAMETER ESTIMATION BY USING PROPERTIES OF SAMPLED SYSTEMS

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Abstract: Consider the problem of estimating the parameters in a continuous-time autoregressive (CAR) model from discrete-time samples. In this paper a simple and computationally efficient method is introduced, and analyzed with respect to bias distribution. The approach is based on replacing the derivatives by delta approximations, forming a linear regression, and using the least squares method. It turns out that consistency can be assured by applying a particular prefilter to the data; the filter is easy to compute and is only dependent on the order of the continuous-time system. Finally, the introduced method is compared to other methods in some simulation studies.

Keywords: system identification; autoregressive process; continuous-time; sampling

1. INTRODUCTION

The problem of continuous-time system identification is of fundamental importance in such areas as economics, astrophysics, control and signal processing (Parzen, 1984; Phadke and Wu, 1974; Sinha and Rao, 1991). There exist a variety of different techniques for identifying a continuous-time system. The surveys (Young, 1981; Unbehauen and Rao, 1990) and the books (Sinha and Rao, 1991; Unbehauen and Rao, 1987) offer broad overviews of many of the available techniques.

One particularly interesting and practical scenario is identification of continuous-time systems using discrete-time data. The objective of this paper is to present a least squares solution to the problem of estimating the parameters in a continuous-time autoregressive (CAR) model from discrete-time measurements. A natural question concerns the relevance of studying processes with a pure AR-structure, as it seems quite restrictive. However, there are several examples of the applicability of AR-models, including: astrophysics (Phadke and Wu, 1974), economics (Göing, 1996) and biomedicine (Mateo and Laguna, 2000). Moreover, the results herein seem to be extendable to the case where an input is present as well. Here we will consider a direct approach, where the differentiation operator is replaced by the well known delta forward operator. In this fashion the model is casted into a discrete-time linear regression. Some possible advantages of such an approach include: it is numerically sound, especially for fast sampling, and it is computationally very efficient.

This setup has been used in various papers, for instance, (Bigi *et al.*, 1994; Söderström *et al.*, 1997*b*; Söderström *et al.*, 1997*c*) and it is well known that an ordinary least squares estimate in general will be severely biased. In the above mentioned papers, this problem is cured by either modifying the least squares method, or by restricting the derivative approximation schemes. However, in this paper it is shown that it is possible to use the simple delta operator, and the ordinary least squares method, and still get consistent estimates by applying a particular prefilter to the data. The filter is easy to compute and it depends only on the order of the continuous-time system. To be more precise, the filter is given by the limiting noise shaping filter for the sampled version of the original continuous-time system. Limiting properties for sampled systems have been treated in several papers; for transfer functions, see, *e.g.*, (Åström *et al.*, 1984; Blachuta, 1999), and for stochastic processes, see, *e.g.*, (Wahlberg, 1988). Moreover, under some weak assumptions we provide some explicit formulas for quantifying the bias for the derived method. This will be helpful when trying to compare the method derived in this paper to other methods. By means of numerical examples the introduced method is compared to some similar methods (Bigi *et al.*, 1994; Söderström *et al.*, 1997*b*; Söderström *et al.*, 1997*c*), and shown to work well. The main differences between our approach and other direct methods available in the literature, see, *e.g.*, (Sinha and Rao, 1991; Unbehauen and Rao, 1987) are twofold: Firstly, they differ in how the system of parameter estimation equations is formulated from the original continuous-time system. Here we use the δ-operator, while other choices might be to use some integral approach or some modulating functions approach. Secondly, in this paper we treat the case of a continuous-time white noise source, while many other methods simply ignore the noise, or assume it to be discrete-time white noise.

