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ROBUST STABILITY OF AN AUTONOMOUS MOBILE ROBOT BASED ON A PARAMETRIC APPROACH

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Abstract: In this paper the robust stability of an autonomous mobile robot based on a parametric approach is presented. The closed-loop control consists of Multi-Input / Multi-Output (MIMO) uncertain plant (mobile robot) and a MIMO Proportional Integral (PI) controller. Using Kharitonov's Theorem and Zero Exclusion Condition the closed-loop system is proved to be robustly stable in the presence of parameter variations or the system dynamics which are sensitive with respect to these parameters (Uncertainty). Simulation results are presented to demonstrate the robust analysis and to prove the robust stability of the closed-loop system. *Copyright © 2002 IFAC*

Keywords: Autonomous mobile robots, MIMO, Uncertainty, Interval polynomials, Robust control, Robust stability, Robust analysis.

1. INTRODUCTION

Almost all-dynamic systems depend on varying or uncertain parameters and this is certainty true for small mobile robots. For instance, consider the velocity of a mobile robot (i.e. due to the battery variations), or the mass of a mobile robot (i.e. adding or removing components) all these parameters may vary more or less significantly within certain bounds and they influence the system dynamics. Traditional control design approaches consider a fixed operating point in which the controller (compensator) is robust enough to stabilise the plant for different operating conditions. These approaches produce good results if the parameter variations are small or the system dynamics are not too sensitive with respect to these parameters. For significant (large) parameter variations these control design methods reach their performance limits. Robust control theory based on interval polynomials is an effective approach when considering plant uncertainty. The interval polynomial problem was first posed by Faedo (1953) who attempted to solve it using the Routh-Hurwitz conditions. Kharitonov (1978) gave the complete solution with his theorem for real polynomials, which then he extended to the complex case. Since then many papers have been published based on parametric approach regarding robust stability of uncertain plants (Siljak, 1989; Kontogiannis and Munro, 1996).

The main objective of this paper is to show that the closed-loop system (mobile robot and controller) is robustly stable to varying or uncertain parameters. The parametric approach based on interval polynomials using Kharitonov theorem was chosen due to its simplicity and its suitability when considering uncertainty of interval polynomials.

This paper is organised into 5 sections. The control of the MIABOT V2 mobile robot is described in section 2. Section 3 presents the robust stability analysis for interval polynomials together with a number of simulations to verify the robust stability of the closed-loop system. Some discussions of the work are given in section 4. Finally, section 5 contains the conclusions of the work presented.

2. CONTROL OF MIABOT V2 MOBILE ROBOT

MIABOT V2 mobile robots shown in Fig 1 are a small sized ($8cm^3$), two-wheeled autonomous mobile robots, which have the ability to achieve speeds up to 1-1.5m/sec by driving each wheel independently (two DC motors). A Multi-Input / Multi-Output (MIMO) Proportional Integral (PI) controller has been designed for accurate speed control.

Fig 2 shows the overall system structure of the closed-loop control. The open-loop robot model $G(s)$ consists of two inputs, and two outputs. The inputs are left and right voltages of the left and right wheel respectively. Outputs are the speed of the left and right wheel. The second-order dynamic model of the mobile robot is described in the following *transfer function matrix* (TFM) form:

$$G(s) = \begin{bmatrix} \dfrac{10.64s + 554.6}{s^2 + 108s + 2835} & \dfrac{2.245s + 25.42}{s^2 + 108s + 2835} \\ \dfrac{2.245s + 25.42}{s^2 + 108s + 2835} & \dfrac{10.64s + 554.6}{s^2 + 108s + 2835} \end{bmatrix} \quad (1)$$

Similarly the MIMO PI controller $G_c(s)$ is described in the following TFM form:

$$G_c(s) = \begin{bmatrix} \dfrac{1.354s + 103}{s} & \dfrac{-1.577s - 25.19}{s} \\ \dfrac{-1.577s - 25.19}{s} & \dfrac{1.354s + 103}{s} \end{bmatrix} \quad (2)$$

Fig. 1. MIABOT V2 mobile robots.

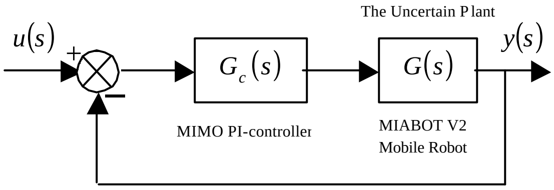

Fig. 2. Closed-loop system.

