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CONTROL OF SEPARATRIX-CROSSING MOTION IN NEAR-HAMILTONIAN SYSTEMS PERTURBED BY WEAK NOISE

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Abstract. The paper develops a control procedure for a noisy near-Hamiltonian system. The control task is to prevent the system from escape from a reference region bounded by a lobe of the separatrix. Motion near the separatrix is presented as a sequence of encirclements over the separatrix lobe between two consecutive vertices, and the problem of avoiding escape through the separatrix is reduced to maximization of energy losses during one encirclement. It is shown that the energy difference during an encirclement can by approximated by the stochastic Melnikov integral. This allows extension of the stochastic Melnikov method to optimal control problems. An approximate solution is constructed as a time-invariant feedback, which is proved to be a nearly-optimal control for the original nonstationary problem. *Copyright © 2002 IFAC.*

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1. INTRODUCTION

The paper considers control against escape from a reference region for a class of stochastic oscillatory systems. The system is supposed to operate near a boundary of the safe region, and weak perturbations may result in escape from this domain. The control goal is to avoid escape or to drive the escaped system into the safe region.

The reference mode of operation for a near-Hamiltonian system is usually associated with oscillation (libration) within the potential well, and escape of this region is associated with the phase transition due to overcoming the potential barrier. In the phase plane, the reference region is bounded by a lobe of the separatrix of the generic non-perturbed Hamiltonian system. Investigation of near-separatrix dynamics requires a special approach. Long before and long after reaching the separatrix, motion can be studied by the well-developed stochastic averaging method (Kushner, 1990, Kovaleva, 1999). This procedure fails near the separatrix where the period tends to infinity. Dunyak and Freidlin (1998) proposed gluing conditions to match the averaged equations in the inner and outer domains. However, this approach does not consider separatrix-crossing transitions.

The asymptotic method most commonly used for analysis of near-separatrix motion is the Melnikov method. The method has been developed for deterministic systems with periodic (Melnikov, 1963) or quasi-periodic excitation (Wiggins, 1990), and then extended to systems with stochastic perturbations (Gundlach, 1995, Simiu, 2002). In the paper we present a nearly-optimal control procedure based on this method.

This approach was used earlier for constructing feedforward control against escape out of the reference region (Frey, 1996, Simiu, 2002). The control task was to suppress irregular escape due to counteracting perturbations and decreasing the effective excitation level. Implementation of the feedforward control requires a quite complicated scheme. This paper develops feedback control scheme. The control goal is to maximize the probability of capture into the safe region.

The equations of motion and main cost criteria are presented in Section 2. We define the probability of capture into the safe region and the probability of escape out of this region as performance measures of the system and describe shortcomings of these criteria from the computational perspective. These shortcomings can be avoided by considering the mean energy difference as a proper cost criterion. A required value of the probability of capture is introduced as an additional constraint assigning limiting control performance. Since we are interested in the system's behaviour in the vicinity of the separatrix, we use the Melnikov method to obtain asymptotic estimation. It is proved that the energy difference is approximated (in the weak sense) by the Melnikov integral $M(\tau)$, and maximization of the performance criterion is reduced to minimization of the expectation $\mathbf{E}M(\tau) < 0$. The inequality $M(\tau) < 0$ implies that the system's energy is insufficient to produce escape, and the probability $P_0 = P\{M(\tau) < 0\}$ can be interpreted as an asymptotic estimate of the probability of capture. In Section 3 we construct a nearly-optimal control. We show that the asymptotic solution leads to a time-independent feedback of a simple structure. This control is proved to be nearly-optimal for the original non-stationary problem. As an example we consider the controlled Duffing equation. We find a nearly-optimal control and evaluate limiting control performance guaranteeing the necessary value of the capture probability.

2 SETTING OF THE PROBLEM

In this section, we introduce cost criteria applicable for the problem in question. For the sake of simplicity, we study in detail a planar system with a two-lobe separatrix; the restriction to R^2 instead of R^{2n} is not essential and allows us to avoid some technical details.

