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EXPONENTIAL STABILIZATION FOR NONLINEAR SYSTEMS WITH APPLICATIONS TO NONHOLONOMIC SYSTEMS

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Abstract: The paper proposes a general framework to study the exponential stabilization problem for a class of nonlinear systems. By using the σ-processing, the exponential stabilization problem can be solved provided that all solutions of an augmented system are uniformly globally bounded. Based on this result, a simple and general stability criterion is presented. Two well-known nonholonomic systems are shown that they fall into the considered class and satisfy the proposed criterion. In particular, the exponential stability can be achieved for these systems. These examples validate the effectiveness of the proposed results. *Copyright © 2002 IFAC*

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1. INTRODUCTION

The paper studies the exponential stabilization problem for the following cascade nonlinear systems

$$\dot{x}_1 = A_1\, x_1 + B_1\, u_1 \tag{1}$$

$$\dot{x}_2 = A_2(x_1,u_1)\, x_2 + B_2(x_1,u_1)\; u_2\,, \tag{2}$$

where $x_i \in \Re^{n_i}$ and $u_i \in \Re^{m_i}$, $\forall\, i = 1,2$; A_1 and B_1 are matrices with suitable dimensions; $A_2(x_1,u_1)$ and $B_2(x_1,u_1)$ are smooth matrix-valued functions. The goal of this paper is two-fold. Firstly, we attempt to propose a simple and general criterion to solve the exponential stabilization problem for the system (1)-(2). Secondly, it will be shown that many well-known nonholonomic systems can be described or transformed into systems of the form (1)-(2) and satisfy the proposed stability criterion, e.g., the chained systems (Murray and Sastry, 1993), the power form systems (Luo and Tsiotras, 1998) and a planar rigid body with a point mass (Reyhanoglu, et al., 1998) , e.t.c.. Due to a limited space, only two examples are illustrated. Other examples can be treated similarly.

Our approach has two stages. Firstly, system (1)-(2) will be transformed into an augmented system by applying the σ -processing introduced by Astolfi (1996). More explicitly, for any $r \in \Re^{n_2}$, $s \in \Re^{m_2}$ and $k > 0$, the following augmented system

$$\dot{\bar{x}}_1 = (kI + A_1)\,\bar{x}_1 + B_1\,\bar{u}_1 \tag{3}$$

$$\dot{\bar{x}}_2 = (kD^r + A_2^{rr}(\bar{x}_1,\bar{u}_1,\lambda)\,)\bar{x}_2 + B_2^{rs}(\bar{x}_1,\bar{u}_1,\lambda)\;\bar{u}_2 \tag{4}$$

will be defined where I is the identity matrix, D^r is a diagonal matrix associated with the vector r , $\lambda = \mu(x_1(0), x_2(0))e^{-kt}$ for some function μ and A_2^{rr} and B_2^{rs} are two matrix-valued functions associated with r , s , A_2 and B_2 . According to the σ -processing, the exponential stabilization problem is solvable for the system (1)-(2) provided that all solutions of the closed-loop system of the transformed system (3)-(4) are uniformly globally bounded by applying some controller. The transformation is very useful since checking the boundedness of solutions is more easily than checking the exponential stability in general. The second stage in this paper is to show that there will be many possibilities for choosing a controller such that all solutions of the closed-loop system of (3)-(4) are uniformly globally bounded if, under some regularity condition the following hypothesis holds:

(H1) Suppose $(kI + A_1, B_1)$ is stabilizable and $(\,kD^r + A_2^{rr}(a,b,0), B_2^{rs}(a,b,0)\,)$ is stabilizable for some vector $(a,b) \in \Re^{n_1} \times \Re^{m_1}$ with $(kI + A_1)\,a + B_1\,b = 0$.

Then, the exponential stabilization can be guaranteed for any system in the form (1)-(2) and satisfying hypothesis (H1).

A similar method was used to study the exponential stabilization problem of nonholonomic systems in present literature (Astolfi, 1996; Laiou and Astolfi, 1999; Luo and Tsiotras, 1998; Luo and Tsiotras, 2000; Tian and Li, 2000; Reyhanoglu, et al., 1998).