Equations (3a, 3b, 3c and 3d) describe the closed-loop system illustrated in Fig 2.

$$\frac{Y_1}{U_1}(s) = \frac{10.87\,s^3 + 1679\,s^2 + 7.806e004\,s + 1.079e006}{s^4 + 129.7\,s^3 + 6265\,s^2 + 1.345e005s + 1.079e006} = g_{11}(s) \quad (3a)$$

$$\frac{Y_1}{U_2}(s) = \frac{-13.75\,s^3 - 877.2\,s^2 - 1.135e004\,s - 0.01564}{s^4 + 129.7\,s^3 + 6265\,s^2 + 1.345e005s + 1.079e006} = g_{12}(s) \quad (3b)$$

$$\frac{Y_2}{U_1}(s) = \frac{-13.75\,s^3 - 877.2\,s^2 - 1.135e004\,s - 0.01564}{s^4 + 129.7\,s^3 + 6265\,s^2 + 1.345e005s + 1.079e006} = g_{21}(s) \quad (3c)$$

$$\frac{Y_2}{U_2}(s) = \frac{10.87\,s^3 + 1679\,s^2 + 7.806e004\,s + 1.079e006}{s^4 + 129.7\,s^3 + 6265\,s^2 + 1.345e005s + 1.079e006} = g_{22}(s) \quad (3d)$$

Fig 3 shows the transfer function description expanded for the MIMO system in Fig 2.

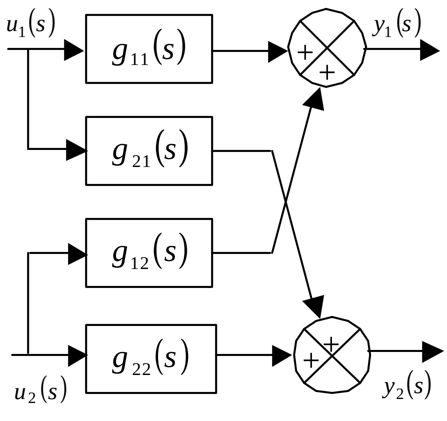

Fig. 3. The transfer function matrix description expanded for the 2x2 system (plant and controller)

The characteristic equation given in Equation (4) will be used for the testing of the robust stability of the closed-loop system given Fig 2.

$$s^4 + 129.7s^3 + 6265\,s^2 + 134500\,s + 1079000 \quad (4)$$

3. ROBUST STABILITY ANALYSIS FOR INTERVAL POLYNOMIALS

In this section definitions and theorems related to robust stability analysis for interval polynomials are given (Bhattacharyya, *et al*., 1995). First the description of uncertainty structure is given through several definitions following by both definitions and theorems regarding value sets and zero exclusion condition. Then the Kharitonov's theorem (Barmish, 1994) is described in brief together with results of the application (closed-loop control). Finally, the robust stability of the closed-loop control is demonstrated using graphical techniques.

3.1 Description of uncertainty structure.

Definition 1 *(Uncertainty Bounding Set)*: The uncertainty bounding set Q is the set

$$Q = \left\{ \mathbf{q} \in R^l \middle| q_i \in R \quad for \quad i = 1,2,\ldots,l \right\} \tag{5}$$

Note that q_i's and therefore need not be connected. However, connected sets will be used since much of the results in literature apply only to connected sets. This assumption is not restrictive because most of the physical parameters (such as viscous friction coefficients, material properties, lengths, etc) entering the uncertainty vector vary continuously over a bounded interval of the real line.

Frequently, each element q_i of $\mathbf{q}$ is described by its lower and upper bounds q_i^- and q_i^+, respectively. Then the uncertainty set is the *box*

$$Q = \left\{ \mathbf{q} \in R^l \middle| q_i^- < q_i < q_i^+ \quad for \quad i = 1,2,\ldots,l \right\} \tag{6}$$

Definition 2 *(Family)*: An uncertain function together with its uncertainty bounding set is called a *family* i.e.

$$F(.,Q) = \left\{ f(.,\mathbf{q}) \middle| \mathbf{q} \in Q \right\} \tag{7}$$

For example, an uncertain plant $G(s,\mathbf{q})$ and its uncertainty bounding set Q form a family of plants denoted by $G(s,Q) = \{G(s,\mathbf{q}) | \mathbf{q} \in Q\}$. Similarly, it can be written $n(s,Q) = \{N(s,\mathbf{q}) | \mathbf{q} \in Q\}$ for the family of numerators and $d(s,Q) = \{D(s,\mathbf{q}) | \mathbf{q} \in Q\}$ for the family of denominators.