We will consider a near-Hamiltonian system in the form

$$\frac{dq}{dt} = p$$
$$\frac{dp}{dt} = -V'(q) + \varepsilon f(t+\tau, p, q, u) \qquad (1)$$

where $V(q)$ is the potential, the derivative $V'(q) = dV/dq$, τ is a phase shift of the external excitation against initial conditions, $0 < \varepsilon << 1$ is a small parameter. The unperturbed subsystem ($\varepsilon = 0$)

$$\frac{dq_0}{dt} = p_0, \quad \frac{dp_0}{dt} = -V'(q_0) \qquad (2)$$

possesses the energy integral

$$H(p_0, q_0) = \frac{1}{2}p_0^2 + V(q_0) \qquad (3)$$

Since (2) is autonomous, its solution is defined up to an arbitrary phase shift and can be written in the form $p_0 = p_0(t-\tau)$, $q_0 = q_0(t-\tau)$. A parameterization in (1) is obtained by changing $t \to t+\tau$, so that $p_0 = p_0(t)$, $q_0 = q_0(t)$. The coefficient takes the form

$$f(t+\tau, p, q, u) = g(p, q, u) + r(p, q)\sigma\xi(t+\tau) \qquad (4)$$

where control u should be found from optimality conditions. Coefficients $g(p,q,u)$ and $r(p,q)$ are assumed to be sufficiently smooth for any admissible control $u = u(t,p,q)$ taking values in a compact set $U \subset R^1$. Noise $\xi(t)$ is a zero-mean uniformly bounded in the mean square stationary process with unit variance.

To specify the problem, we consider a system with a bistable (two-well) potential $V(q)$ and a two-lobe (homoclinic) separatrix. The threshold $H = 0$ corresponds to the separatrix, the energy in the domains A_- and A_+ inside the separatrix lobes S_- and S_+ is negative. The domains A_- and A_+ correspond to oscillations around each of the stable equilibrium (motion within a potential well), the right domain A_+ will be considered as a reference region. The outer domain B beyond the separatrix corresponds to motion overlapping both of stable positions (Fig. 1).

As in deterministic systems (Neishtadt, 1975, Cary *et al.*, 1986), dynamics in the ε-vicinity of the separatrix is studied by breaking up the motion into a sequence of steps. Each step is considered as an encirclement between two consecutive vertices. Let motion begin at the vertex a_1 in the ε - vicinity of the saddle point in the domain A_+ (Fig.1). The control task is to drive the system away from the separatrix and "to deepen" it into the safe region during an encirclement between two consecutive vertices a_1 and a_2. Note that the dotted and dashed encirclement lines in Fig. 1 depict averaged orbits. A real near-separatrix trajectory is a fast varying non-smooth process (Wiggins, 1990, Simiu, 2002),

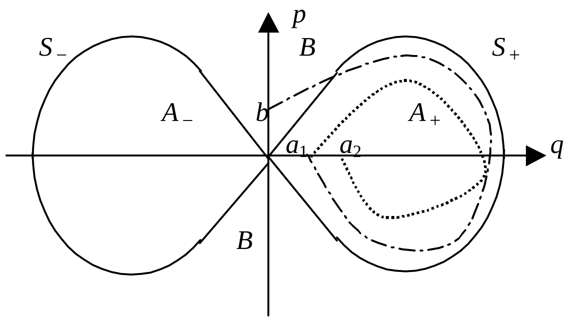

Fig. 1. Scheme of near-separatrix motion

2.1. Performance indices of the system.

As shown in (Kifer, 1988), escape from the vicinity of the saddle point in a weakly perturbed systems follows the deterministic scenario. A path becomes "trapped" in a neighbourhood of the saddle point, and, as $\varepsilon \to 0$, the mean escape time becomes independent of noise. Hence, performance measures based on the mean escape time criterion are not informative for the problem in question.

From a physical point of view, the drift into the core of the reference region corresponds to dissipation of the full energy during an encirclement. In stochastic systems, this condition is treated in a probabilistic sense. Let $H^* < 0$, $H^{**} < 0$ be the energy levels corresponding to the vertices a_1 and a_2, respectively. Then the probability of capture into the safe region A_+ can be written in the form

$$P_c = \Pr\{H^{**} - H^* < 0\,;\, H^* \le 0\} \tag{5}$$

Let the orbit begin at the vertex b in the "bad" region B, and the control task is to "capture" the system into the safe region during one encirclement between consecutive vertices b and a_1 (Fig. 1). In this case the capture probability can be written as

$$P_c = \Pr\{H^{**} < 0\,;\, H^* \ge 0\} \tag{6}$$

where $H^* > 0$, $H^{**} < 0$ are the energy levels corresponding to the vertices b and a_1, respectively.