2. PROBLEM FORMULATION

Consider a stable continuous-time autoregressive (CAR) process

$$\begin{aligned} A_c(p)y(t) &= e_c(t) \\ E\{e_c(t)e_c(s)\} &= \lambda_0^2\delta(t-s) \end{aligned} \tag{1}$$

where $y(t)$ is the output, $e_c(t)$ is a continuous-time white noise source with intensity λ_0^2, $\delta(\tau)$ is Dirac's δ-function and $A_c(p)$ is defined as

$$\begin{aligned} A_c(p) &= p^n + a_1p^{n-1} + \cdots + a_n \\ &= \prod_{i=1}^{n}(p-p_i) \end{aligned} \tag{2}$$

Here $p = \frac{d}{dt}$ denotes the differentiation operator and $\{p_i\}$ are the zeros of $A_c(p)$. Note that by (1) is meant a stochastic process with power spectrum

$$\Phi_c(s) = \frac{\lambda_0^2}{A_c(s)A_c(-s)} \tag{3}$$

For a rigorous treatment of continuous-time stochastic processes see, *e.g.*, (Åström, 1970; Söderström, 1994). The time series (1) is observed at $t_k = hk$, for $k = 1, 2, \ldots, N$. The model order n is assumed to be known. The problem is how to estimate the true parameter vector

$$\boldsymbol{\theta}_0 = [a_1 \cdots a_n]^T \tag{4}$$

from the available discrete-time data in a simple and computationally efficient manner.

Before we describe the approach here taken and state the main results, some preliminary issues concerning sampling of continuous-time AR-processes will be addressed; this in order to provide some answers required in the analysis to follow in Section 4.

3. PRELIMINARIES

Assume that the time series (1) is instantaneously sampled at $t_k = hk$, for $k = 1, 2, \ldots, N$. The discrete-time stochastic process $\{y(t_k)\}_{k=1}^N$ is then a sequence of stochastic variables with the same second order properties as the continuous-time system (1) at the time instances $t = t_k$. The discrete-time system representation, corresponding to the continuous-time system (1), turns out to be given by an autoregressive moving average (ARMA) process, see, *e.g.*, (Åström, 1970; Wahlberg *et al.*, 1993)

$$\begin{aligned} A_d(q)y(t_k) &= C_d(q)e(t_k) \\ E\{e(t_k)e(t_s)\} &= \lambda_h^2\delta_{k,s} \end{aligned} \tag{5}$$

where $e(t_k)$ is discrete-time white noise with variance λ_h^2, and $C_d(q)$ and $A_d(q)$ are stable polynomials of degree n given as

$$C_d(z) = z\prod_{i=1}^{n-1}(z-z_i) \tag{6}$$

$$A_d(z) = \prod_{i=1}^{n}(z-q_i) \tag{7}$$

Note that even though the continuous-time system has relative degree n, the sampled version will have relative degree zero (or, relative degree one if a time delay is included in $C_d(z)$). For an interesting discussion around this issue, see (Wahlberg, 1990).

Given a continuous-time system (1), it is in general a tedious procedure to find the discrete-time representation (5). Hence, it would be desirable to have some simple mappings between the system descriptions (1) and (5). In particular, we would like to be able to express the polynomials $A_d(q)$ and $C_d(q)$ in terms of the parameters of the continuous-time system (1).

It is well known that the zeros of $A_d(q)$ and $A_c(p)$ are related according to

$$q_i = e^{p_ih} \tag{8}$$

By introducing the delta operator

$$\delta = \frac{q-1}{h} \tag{9}$$

the above relation allow us to establish a simple, but useful mapping between $A_c(\delta)$ and $A_d(q)$. It holds that (Larsson, 2001; Larsson and Söderström, 2001)

$$A_d(q) = h^n A_c(\delta) + O(h^2) \tag{10}$$

Unfortunately, there are in general no simple closed form expressions for the zeros of $C_d(q)$. The limiting case for small sampling periods ($h \to 0$) can, however, be characterized. It turns out that for a system with relative degree n, the $C_d(q)$ polynomial converges to a constant polynomial $C_n^*(q)$, which is only dependent on the order of the system and *not* on the continuous-time system parameters as the sampling interval goes to zero (Wahlberg, 1988). In (Larsson, 2001; Larsson and Söderström, 2001) this result is extended to include expressions for the h and h^2 coefficients in a series expansion of the zeros of $C_d(q)$, in terms of the parameters of the continuous-time system. In particular, as a result of Lemma 3.1 in (Larsson, 2001; Larsson and Söderström, 2001), it follows that $C_d(z)$, defined in (6), can be written as