The pioneer work of Astolfi in 1996 shown that the exponential stability of a nonholonomic chained system can be guaranteed by using a discontinuous feedback law. Later on, it was extended to high-order generalized chained systems in (Laiou and Astolfi, 1999). The discontinuous feedback method was also applied to studying a nonholonomic underactuated mechanic system in (Reyhanoglu, et al., 1998). Although the discontinuous feedback method is a powerful tool, it has some weakness. For example, there is a singular hyperplane such that the controller cannot be defined when the initial state is on the hyperplane. Moreover, the control effect will become very large when the initial state is close to the singular hyperplane. Thus, a modification is necessary. In (Luo and Tsiotras, 2000), a switching controller approach was proposed to overcome this problem. For chained systems, another possibility was given in (Tian and Li, 2000) by using a time-varying smooth controller modified from the discontinuous feedback law. Roughly speaking, these results are all using the σ-processing to transform the studied system into a system of the form (3)-(4). The difference is the choice of the function μ. For the discontinuous feedback method, the function μ was usually chosen as a linear function. On the contrast, the time-varying smooth feedback method was using the constant function $\mu \equiv 1$.

In this paper, we attempt to give a unified framework and extend the results given in present literature to more general nonlinear systems of the form (1)-(2). The proposed criterion as stated in hypothesis (H1) is very easily verified. Moreover, it can be applied to many nonlinear systems rather than some special nonholonomic systems. Two illustrated examples will be given to validate the effectiveness of our approaches.

2. PRELIMINARIES

2.1 σ-processing: dilation and augmented systems

In this subsection, the definition of dilation on a Euclidean space will be reviewed and extended to the case of matrices. We will use it to define the augmented system given in (3)-(4). Throughout this paper, let $\Re^{n\times m}$ denote the set of all $n\times m$ matrices and $D^r = diag(r_1, r_2, \cdots, r_n)$ denote the diagonal matrix with respect to a vector $r = (r_1, r_2, \cdots, r_n) \in \Re^n$. In the following, let us recall and extend the definition of the dilation given in present literature to the case of matrices.

Definition 1: Let $v = (v_1, v_2, \cdots, v_n)^T \in \Re^n$. A dilation $\Delta_\lambda^r : \Re^n \to \Re^n$ on $\Re^n$ is defined by assigning n real numbers $r = (r_1, r_2, \cdots, r_n)$ and a nonzero real number ζ such that $\Delta_\zeta^r v = (\zeta^{r_1} v_1, \zeta^{r_2} v_2, \cdots, \zeta^{r_n} v_n)$. Similarly, let $A = (a_{ij}) \in \Re^{n\times m}$. A dilation $\Delta_\lambda^{rs} : \Re^{n\times m} \to \Re^{n\times m}$ on $\Re^{n\times m}$ is defined by assigning $n+m$ real numbers $r = (r_1, r_2, \cdots, r_n)$ and $s = (s_1, s_2, \cdots, s_m)$, and a real number $\zeta \neq 0$ such that $\Delta_\zeta^{rs} A = (\zeta^{r_i - s_j} a_{ij})$. ■

The following lemma gives a basic property of the dilation. The proof is omitted since it is straightforward.

Lemma 1: Let $v \in \Re^n$, $w \in \Re^m$ and $A \in \Re^{n\times m}$ such that $Aw = v$. Then, for any vector $(r,s) \in \Re^n \times \Re^m$, the following equation holds:

$$(\Delta_\zeta^{rs} A)\, \Delta_\zeta^s w = \Delta_\zeta^r v . \tag{5}$$

In the following, we would like to define a augmented system of the form (3)-(4) for the system (1)-(2) using the concept of dilation. Consider the system (1)-(2). Let $\mu : \Re^{n_1} \times \Re^{n_2} \to [0,\infty)$ be any continuous function. For any initial state $(x_1(0), x_2(0))$ and any positive constant k, define the function λ by

$$\lambda(t) = \mu(x_1(0), x_2(0)) e^{-kt} . \tag{6}$$

Thus, $\dot\lambda = -k\lambda$ by the direct computation. Now, let us apply the σ-processing for the system (1)-(2) to derive a augmented system of the form (3)-(4) (Astolfi, 1996). Assume that $\mu(x_1(0), x_2(0)) \neq 0$ temporarily. Then, $\lambda(t) \neq 0, \forall t \geq 0$. Let $(r,s) \in \Re^{n_2} \times \Re^{m_2}$ be any vector. Define new state variables $(\bar x_1, \bar x_2)$ and new control variables $(\bar u_1, \bar u_2)$ as in the following:

$$\bar x_1 = x_1/\lambda, \quad \bar u_1 = u_1/\lambda, \quad \bar x_2 = \Delta_{1/\lambda}^r x_2, \quad \bar u_2 = \Delta_{1/\lambda}^s u_2 . \tag{7}$$