Definition 3 *(Independent Uncertainty Structure):* An uncertain polynomial

$$p(s,\mathbf{q}) = \sum_{i=0}^{n} a_i(\mathbf{q}) s^i \tag{8}$$

is said to have an *independent uncertainty structure* if each component q_i of $\mathbf{q}$ enters into only one coefficient.

Definition 4 *(Affine Linear Uncertainty Structure):* An uncertain polynomial $p(s,\mathbf{q})$ is said to have an *affine linear uncertainty structure* if each coefficient function $a_i(\mathbf{q})$ is of the form

$$a_i(\mathbf{q}) = a_i^T \mathbf{q} + b_i \tag{9}$$

where a_i is a column vector and b_i is a scalar

Definition 5 *(Multilinear Uncertainty Structure):* An uncertain polynomial $p(s,\mathbf{q})$ is said to have a *multilinear uncertainty structure* if each function $a_i(\mathbf{q})$ is a multilinear function in the components of $\mathbf{q}$. That is, if all but one uncertain parameter is kept constant, then $a_i(\mathbf{q})$ is affine linear in the remaining component of $\mathbf{q}$.

Definition 6 *(Polynomic Uncertainty Structure):* An uncertain polynomial $p(s,\mathbf{q})$ is said to have a *polynomic uncertainty structure* if each coefficient function $a_i(\mathbf{q})$ is a multivariable polynomial in the components of $\mathbf{q}$.

3.2 Value sets and zero exclusion condition.

Definition 7 *(Value Set):* The *value set* is the subset of the complex plane consisting of all values which can be assumed by $p(jw,\mathbf{q})$ as $\mathbf{q}$ ranges over Q (w is a fixed frequency).

Theorem 1 *(Zero Exclusion Condition):* A polynomial family $P(s,Q)$ having invariant degree with associated uncertainty bounding set Q, which is pathwise connected, continuous coefficient functions $a_i(\mathbf{q})$ for $i = 0,1,2,\ldots,n$ and at least one stable member $p(s,\mathbf{q}^*)$ is robustly stable if and only if the origin of the complex plane is excluded from the value set $P(jw,Q)$ at all nonnegative frequencies i.e.

$$0 \in p(jw,\mathbf{q})$$

for all frequencies $w \geq 0$ and $\mathbf{q} \in Q$.

Definition 8 *(Robust Stability):* An uncertain system with the characteristic polynomial $p(s,\mathbf{q})$ is *robustly stable* if and only if $p(s,\mathbf{q})$ is stable for all $\mathbf{q} \in Q$, where Q is the uncertainty bounding set.

Definition 9 *(Interval Polynomial Family):* A family of polynomials $P(s,Q)=\{p(s,\mathbf{q})|\mathbf{q}\in Q\}$ is said to be an *interval polynomial family* if $p(s,\mathbf{q})$ has an independent uncertainty structure, each coefficient depends continuously on $\mathbf{q}$ and the uncertainty bounding set Q is a $n-$ dimensional box.

For brevity, is also referred to $P(s,Q)$ as an interval polynomial. Similarly, a family of uncertain plants $G(s,Q)=\{G(s,\mathbf{q})=N(s,\mathbf{q})/D(s,\mathbf{q})|\mathbf{q}\in Q\}$ is said to be an *interval plant family* if both $N(s,\mathbf{q})$ and $D(s,\mathbf{q})$ are interval polynomials.

3.3 Kharitonov's theorem.

Definition 10 *(Kharitonov Polynomials):* Associated with the interval polynomial family

$$P(s,Q)=\left\{p(s,\mathbf{q})=\sum_{i=0}^{n}a_i(\mathbf{q})s^i|\mathbf{q}\in Q\right\} \qquad (10)$$

are four fixed Kharitonov polynomials

$$K_1(s)=a_0^-+a_1^-s+a_2^+s^2+a_3^+s^3+a_4^-s^4+a_5^-s^5+... \qquad (11)$$
$$K_2(s)=a_0^++a_1^+s+a_2^-s^2+a_3^-s^3+a_4^+s^4+a_5^+s^5+... \qquad (12)$$
$$K_3(s)=a_0^++a_1^-s+a_2^-s^2+a_3^+s^3+a_4^+s^4+a_5^-s^5+... \qquad (13)$$
$$K_4(s)=a_0^-+a_1^+s+a_2^+s^2+a_3^-s^3+a_4^-s^4+a_5^+s^5+... \qquad (14)$$

Theorem 2 *(Kharitonov's Theorem):* An interval polynomial family $P(s,Q)$ with invariant degree is robustly stable if and only if its four Kharitonov polynomials are stable.