$$C_d(z) = C_n^*(z) + \tilde{C}(z) \tag{11}$$

where

$$C_n^*(z) = z \prod_{i=1}^{n-1} (z - z_i^*) \tag{12}$$

and $\tilde{C}(z) = O(h^2)$. The interpretation of $\tilde{C}(z) = O(h^2)$ is that all the coefficients of $\tilde{C}(z)$ are of order $O(h^2)$. Moreover, it turns out that the roots $\{z_i^*\}$ are the $n-1$ roots of $B_{2n-1}(z)$ inside the unit circle, where the $n-1$ order polynomial $B_n(z)$ is defined as:

$$B_n(z) = b_1^n z^{n-1} + b_2^n z^{n-2} + \cdots + b_n^n \tag{13}$$

with the coefficients b_k^n given by

$$b_k^n = \sum_{l=1}^{k} (-1)^{k-l} l^n \binom{n+1}{k-l}, \quad k = 1, \ldots, n \tag{14}$$

or recursively computed as (Åström *et al.*, 1984)

$$b_1^n = b_n^n = 1 \tag{15}$$

$$b_k^n = k b_k^{n-1} + (n-k+1) b_{k-1}^{n-1} \tag{16}$$

4. LS PARAMETER ESTIMATION

Reconsider the continuous-time AR process defined in (1)-(3), where the time series is observed at $t_k = hk$, for $k = 1, 2, \ldots, N$. It is of interest to estimate the parameter vector (4) from the available data.

As a first step the outputs $y(t_k)$ are filtered by means of an IIR filter

$$y^F(t_k) = \frac{1}{F(q)} y(t_k), \qquad k = 1, \ldots, N \tag{17}$$

where $F(q)$ is a stable polynomial of degree n, yet to be specified. Next, the differentiation operator p^j in (2) is approximated by the delta operator

$$p^j \approx \delta^j \triangleq [\frac{q-1}{h}]^j, \quad j = 1, \ldots, n \tag{18}$$

Notice that this approximation fulfills what is referred to as the *natural conditions* in *e.g.*, (Söderström *et al.*, 1997*b*). In other words, the approximation error in (18) is of order $O(h)$. After substituting the derivatives in (2) by approximations (18) and filtering the outputs (17), the following linear regression model can be formed

$$\begin{aligned} w_F(t_k) &= \boldsymbol{\varphi}_F^T(t)\boldsymbol{\theta} + \varepsilon(t) \\ w_F(t_k) &= \delta^n y^F(t_k) \\ \boldsymbol{\varphi}_F^T(t_k) &= [-\delta^{n-1} y^F(t_k) \ldots - \delta^0 y^F(t_k)] \\ \boldsymbol{\theta} &= [a_1 \cdots a_n]^T \end{aligned} \tag{19}$$

The parameter vector $\boldsymbol{\theta}$ is estimated by a standard least squares method,

$$\hat{\boldsymbol{\theta}}_N = \boldsymbol{R}_N^{-1} \Big[\frac{1}{N} \sum_{t=1}^{N} \boldsymbol{\varphi}_F(t_k) w_F(t_k)\Big] \tag{20}$$

$$\boldsymbol{R}_N \triangleq \frac{1}{N} \sum_{t=1}^{N} \boldsymbol{\varphi}_F(t_k) \boldsymbol{\varphi}_F^T(t_k) \tag{21}$$

which in the asymptotic case ($N \to \infty$) can be written as, *cf.* (Söderström and Stoica, 1989)

$$\hat{\boldsymbol{\theta}} = \boldsymbol{R}^{-1} \boldsymbol{r} \tag{22}$$

where

$$\boldsymbol{R} \triangleq E\{\boldsymbol{\varphi}_F(t_k)\boldsymbol{\varphi}_F^T(t_k)\} \tag{23}$$