Then, by the direct computation, we have

$$\dot{\bar x}_1 = -\frac{\dot\lambda}{\lambda}\frac{x_1}{\lambda} + \frac{\dot x_1}{\lambda} = (kI + A_1)\bar x_1 + B_1 \bar u_1 . \tag{8}$$

in view of equation (1) and the fact $\dot\lambda = -k\lambda$. Let A_2^{rr} and B_2^{rs} be two matrix-valued functions defined as in the following:

$$A_2^{rr}(v,w,\zeta) = \Delta_{1/\zeta}^{rr} A_2(\zeta v, \zeta w),$$
$$B_2^{rs}(v,w,\zeta) = \Delta_{1/\zeta}^{rs} B_2(\zeta v, \zeta w), \tag{9}$$

for all $v \in \Re^{n_1}, w \in \Re^{m_1}, \zeta \in \Re - \{0\}$. Then, it can be checked that the following equations hold:

$$\begin{aligned}\dot{\bar x}_2 &= -\frac{\dot\lambda}{\lambda} D^r (\Delta_{1/\lambda}^r x_2) + \Delta_{1/\lambda}^r \dot x_2 = kD^r \bar x_2 \\ &+ (\Delta_{1/\lambda}^{rr} A_2(x_1,u_1))(\Delta_{1/\lambda}^r x_2) + (\Delta_{1/\lambda}^{rs} B_2(x_1,u_1))(\Delta_{1/\lambda}^s u_2) \\ &= (kD^r + A_2^{rr}(\bar x_1, \bar u_1, \lambda))\bar x_2 + B_2^{rs}(\bar x_1, \bar u_1, \lambda)\bar u_2,\end{aligned} \tag{10}$$

by Lemma 1 and the definitions of A_2^{rr} and B_2^{rs}. In particular, an augmented system of the form (3)-(4) is derived. We summarize the previous discussion into the following lemma.

Lemma 2: Consider the system (1)-(2). Let

$(r,s) \in \Re^{n_2} \times \Re^{m_2}$ be any vector. Using the coordinate transformation (7), it can be transformed into a system of the form (3)-(4) for any initial state satisfying $\mu(x_1(0), x_2(0)) \neq 0$ where A_2^{rr} and B_2^{rs} are the matrix-valued functions defined in (9). ■

2.2 A preliminary result

In this subsection, the exponential stabilization for the system (1)-(2) will be guaranteed by employing the augmented system (3)-(4). This is a preliminary study for our main results. To simplify the discussion and avoid the singularity, the function μ is always chosen as $\mu \equiv 1$ in the remainder of this paper. Note that $\lambda(t) > 0, \forall t \geq 0$, in this case. Thus, the coordinate transformation (7) can be performed for all initial state. In the following, a definition about the κ -exponential stability is reviewed (Sordalen and Egeland, 1995).

Definition 2: The equilibrium point $x = 0$ of $\dot{x} = f(t,x)$ is weakly globally κ -exponentially stable if there exist a strictly increasing continuous function $\alpha : [0,\infty) \to [0,\infty)$ and a positive constant σ such that

$$|x(t)| \leq \alpha(|x(0)|)e^{-\sigma t}, \quad \forall t \geq 0, \forall x(0) \in \Re^n. \qquad (11)$$

In addition to $\alpha(0) = 0$, it is said that the equilibrium point $x = 0$ is globally κ -exponentially stable. ■

In the following, we want to show that the weak global κ -exponential stability can be achieved for the system (1)-(2). To this end, we need the following hypothesis for augmented systems of the form (3)-(4).

(H2) Suppose there exist a constant $k > 0$, $n_2 + m_2$ positive real numbers $r = (r_1, r_2, \cdots, r_{n_2})$ and $s = (s_1, s_2, \cdots, s_{n_2})$, with $r_i \geq 1$ and $s_j \geq 1$ for all i,j such that with the controller chosen as $(\bar{u}_1, \bar{u}_2) = (\beta_1(\bar{x}_1, \bar{x}_2), \beta_2(\bar{x}_1, \bar{x}_2))$, all solutions of the closed-loop system of the augmented system (3)-(4) are uniformly globally bounded where A_2^{rr} and B_2^{rs} are the matrix-valued functions defined in (9).