Strictly speaking, the perturbed system has a chaotic near-separatrix layer of width $d \sim O(\varepsilon)$ arising due to crossings of stable and unstable manifolds of the system (Wiggins, 1990). This implies that the capture into the safe region can be considered if the internal vertex lies not only inside the safe region but also beyond the chaotic layer. Omitting details, we investigate non-chaotic capture into the safe region.

An opposite process of escape out of the potential well is associated with the probability of escape

$$P_{es} = \Pr\{H^{**} > 0;\, H^* \le 0\} \tag{7}$$

where H^*, H^{**} are the energy levels corresponding to the vertices a_1 and b, respectively.
Maximization of (5) or (6) as well as minimization of (7) may be considered as a control task. However, direct application of these criteria has computational shortcomings associated, in particular, with their great sensitivity to the noise intensity. In this paper we consider a simplified criterion of the mean energy difference

$$J = \mathrm{E}(H^{**} - H^*) \tag{8}$$

The drift into the core of the safe region may arise if the mean energy difference $J = \mathrm{E}(H^{**} - H^*) < 0$. Furthermore, if $J < 0$, minimization of criterion (8) corresponds to the maximum deepening into the safe region during an encirclement. This cost criterion is more sensitive to the change of the drift than to noise fluctuations, and the feedback control is expected to be an effective tool for controlling the system. The requisite level of the probability of capture can be interpreted as an additional constraint.

2.2. Asymptotic analysis

Asymptotic analysis is based on the small parameter expansions in the vicinity of the separatrix. From (1), (3) we obtain the equation

$$\frac{dH}{dt} = \frac{\partial H}{\partial p}\frac{dp}{dt} + \frac{\partial H}{\partial q}\frac{dq}{dt} = \varepsilon p f(t+\tau,\ p,q,u) \tag{9}$$

where $u = u(t,p(t,\tau),q(t,\tau))$. To evaluate H in an ε-vicinity of the separatrix, construct the expansions

$$\begin{aligned} H(t,\tau,\varepsilon) &= H_0 + \varepsilon h(t,\tau) + \varepsilon^2 R(t,\tau,\varepsilon) \\ p(t,\tau,\varepsilon) &= p_0(t) + \varepsilon \rho_1(t,\tau,\varepsilon) \\ q(t,\tau,\varepsilon) &= q_0(t) + \varepsilon \rho_2(t,\tau,\varepsilon) \end{aligned} \tag{10}$$

Here $H_0 = H(p_0(t),q_0(t)) = 0$ and $p_0(t)$, $q_0(t)$ is the solution of the generic system (2) taken along the separatrix. The first approximation h satisfies the linear equation

$$\frac{dh}{dt} = \varepsilon p_0(t) f^{u}(t,\tau) \tag{11}$$

where $f^{u}(t,\tau) = f\,[t+\tau,\ p_0(t),\ q_0(t),\ u(t,p_0(t),\ q_0(t))]$ for any admissible control u.

If ε is small enough, expansions (10) converge (weakly) for all $t \in (-\infty, \infty)$ (Kovaleva, 1998a, 2001). This result is interpreted as a stochastic counterpart of asymptotic estimation for deterministic systems (Sanders, 1982). Due to the weak convergence $H \to h$, control problems for (1), (9) can be reduced to similar problems for the approximating system (11); the requisite asymptotic estimation can be obtained as in (Kushner, 1990, Kovaleva, 1999). In general, control problems for linear system (11) can be considered with standard procedures. An effective asymptotic solution is based on the Melnikov method

For motion in the ε–vicinity of the separatrix we let $H^* = \varepsilon h^*$, $H^{**} = \varepsilon h^{**}$. From (9), we obtain

$$h^{**} = h^* + \eta, \quad \eta = \varepsilon^{-1} \int_C dH \tag{12}$$

integral is taken along the orbit C between vertices a_1 and a_2 . It was proved (Kovaleva, 1998a, 2001) that

$$\eta = M^{u}(\tau) + \gamma(\varepsilon), \quad h^{**} = h^* + M^{u}(\tau) + \gamma(\varepsilon) \tag{13}$$

where

$$M^{u}(\tau) = \int_{S_+} dh = \int_{-\infty}^{\infty} [p_0(t), f^{u}(t,\tau)]\mathrm{d}t \tag{14}$$

is the Melnikov integral of (1) taken over the right lobe of the separatrix (Wiggins, 1990), the residual term $\gamma(\varepsilon) \sim \varepsilon \ln\varepsilon \to 0$, as $\varepsilon \to 0$. Relationships (13) are considered in the weak sense, that is

$$\Pr\{\,|\,h^{**} - h^* - M^{u}(\tau)\,| \geq \gamma(\varepsilon)\,|\,\} \to 0, \quad \varepsilon \to 0 \tag{15}$$