$$\boldsymbol{r} \triangleq E\{\boldsymbol{\varphi}_F(t_k) w_F(t_k)\} \tag{24}$$

The least squares estimate (22) will in general give rise to estimates with a severe bias, also for h small. In the case where $F(q) \equiv 1$ there are several papers published that deal with this problem, see, *e.g.*, (Söderström *et al.*, 1997*b*; Söderström *et al.*, 1997*c*). In those papers consistency (an estimate of $\boldsymbol{\theta}$ with a small bias of order $O(h)$) is assured by either modifying the LS-method, or by introducing some restrictions on the derivative approximation schemes. Here we will show that it is possible to use the simple delta approximation and the conceptually clear LS method, and still get consistent estimate of $\boldsymbol{\theta}$ by choosing the $F(q)$ polynomial in a proper way. The result is given in the following lemma:

Lemma 1. If the polynomial $F(q)$ in (17) is chosen as

$$F(q) = C_n^*(q) \tag{25}$$

where $C_n^*(q)$ is defined in equation (12), then the least squares estimate (20) of $\boldsymbol{\theta}$ fulfills

$$\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}_0 + O(h) \tag{26}$$

as $N \to \infty$.

PROOF. See (Larsson and Söderström, 2001; Larsson, 2001). □

An intuitive explanation of the method is as follows: as seen from (5) and (11) the choice $F(q) = C_n^*(q)$ corresponds to the limiting ($h \to 0$) noise shaping filter for the data generating system (5). By recalling that $A_d(q) = h^n A_c(\delta) + O(h^2)$, this means that the filtered outputs obey

$$h^n A_c(\delta) y^F(t_k) = e(t_k) + O(h^2) \tag{27}$$

where $e(t_k))$ is a discrete-time white noise sequence. Hence, we observe that the troublesome $C_d(q)$ polynomial has "disappeared", and therefore should an ordinary LS method give consistent estimates. What is not directly clear from (27), is how the $O(h^2)$ term will effect the estimate.

In order to measure the performance of the method, and to be able to compare it to other methods it would be desirable to have an analytic expression for the dominating bias term. Note that by series expansions in h we can write

$$\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}_0 + \tilde{\boldsymbol{\theta}} h + O(h^2) \tag{28}$$

$$\boldsymbol{R} = \boldsymbol{R}_0 + O(h) \tag{29}$$

$$\boldsymbol{r} = \boldsymbol{r}_0 + O(h) \tag{30}$$

$$\boldsymbol{\varepsilon} \triangleq \boldsymbol{R}\boldsymbol{\theta}_0 - \boldsymbol{r} = \boldsymbol{\varepsilon}_0 h + O(h^2) \tag{31}$$

The following lemma then provides the answer:

Lemma 2. Let $\{p_i\}_{i=1}^n$ denote the zeros of $A_c(p)$, defined in (2). It holds that

$$\tilde{\boldsymbol{\theta}} = -\boldsymbol{R}_0^{-1} \boldsymbol{\varepsilon}_0 \tag{32}$$

where, under the assumption that the zeros $\{p_i\}$ are *distinct*, the $(n-i, n-j)th$ element of $\boldsymbol{R}_0$, and the $(n-j)th$ element of $\boldsymbol{\varepsilon}_0$ fulfills

$$[\boldsymbol{R}_0]_{n-i,n-j} = (-1)^{n+j} \frac{\gamma_n}{2} \sum_{l=1}^{n} \frac{p_l^{i+j-1}}{\prod_{k \neq l}(p_l^2 - p_k^2)} \tag{33}$$

$$[\boldsymbol{\varepsilon}_0]_{n-j} = -\frac{\gamma_n}{2} \sum_{l=1}^{n} \frac{p_l^j}{\prod_{k \neq l}(p_l - p_k)} \sum_{k=1}^{n} \frac{p_k^2}{p_k + p_l} \tag{34}$$

for $0 \le i, j \le n-1$, and with

$$\gamma_n = \frac{1}{\prod_{i=1}^{n-1}(1 - z_i^*)^2} \tag{35}$$

where $\{z_i^*\}$ are the $n-1$ roots of $B_{2n-1}(z)$ inside the unit circle, see (13).