Definition 11 *(Kharitonov Rectangle):* Associated with the four Kharitonov polynomials $K_1(s)$, $K_2(s)$, $K_3(s)$ and $K_4(s)$ is a rectangle (the *Kharitonov rectangle*) whose four vertices are obtained by evaluating the four Kharitonov polynomials at $s=jw_0$. Therefore given an interval polynomial family $P(s,Q)$ and a fixed frequency $w=w_0$, the value set $P(jw_0,Q)$ is a rectangle whose vertices are given by $K_i(jw_0)$ for $i=1,2,3,4$.

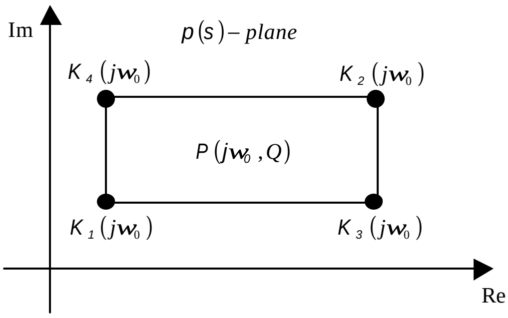

Fig. 4. The Kharitonov rectangle for $w_0\geq 0$.

Fig. 4 shows a generic Kharitonov rectangle. Note that the size and the position of the Kharitonov rectangle change with w, while its sides always remain parallel to the respective real and imaginary axes. Therefore, as we sweep the frequancy over a certain polynomial, we can observe the motion of the Kharitonov rectangle.

3.4 The application (robust stability of mobile robot)

According to the definitions and theorems from subsection 3.1, 3.2 and 3.3 the robust stability of the closed-loop system of Equation (4) can be proved. Equation (4) can be written as interval polynomial in the following form:

$$p(s,\mathbf{q})=q_4s^4+q_3s^3+q_2s^2+q_1s+q_0 \qquad (15)$$

where

$$\mathbf{q}=[q_4,q_3,q_2,q_1,q_0] \qquad (16)$$

is the vector of uncertain parameters, and assume that

$q_4\in[1,2]$, $q_3\in[123.2,136.2]$, $q_2\in[5951,6579]$, $q_1\in[127780,141230]$, $q_0\in[1025100,1133000]$

then the uncertainty bounding set (Definition 1) is

$$Q=\left\{\mathbf{q}\left|\begin{array}{l}q_0\in[1025100,1133000],q_1\in[127780,141230],\\ q_2\in[5951,6579],q_3\in[123.2,136.2],q_4\in[1,2]\end{array}\right.\right\} \qquad (17)$$

The above interval polynomial family is denoted by writing an interval polynomial family of the form:

$$p(s,\mathbf{q})=[1,2]s^4+[123.2,136.2]s^3+[5951,6579]s^2+[127780,141230]s+[1025100,1133000] \qquad (18)$$

where $0\notin[q_4^-,q_4^+]=[1,2]\Rightarrow$

interval polynomial family $P(s,Q)$ has invariant degree.

From Definition 10 the four fixed Kharitonov polynomials are derived as follows:

$$K_1(s)=s^4+136.2s^3+6579s^2+127780s+1025100 \qquad (19)$$
$$K_2(s)=2s^4+123.2s^3+5951s^2+141230s+1133000 \qquad (20)$$
$$K_3(s)=2s^4+136.2s^3+5951s^2+127780s+1133000 \qquad (21)$$
$$K_4(s)=s^4+123.2s^3+6579s^2+141230s+1025100 \qquad (22)$$

Using Routh criterion it is easy to verify that all four Kharitonov polynomials are stable (Routh column is positive in all cases). Hence it can be concluded that the closed-loop control system is robustly stable. The same conclusions can be drawn using the Zero Exclusion Condition in subsection 3.5 below.

3.5 Robust stability testing via graphics.

The Kharitonov rectangle provides a very handy graphical means to test the robust stability of physical systems. Plots of successive Kharitonov rectangles over the frequency interval $w \in [0, \infty)$, can produce observation of their motion in the complex plane. This plot together with the following theorem enables checking the stability of interval polynomials.