Proposition 1: Consider the system (1)-(2). Let λ be chosen as in the equation (6) with $\mu \equiv 1$. Suppose hypothesis (H2) holds. Choose the controller as in the following

$$u_1 = \lambda \beta_1(\frac{x_1}{\lambda}, \Delta^r_{1/\lambda} x_2) \text{ and } u_2 = \Delta^s_\lambda \beta_2(\frac{x_1}{\lambda}, \Delta^r_{1/\lambda} x_2). \qquad (12)$$

Then, the origin of the closed-loop system is weakly globally κ -exponentially stable.

Proof: Note that $\Delta^s_{1/\lambda}\Delta^s_\lambda w = w, \forall w \in \Re^m$, by the definition of dilation. From this and using the coordinate transformation (7), we have

$$\bar{u}_1 = \frac{u_1}{\lambda} = \beta_1(\bar{x}_1, \bar{x}_2) \text{ and } \bar{u}_2 = \Delta^s_{1/\lambda} u_2 = \beta_2(\bar{x}_1, \bar{x}_2).$$

By Lemma 2, the system (1)-(2) can be transformed into the augmented system (3)-(4). In view of hypothesis (H2), all solutions of the closed-loop system of (3)-(4) are uniformly globally bounded. This implies that there is a strictly increasing continuous function $\alpha : [0,\infty) \to [0,\infty)$ such that

$$|(\bar{x}_1(t), \bar{x}_2(t))| \leq \alpha(|(\bar{x}_1(0), \bar{x}_2(0))|), \forall t \geq 0, \quad (13)$$

see (Khalil, 1996) for example. Since $\lambda(t) = e^{-kt} \leq 1$ and $r_i \geq 1$ by definitions, the inequality $\lambda(t)^{r_i} \leq \lambda(t)$ holds, $\forall t \geq 0$, $\forall i = 1,2,\cdots,n_2$. This results in

$$|(x_1(t), x_2(t))| = |(\lambda \bar{x}_1(t), \Delta^r_\lambda \bar{x}_2(t))| \leq |(\bar{x}_1(t), \bar{x}_2(t))|\lambda$$
$$\leq \alpha(|(x_1(0), x_2(0))|)e^{-kt}, \forall t \geq 0. \qquad (14)$$

in view of the inequality (14) and the fact $(\bar{x}_1(0), \bar{x}_2(0)) = (x_1(0), x_2(0))$. Thus, the origin of the closed-loop system is weakly globally κ -exponentially stable. This completes the proof. ■

Remark 1: In next section, it will be shown that a linear controller $(\bar{u}_1, \bar{u}_2) = (\beta_1(\bar{x}_1, \bar{x}_2), \beta_2(\bar{x}_1, \bar{x}_2))$ can be given such that Hypothesis (H2) holds. More explicitly, it will take the form:

$$\beta_1 = c + K_1 \bar{x}_1 \text{ and } \beta_2 = K_2 \bar{x}_2, \qquad (15)$$

for some $c \in \Re^{n_1}, K_1 \in \Re^{m_1 \times n_1}$ and $K_2 \in \Re^{m_2 \times n_2}$. Then, the controller (u_1, u_2) of the original system can be explicitly given by

$$u_1 = c\lambda + K_1 x_1$$

and $$u_2 = \Delta^s_\lambda(K_2 \Delta^r_{1/\lambda} x_2) = (\Delta^{sr}_\lambda K_2)x_2. \qquad (16)$$

in view of the equation (12) and Lemma 1. ■

Remark 2: In Proposition 1, the function μ is chosen as $\mu \equiv 1$. For the other choice of μ, a similar result also holds for all initial states satisfying $\mu(x_1(0), x_2(0)) \neq 0$. However, the controller given in (12) cannot be defined on the set $S = \{(x_1, x_2) | \mu(x_1, x_2) = 0\}$ in general. ■

3. GLOBAL EXPONENTIAL STABILITY

We have shown that the exponential stabilization of system (1)-(2) can be guaranteed using the σ-processing and the hypothesis (H2) in previous section. As it was introduced in Section 1, the hypothesis (H1) will be used to check the hypothesis (H2). Note that, the functions A_2^{rr} and B_2^{rs} given in (9) can only defined on $\Re^{n_1} \times \Re^{m_1} \times \Re - \{0\}$. Thus, the following regularity hypothesis is necessary to employ hypothesis (H1).