PROOF. See (Larsson, 2001). □

Remark 1. Notice that (32) still holds when the zeros $\{p_i\}$ are not distinct. However, the expressions (33)-(34) have to be replaced. This rederivation will be more messy, at least taking the approach used in (Larsson and Söderström, 2001; Larsson, 2001). Moreover, it can be shown that $[\boldsymbol{R}_0]_{n-i,n-j} = 0$ when $i + j$ is odd.

5. NUMERICAL EXAMPLES

In this section the performance of the method introduced in Section 4 (referred to as **M1**) is considered in some simulation studies.

Data were generated by instantaneous sampling of the second order process

$$(p^2 + a_1 p + a_2) y(t) = e(t) \tag{36}$$

where $e(t)$ is a continuous-time white noise source with unit incremental variance. Both a_1 and a_2 were equal to 2 and the process was observed $N = 10000$ times. Each trial was repeated 200 times. In all simulations the sampling interval, h, varied between 0.01 and 0.1.

Example 1. In the first example the parameters in (36) were estimated using the method proposed in Section 4. According to Lemma 1 the polynomial $F(q) = C_2^*(q) = q[q - (-2 + \sqrt{3})]$. The theoretical estimate is found by solving the normal equations (22) for fixed h. The estimate of a_1 is presented in Figure 1, while the estimate of a_2 is presented in Figure 2. The results indicates that the bias for $\hat{a}_1$ is smaller than the bias for $\hat{a}_2$; a result that is theoretically supported by Lemma 2. In fact, it turns out that the dominating bias term when $a_1 = a_2 = 2$ is

$$\tilde{\boldsymbol{\theta}}_{\mathbf{M1}} = \begin{bmatrix} -\frac{1}{2}a_1^2 + a_2 \\ -\frac{1}{2}a_1 a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \tag{37}$$

It is also clear from the figures that there is a good match between the theoretical, and the experimental values. Another observation is that the estimation error variance is a monotonically decreasing function with respect to h, as expected. It is important to keep in mind that the total length of identification measured in continuous-time is Nh. Thus, for a fixed N, a larger h will mean that we gain more insight of the system, and therefore we can expect estimates with better accuracy.

Example 2. In this example the method proposed in this paper (**M1**) is compared to some existing methods for direct identification of CAR models. The other considered methods are:

M2 An *instrumental variable* (IV) method, with delayed values of the output signal as instruments. The method is described in (Bigi *et al.*, 1994).

M3 A least squares scheme, called the *shifted least squares* (SLS) method, with a shift structure for estimating the nth order derivative of $y(t)$. The method is described in (Söderström *et al.*, 1997*b*).

M4 A least squares scheme, with a bias compensation feature. The method is commonly refereed to as the *bias compensated least squares* (BCLS), see (Söderström *et al.*, 1997*c*).

The setup for the different methods are the same as described in (Söderström *et al.*, 1997*a*).

The estimate of a_1 is presented in Figure 3, while the estimate of a_2 is presented in Figure 4. It is clear from the figures that **M1** and **M2** give the best estimates, while **M3** seems to produce the worst estimate. This is something that have been theoretically supported. In this case it can be shown that **M1** and **M2** have identically the same dominating bias contribution. Moreover it is seen from Figure 5 that all the methods give rise to a similar, and small estimation error variance. Indeed it has been shown (Söderström, 1999) that the methods **M2** - **M4** are asymptotically statistically efficient, *i.e.*, they reach the CRB in the limiting case when the sampling interval tends to zero.