Theorem 3 *(Zero Exclusion for Interval Families):* An interval polynomial family $P(s,Q) = \{p(s,\mathbf{q}) | \mathbf{q} \in Q\}$ having invariant degree and at least one stable member $p(s,\mathbf{q}^*)$ is robustly stable if and only if the origin of the complex plane is excluded from the Kharitonov rectangle at all nonnegative frequencies i.e. $0 \notin P(jw_0, Q)$ for all frequencies $w \geq 0$.

In practice, there is not need to plot the Kharitonov rectangles for all $w \geq 0$. A cut-off frequency $w_c > 0$ can be obtained such that $0 \notin P(jw_0, Q)$ for all $w \geq w_c$. One such estimate, suggested from the classical bounds on the roots of a polynomial, provides an appropriate cut-off frequency as given by

$$w_c = 1 + \frac{\max\{q_0^+, q_1^+, ..., q_{n-1}^+\}}{q_n^-} \qquad (23)$$

for the interval polynomial $p(s,\mathbf{q})$ with $q_n^- > 0$ (n is the order of the polynomial).

Instead of generating two-dimensional Kharitonov rectangles, examination of the plot of the scalar function $H(w)$ (*Frequency Sweeping Function*) is determine if the family of polynomials P is robustly stable.

Theorem 4 *(Frequency Sweeping Function for Robust Stability):* Let P be an interval polynomial family with interval degree, at least one stable member and associated Kharitonov polynomials $K_1(s)$, $K_2(s)$, $K_3(s)$ and $K_4(s)$. Then with

$$H(w) = \max\left\{\begin{matrix} \mathrm{Re}\, K_i(jw), -\mathrm{Re}\, K_2(jw), \\ \mathrm{Im}\, K_3(jw), -\mathrm{Im}\, K_4(jw) \end{matrix}\right\} \qquad (24)$$

it follows that P is robustly stable if and only if $H(w) > 0$ for all frequencies $w \geq 0$.

3.6 Verification of the closed-loop robust stability

To verify that the closed loop control system is robustly stable further testing using graphics is performed using the Theorem 3 and 4. The characteristic equation of the closed-loop system is given in Equation (4). Equations (15) and (16) both provide the uncertainty vector $\mathbf{q}$ and the uncertainty bounding set Q. It was already shown that the interval polynomial family $P(s,Q)$ has invariant degree. In accordance with Theorem 3, the first step in the graphical test for robust stability requires that at least one polynomial in $P(s,Q)$ that is stable. Using the midpoint of each interval from Equation (17) $\mathbf{q}^*$ is obtained as follows:

$$\mathbf{q}^* = (1.5, 129.7, 6265, 134505, 1079050) \qquad (25)$$

then

$$p(s,\mathbf{q}^*) = 1.5s^4 + 129.7s^3 + 6265s^2 + 134505s + 1079050 \qquad (26)$$

Using the Routh criterion it is easy to verify that $p(s,\mathbf{q}^*)$ is stable. The cut-off frequency can be calculated from Equation (22) as follows:

$$w_c = 1 + \frac{\max\{1133000, 141230, 6579, 136.2\}}{1} = 1133001 \ rad/s \qquad (27)$$

The Kharitonov rectangles can be plotted to verify the stability of the closed loop system for frequency range $w \in [0, 1133001] rad/s$. For more convenient frequency range $w \in [0, 100] rad/s$ plot is shown in order the zero point in the graph to be visible. Fig 5 shows the Kharitonov rectangles for the closed-loop system. Fig 6 shows the plot of the frequency sweeping function $H(w)$.

Since the origin is excluded from the Kharitonov rectangles (Fig 5) it is concluded that the closed-loop control system is robustly stable. The same conclusion can be obtained from Fig 6 because it can be observed that the frequency sweeping function $H(w)$ is positive for all $w \in [0, 100] rad/s$.

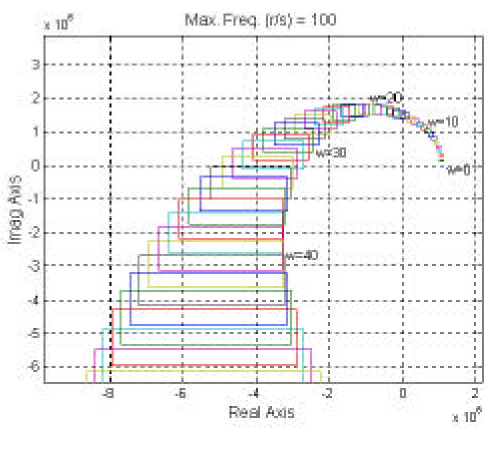

Fig. 5. Kharitonov rectangles for the controlled closed-loop control system.