(H3) Suppose there exist $n_2 + m_2$ positive real numbers $r = (r_1, r_2, \cdots, r_{n_2})$ and $s = (s_1, s_2, \cdots, s_{n_2})$, with $r_i \geq 1$ and $s_j \geq 1$ for all i,j, such that the limits

$$\lim_{\zeta \to 0, x_1 \to v, u_1 \to w} A_2^{rr}(x_1, u_1, \zeta) \, (= A_2^{rr}(v, w, 0))$$

and $\lim_{\zeta\to 0, x_1\to v, u_1\to w} B_2^{rs}(x_1,u_1,\zeta)\ (=B_2^{rs}(v,w,0))$

exist, for all $v\in\Re^{n_1}$ and $w\in\Re^{m_1}$.

Remark 3: Let us give a brief discussion about the verification of (H3). On practical applications, it is usually appeared that functions $A_2(x_1,u_1)$ and $B_2(x_1,u_1)$ both are analytic matrix-valued functions. Then, they can be decomposed as

$$A_2 = A_2^l + A_2^h \text{ and } B_2 = B_2^l + B_2^h,$$

by using their Taylor expansions at (0,0) where the matrix-valued functions $A_2^l(x_1,u_1)$ and $B_2^l(x_1,u_1)$ are consisting of the lowest order terms appearing in the Taylor expansion, respectively. Thus, it can be seen that elements of $A_2^l(x_1,u_1)$ and $B_2^l(x_1,u_1)$ are all homogeneous polynomials. Let $d_{ij}(A_2^l)$ and $d_{ij}(B_2^l)$ denote the degrees of the (*i,j*) entry of $A_2^l(x_1,u_1)$ and $B_2^l(x_1,u_1)$, respectively. We define $d_{ij}=\infty$ for the zero function. Then, it is possible to show that the hypothesis (H3) holds if and only if the inequalities $r_i - r_j \le d_{ij}(A_2^l)$ and $r_i - s_{\tilde{j}} \le d_{i\tilde{j}}(B_2^l)$ hold for all $1\le i,j\le n_2$, $1\le \tilde{j}\le m_2$. ■

In view of hypothesis (H3), A_2^{rr} and B_2^{rs} are continuous functions defined on $\Re^{n_1}\times\Re^{m_1}\times\Re$ now. Before state the main theorem, the following technique lemma is necessary. The proof is standard and we refer readers to the book of Khalil (1996) for details.

Lemma 3: Consider the following time-varying system

$$\dot{x} = (A + B(x,t))x. \tag{17}$$

Let $h:[0,\infty)\times[0,\infty)\to[0,\infty)$ be a function such that, for each fixed t, the function $h(\eta,t)$ is increasing w.r.t. η and, for each fixed η, the function $h(\eta,t)$ is decreasing w.r.t. t and $h(\eta,t)\to 0$ as $t\to\infty$. Suppose A is a stable matrix and $|B(x(t),t)|\le h(|x(0)|,t)$, for all $t\ge 0$ and all solutions $x(t)$ of (17). Then, all solutions of (17) are uniformly globally bounded. ■

Now, we are in a position to give the main result.

Theorem 1: Consider the system (1)-(2). Suppose there exists a positive constant k such that hypotheses (H1) and (H3) hold where A_2^{rr} and B_2^{rs} are the matrix-valued functions defined as in (9). Let $\lambda = e^{-kt}$. Then, there exist a vector $c\in\Re^{n_1}$, two matrices $K_1\in\Re^{m_1\times n_1}$ and $K_2\in\Re^{m_2\times n_2}$ such that with the controller (u_1,u_2) chosen as in (16), the origin of the closed-loop system is weakly globally κ-exponentially stable.

Proof: Let a, b, r and s be the vectors given in hypotheses (H1) and (H3). Consider the augmented system (3)-(4). We want to show that hypothesis (H2) holds with the functions β_1 and β_2 taken the form (15). Let us define a new coordinate by $\hat{x}_1 = \bar{x}_1 - a$ and $\hat{u}_1 = \bar{u}_1 - b$. Replacing $\bar{x}_1$ and $\bar{u}_1$ by $\hat{x}_1$ and $\hat{u}_1$, respectively, the equation (3) still holds in view of the equation $(kI + A_1)a + B_1 b = 0$. Let $\overline{A}_2 = kD^r + A_2^{rr}(a,b,0)$ and $\overline{B}_2 = B_2^{rs}(a,b,0)$. By hypothesis (H1), there exist two matrices $K_1\in\Re^{m_1\times n_1}$ and $K_2\in\Re^{m_2\times n_2}$ such that the matrices $kI + A_1 + B_1K_1$ and $\overline{A}_2 + \overline{B}_2K_2$ are both stable matrices. Choose the controller $(\hat{u}_1,\bar{u}_2)$ as $(\hat{u}_1,\bar{u}_2) = (K_1\hat{x}_1, K_2\bar{x}_2)$. Then, the controller $(\bar{u}_1,\bar{u}_2)$ is in the form of (15) with $c = b - K_1 a$. Thus, the controller (u_1,u_2) can be given as in (16) by the discussion in Remark 1. Define the state $x = (\hat{x}_1,\bar{x}_2)$ and two matrices in the following