6. CONCLUSIONS

The problem of estimating continuous-time autoregressive models from discrete-time data, by using limiting properties of sampled systems, has been the focus of this paper. The approach taken here consists of replacing the differentiation operator by the delta forward operator, forming a linear regression model and solve it by using an ordinary least squares scheme. To get parameter estimates with negligible bias, we found that a sufficient mean was to prefilter the discrete-time data by a particular IIR filter. It turned out that the filter was given by the limiting noise shaping filter for the sampled version of the original continuous-time system.

Some of the more pronounced advantages of the introduced method, beside its intuitively clear explanation, include: it is very easy to apply, straightforward to implement and computationally very efficient. The method has been theoretically analyzed with respect to bias distribution, and the derived expressions have been verified in some simulation examples. Moreover, the method has been compared with some other methods in some simulation studies. The results indicate that all the methods in general behaves similarly with respect to bias distribution and estimation error accuracy.

7. ACKNOWLEDGEMENT

This work was supported by Swedish Research Council for Engineering Science under contract 98 – 654.

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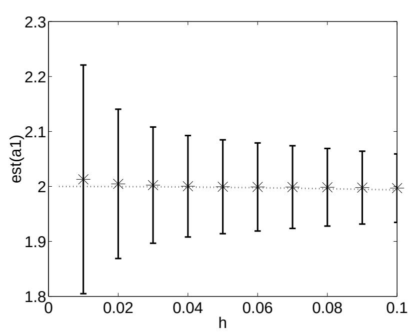

Fig. 1. Estimation results for a_1. True $a_1 = 2$. $(\cdots)$ - theoretical estimates. $(*)$ - experimental estimates. The standard deviation of the estimates are shown by the vertical lines.

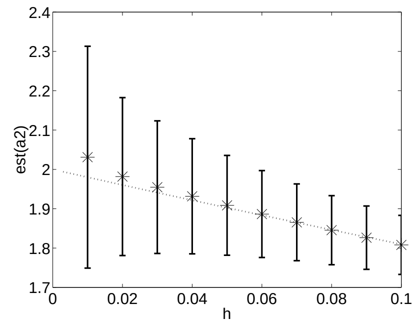

Fig. 2. Estimation results for a_2. True $a_2 = 2$. $(\cdots)$ - theoretical estimates. $(*)$ - experimental estimates. The standard deviation of the estimates are shown by the vertical lines.

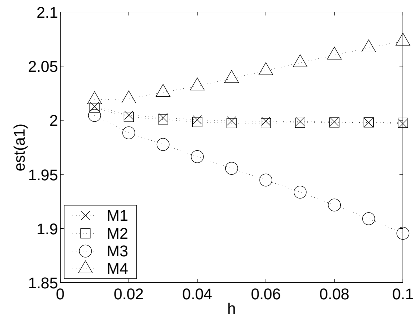

Fig. 3. The estimate of a_1 for the four different methods described in Example 2 as a function of the sampling interval. True $a_1 = 2$.

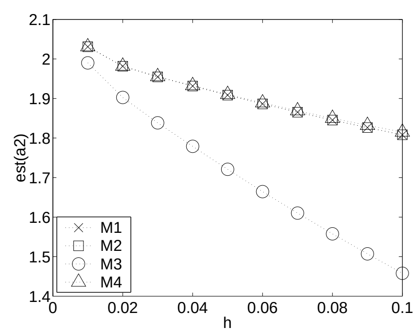

Fig. 4. The estimate of a_2 for the four different methods described in Example 2 as a function of the sampling interval. True $a_2 = 2$.

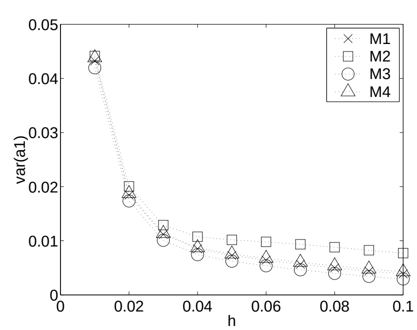

Fig. 5. Estimation error variance for estimating the parameter a_1 for the four different methods described in Example 2.