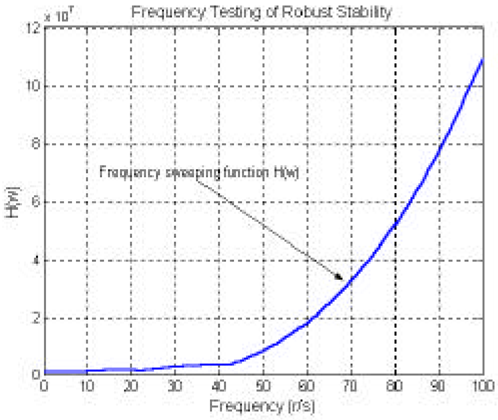

Fig. 6. A plot of $H(w)$ versus w.

4. DISCUSSIONS

The uncertainty of the closed-loop system was modelled by replacing the coefficients of the closed-loop characteristic equation of the MIMO system with closed interval polynomials. Although the robust stability was proven, a question remains of how to map a closed-loop characteristic equation of system to the system's physical parameters. For example, the mobile robot for which the robust analysis took place weighs 0.5 kg. If there was a need for 10% increase of its mass (i.e. adding more sensing elements) how the coefficients of the closed–loop characteristic equation will change is of interest. To map the change of the robot's mass to the change in the coefficients of the closed-loop characteristic equation is very difficult. In order to demonstrate this, consider the modified open-loop robot transfer function matrix in Equation (28), and the closed-loop transfer function of the system in Equation (29a, 29b, 29c, 29d) resulting from the 10% increase in mass. It can be observed that the intervals used for Equation (16) do not include all the variations in coefficients resulting from a 10% increase in mass. Care must therefore be taken in selecting the most suitable interval in order to accommodate the range of the expected parameter variations. The closed-loop control system described in Equations (3a, 3b, 3c, 3d) was tested again for robust stability based on new uncertainty bounding set given in equation (30) and was found to be robustly stable.

$$G(s)=\begin{bmatrix} \dfrac{10.06s+504.2}{s^2+102.3s+2577} & \dfrac{1.659s+23.11}{s^2+102.3s+2577} \\ \dfrac{1.659s+23.11}{s^2+102.3s+2577} & \dfrac{10.06s+504.2}{s^2+102.3s+2577} \end{bmatrix} \quad (28)$$

$$\frac{Y_1}{U_1}(s)= \quad (29a)$$

$$\frac{11s^3+1576s^2+7.096e004s+9.814e005}{s^4+124.3s^3+5793s^2+1.223e005s+9.814e005}=g_{11}(s)$$

$$\frac{Y_1}{U_2}(s)= \quad (29b)$$

$$\frac{-13.62s^3-846.4s^2-1.032e004s}{s^4+124.3s^3+5793s^2+1.223e005s+9.814e005}=g_{12}(s)$$

$$\frac{Y_2}{U_1}(s)= \quad (29c)$$

$$\frac{-13.62s^3-846.4s^2-1.032e004s}{s^4+124.3s^3+5793s^2+1.223e005s+9.814e005}=g_{21}(s)$$

$$\frac{Y_2}{U_2}(s)= \quad (29d)$$

$$\frac{11s^3+1576s^2+7.096e004s+9.814e005}{s^4+124.3s^3+5793s^2+1.223e005s+9.814e005}=g_{22}(s)$$

$$Q=\left\{\mathbf{q}\left|\begin{matrix} q_0\in[981400,1133000],\ q_1\in[122300,141230], \\ q_2\in[5793,6579],\ q_3\in[123.2,136.2],\ q_4\in[1,2] \end{matrix}\right.\right\} \quad (30)$$

5. CONCLUSIONS

In this paper the robust analysis of a closed-loop MIMO system based on parametric approach was investigated. Robust stability is vital due to the dynamics of the system. To demonstrate robust stability the Kharitonov theorem was used, based on interval polynomials control theory. The closed-loop control system was shown to be robustly stable under uncertainty based on closed intervals (first arbitrary then specific). Finally the robust stability was verified using graphical techniques based on Zero Exclusion Condition.

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