$$A = \begin{bmatrix} kI + A_1 + B_1K_1 & 0 \\ 0 & \overline{A}_2 + \overline{B}_2K_2 \end{bmatrix},\quad \hat{B}(x,\zeta) = \begin{bmatrix} 0 & 0 \\ 0 & B_{22} \end{bmatrix},$$

where

$$B_{22} = A_2^{rr}(\hat{x}_1 + a, K_1\hat{x}_1 + b,\zeta) - A_2^{rr}(a,b,0) + (B_2^{rs}(\hat{x}_1 + a, K_1\hat{x}_1 + b,\zeta) - B_2^{rs}(a,b,0))K_2.$$

Then, A is also a stable matrix and the closed-loop system of the system (3)-(4) can be written into the equation (17) with $B \equiv \hat{B}(x, e^{-kt})$. Note that $\hat{B}(x,\zeta)$ is only the function of $\hat{x}_1$ and ζ. Since $kI + A_1 + B_1K_1$ is stable, every solution $\hat{x}_1(t)$ of $\dot{\hat{x}}_1 = (kI + A_1 + B_1K_1)\hat{x}_1$ satisfies the inequality $|\hat{x}_1(t)| \le \sigma_2 e^{-\sigma_1 t}|\hat{x}_1(0)|$ for some positive constants σ_1 and σ_2. Define a function $h:[0,\infty)\times[0,\infty)\to[0,\infty)$ as in the following $h(\eta,t) = \sup\left\{\left\|\hat{B}(x,\zeta)\right\| \,\middle|\, |\hat{x}_1| \le \sigma_2 e^{-\sigma_1 t}\eta,\ |\zeta| \le e^{-kt}\right\}$.

By the definition of $\hat{B}$, it is easy to see that $\lim_{\hat{x}_1\to 0,\zeta\to 0}\left\|\hat{B}(x,\zeta)\right\| = 0$. This implies that for each fixed η, $h(\eta,t)\to 0$ as $t\to\infty$. Moreover, for each fixed t, the function $h(\eta,t)$ is increasing w.r.t. η and, for each fixed η, the function $h(\eta,t)$ is decreasing w.r.t. t. By the definitions of h and B, we have $|B(x(t),t)| \le h(|x(0)|,t)$, for all $t\ge 0$. Note that the boundedness of $(\hat{x}_1(t),\bar{x}_2(t))$ is equivalent to the boundedness of $(\bar{x}_1(t),\bar{x}_2(t))$ because of $\hat{x}_1 = \bar{x}_1 - a$ by definition. Thus, all solutions of the closed-loop system of (3)-(4) are uniformly globally bounded in view of Lemma 3. In particular, the hypothesis (H2) holds and the theorem follows from Proposition 1. This completes the proof. ■

Remark 4: In nonlinear systems theory, a well-known theorem says that the local exponential stability of a nonlinear system can be guarantee when its linearized system is stable. Theorem 1 gives a similar criterion. Indeed, consider the linearized system of (3)-(4) (with $\lambda = 0$) at $(\bar{x}_1, \bar{x}_2) = (a, 0)$ and $(\bar{u}_1, \bar{u}_2) = (b, 0)$ in the following

$$\dot{\hat{x}}_1 = (kI + A_1)\hat{x}_1 + B_1 \hat{u}_1 \quad (18)$$

$$\dot{\hat{x}}_2 = (kD^r + A_2^{rr}(a,b,0))\hat{x}_2 + B_2^{rs}(a,b,0)\ \hat{u}_2 . \quad (19)$$

Then, the hypothesis (H1) says that the linearized system (18)-(19) is stabilizable. Thus, roughly speaking, Theorem 1 tell us that the original system is exponentially stabilizable when the linearized system (18)-(19) is stabilizable. In particular, the exponential stability is independent on the nonlinear terms $A_2^{rr}(\bar{x}_1, \bar{u}_1, \lambda) - A_2^{rr}(a,b,0)$, $B_2^{rs}(\bar{x}_1, \bar{u}_1, \lambda) - B_2^{rs}(a,b,0)$. From the point of view above, the proposed controller has the robustness w.r.t. some nonlinear uncertainties. We refer readers to (Laiou and Astolfi, 1999) for a further discussion. ■

In the following, we would like to give a simplified criterion to check the hypothesis (H1). Usually, a stronger condition-controllability than the stability can be guaranteed on practical applications. In this case, it is well-known that (A, B) is controllable if and only if $(kI + A, B)$ is controllable for any constant k. In particular, the hypothesis (H1) holds under the following hypothesis.

(H4) Suppose (A_1, B_1) is controllable and there exists a positive constant k, a vector $(a,b) \in \Re^{n_1} \times \Re^{m_1}$ such that $(kI + A_1)a + B_1 b = 0$ and $(kD^r + A_2^{rr}(a,b,0), B_2^{rs}(a,b,0))$ is controllable.

In view of (H4), the following corollary is readable from Theorem 1.

Corollary 1: The same result as stated in Theorem 1 holds when the hypothesis (H1) is replaced by the hypothesis (H4). ■

Remark 5: So far, it seems that an explicit controller was not given in this paper. In fact, there are many well-established methods can be employed to find a stable controller under the controllability condition in linear system theory, e.g., the pole-placement method, the LQG method, e.t.c.. Then, the controller of systems (1)-(2) can be chosen as in (15) with $c = b - K_1 a$, and K_1 and K_2 are the corresponding stable feedback gain. ■

4. TWO EXAMPLES FROM NONHOLONOMIC SYSTEM

In this section, two well-known examples from nonholonomic systems will be proposed and show that they can be described or transformed into a system of the form (1)-(2). Moreover, the hypotheses (H3) and (H4) hold for these systems. Thus, the exponential stability can be achieved by using Corollary 1. For these systems, (A_1, B_1) are all in the controllable canonical form (CCF). Thus, the main task is to check the controllability of $(kD^r + A_2^{rr}(a,b,0), B_2^{rs}(a,b,0))$.

4.1 Chained systems

Consider the following chained system

$$\begin{aligned} \dot{y}_0 &= u_1 \\ \dot{y}_n &= u_2 \\ \dot{y}_i &= y_{i+1}u_1,\ i = 1, \cdots, n-1, \end{aligned} \quad (20)$$

see (Murray and Sastry, 1993). We want to show that the chained system (20) can be described into a system of the form (1)-(2). Moreover, hypotheses (H3) and (H4) also hold. We divide the verification procedure into the following steps.

Step 1: (Transformation) Let $x_1 = y_0$ and $x_2 = (y_1, y_2, \cdots, y_n)^T$. Then, the system (20) can be transformed into the form (1)-(2) where $A_1 = 0$, $B_1 = 1$, $B_2 = B_0 = (0, \cdots, 0, 1)^T$, $A_2 = u_1 A_0$ and (A_0, B_0) is in CCF.

Step 2: (Choosing parameters) Let $r = (n, \cdots, 2, 1)$, $s = 1$, $a = \eta$ and $b = -k\eta$, $\forall k > 0$, $\forall \eta \neq 0$. Then, $D^r = diag(n, \cdots, 2, 1)$ and $(kI + A_1)a + B_1 b = 0$.

Step 3: (Computing A_2^{rr} and B_2^{rs}, see (9)) By direct computation, $A_2^{rr}(v, w, \zeta) = \Delta_{1/\zeta}^{rr} A_2(\zeta v, \zeta w) = wA_0$ and $B_2^{rs}(v, w, \zeta) = \Delta_{1/\zeta}^{rs} B_2(\zeta v, \zeta w) = B_0$. Then, hypothesis (H3) holds.

Step 4: (Checking the controllability) Let $\bar{A}_2 = kD^r + A_2^{rr}(a,b,0) = kD^r - k\eta A_0$ and $\bar{B}_2 = B_2^{rs}(a,b,0) = B_0$. The determinant of controllability matrix can be computed as $\det([\bar{B}_2, \bar{A}_2\bar{B}_2, \cdots, \bar{A}_2^{n-1}\bar{B}_2]) = (-k\eta)^{n(n-1)/2} \neq 0$. Thus $(\bar{A}_2, \bar{B}_2)$ is controllable.

Particularly, the hypothesis (H4) holds. Thus, the exponential stability can be achieved by using Corollary 1. A similar procedure will be applied to another example. Due to a limit space, only steps 1-4 are listed. The detailed discussion is omitted.

4.2 A planar rigid body with a point mass

Consider a planar rigid body with a point mass (Reyhanoglu, et al., 1998):

$$\begin{aligned} \ddot{\theta} &= u_1 \\ \ddot{x} &= v_1 \\ \ddot{y} &= v_2 \\ \ddot{s} &= -v_1\cos(\theta) - v_2\sin(\theta) + s\dot{\theta}^2 . \end{aligned} \quad (21)$$

Step 1: (Transformation) Let $x_1 = (\theta, \dot{\theta})^T$ and

$$U(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \forall \theta \in \Re.$$

Define $x_2 = (x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{26})^T$ where $(x_{21}, x_{22})^T = U(\theta)\,(x, y)^T$, $(x_{23}, x_{24})^T = U(\theta)\,(\dot{x}, \dot{y})^T$ and $(x_{25}, x_{26})^T = (s, \dot{s})^T + (x_{21}, x_{23})^T$.

Let $u_2 = U(\theta)(v_1, v_2)^T$. Then, the system (21) is feedback equivalent to the system (1)-(2) where $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_1 = (0,1)^T$, the matrix-valued functions A_2 and B_2 are given in the following:

$$A_2(\dot{\theta}) = \begin{bmatrix} 0 & \dot{\theta} & 1 & 0 & 0 & 0 \\ -\dot{\theta} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dot{\theta} & 0 & 0 \\ 0 & 0 & -\dot{\theta} & 0 & 0 & 0 \\ 0 & \dot{\theta} & 0 & 0 & 0 & 1 \\ -\dot{\theta}^2 & 0 & 0 & \dot{\theta} & \dot{\theta}^2 & 0 \end{bmatrix}, \quad B_2 \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Step 2: (Choosing parameters) Let $r = (2,1,2,1,2,2)$ and $s = (2,1)$, $a = (\eta, -k\eta)^T$ and $b = k^2\eta$, $\forall k > 0$, $\forall \eta \neq 0$. Then, $D^r = diag(2,1,2,1,2,2)$ and $(kI + A_1)\,a + B_1\,b = 0$.

Step 3: (Computing A_2^{rr} and B_2^{rs}) Let $v = (v_1, v_2)^T$. Then, $B_2^{rs} \equiv B_2$ and A_2^{rr} can be computed s the following:

$$A_2^{rr}(v, w, \zeta) = \begin{bmatrix} 0 & v_2 & 1 & 0 & 0 & 0 \\ -v_2\zeta^2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & v_2 & 0 & 0 \\ 0 & 0 & -v_2\zeta^2 & 0 & 0 & 0 \\ 0 & v_2 & 0 & 0 & 0 & 1 \\ -v_2^2\zeta^2 & 0 & 0 & v_2 & v_2^2\zeta^2 & 0 \end{bmatrix}.$$

Then, hypothesis (H3) holds.

Step 4: (Checking the controllability) The matrix $\overline{A}_2 = kD_r + A_2^{rr}(a,b,0)$ can be computed in the following:

$$\overline{A}_2 = \begin{bmatrix} 2k & -k\eta & 1 & 0 & 0 & 0 \\ 0 & k & 0 & 1 & 0 & 0 \\ 0 & 0 & 2k & -k\eta & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & -k\eta & 0 & 0 & 2k & 1 \\ 0 & 0 & 0 & -k\eta & 0 & 2k \end{bmatrix}.$$

Let $\overline{B}_2 = B_2^{rs}(a,b,0) = B_2$, b_1 and b_2 are two column vectors such that $(b_1, b_2) = B_2$. Let $b_{11} = \overline{A}_2 b_1 - 2kb_1$, $b_{21} = \overline{A}_2 b_2 - kb_2 + kab_1$, $b_{22} = \overline{A}_2 b_{21} - kb_{21} + kab_{11}$ and $b_{23} = \overline{A}_2 b_{22}$. Then, $\det(b_1, b_2, b_{11}, b_{21}, b_{22}, b_{23}) = -k^4\eta^2 \neq 0$. This implies that $(\overline{A}_2, \overline{B}_2)$ is controllable. Particularly, the hypothesis (H4) holds. Thus, the exponential stability can be achieved by using Corollary 1.

5. CONCLUSION

The exponential stabilization was studied for a class of nonlinear systems using a systematic approach. A simple and general criterion was proposed to guarantee the exponential stability. Several examples from nonholonomic systems were given to validate the effectiveness of the proposed result. The future work may toward to extending the result given in this paper to more general nonlinear systems and finding a geometry condition to assert that a nonlinear system can be transformed to a system in the form (1)-(2) and satisfying the hypotheses (H3)-(H4).

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