This implies that condition (5) can be reduced to the form

$$\mathrm{P_c} = \mathrm{P_0} + \mathrm{P_\varepsilon}$$

where $\mathrm{P_\varepsilon} \to 0$ for $\varepsilon \to 0$. The main term is written as

$$\mathrm{P_0} = \Pr\{M^{u}(\tau) < 0\} \tag{16}$$

The control task is thus to maximize criterion (16) independent of the precise positions of vertices a_1, a_2 or b.

If an orbit starts at the saddle point corresponding to $h^* = 0$, then $h^{**} = M^{u}(\tau)$ (terms of higher orders are dropped out). In this case 16) can be considered as the probability of capture into the safe region.

From (4), (11) we obtain

$$M^{u}(\tau) = m + \mu(\tau) \tag{17}$$

where $m = \mathrm{E}M^{u}(\tau)$

$$m = \int_{-\infty}^{\infty} p_0(t) g_0(t,u) dt \tag{18}$$

$$g_0(t,u) = g(p_0(t),\ q_0(t),\ u(t,\ p_0(t),\ q_0(t))$$

for any admissible control u, and

$$\mu(\tau) = \int_{-\infty}^{\infty} D(t)\sigma\xi(t+\tau)dt$$

$$D(t) = p_0(t) r(\,p_0(t), q_0(t)) \tag{19}$$

It follows from (17), (18), (19) that $M^{u}(\tau)$ can be interpreted as a stationary process with expectation m and variance d calculated from (19). In case $\xi(t)$ is Gaussian, probability (16) can be written in the explicit form. As seen from the above, in the ε-vicinity of the separatrix control $u(t,p,q)$ only affects the drift m. Hence, in the limit, as $\varepsilon \to 0$, one can replace minimization of criterion (8) with a simpler minimization problem for expectation (18), provided $m < 0$.

The capture probability may be limited by limiting control performance. If a preassigned value of $\mathrm{P_c}$ (or $\mathrm{P_0}$) cannot be achieved only due to feedback control $u(t,p,q)$ with given constraints, control design has to be changed, and feedforward or hybrid control schemes can be employed.

3 ASYMPTOTIC SOLUTION OF OPTIMAL CONTROL PROBLEMS

We use an asymptotic approach for minimization criterion (8). Since the drift into the core of the safe region exists if

$$\mathrm{E}(H^{**} - H^*) = \varepsilon^{-1}\,\mathrm{E}(h^{**} - h^*) < 0 \tag{20}$$

the performance cost is written as

$$J(u) = \mathrm{E}(h^{**} - h^*),\ u \in U : [u_1,\ u_2] \tag{21}$$

Optimal control u_ε is thus defined as

$$J(u_\varepsilon) = \min_{u \in U} J(u) \tag{22}$$

the inequality $J(u) < 0$ is not included in optimality conditions and need to be verified.

Denote $m = m(u)$. The weak convergence (15) implies the condition

$$|J(u) - m(u)| \to 0, \quad \varepsilon \to 0 \tag{23}$$

for any admissible control u (Kovaleva, 1998a, 2001). Owing to this, one can reduce problem (22) to a simpler problem of minimization expectation (18). Let $u_0(t)$ be a control minimizing (18), that is

$$m(u_0) = \min_{u \in U} m(u)$$

$$u_0(t) = \arg\min_{u \in U} [p_0(t) g(p_0(t), q_0(t), u)] = \tag{24}$$

$$= U_0(p_0(t), q_0(t))$$

where arguments $p_0(t)$, $q_0(t)$ are calculated along the right lobe of the separatrix S_+ (Fig.1). One can prove that (23), (24) yield the estimate (Kovaleva, 1999)

$$0 \le J(u_0) - J(u_\varepsilon) \le \mu(\varepsilon) \to 0, \quad \varepsilon \to 0 \tag{25}$$

This implies that control (24) can be interpreted as a nearly-optimal open-loop control (a programme) for problem (22). An equivalent closed-loop control can be written in the form

$$u_*(p,q) = U_0(p, q) \tag{26}$$

where U_0 is defined by (24). As it follows from (18), the programme(24) and the closed-loop control (24) are equivalent in the ε-vicinity of the separatrix. Hence, $u_*(p,q)$ admits estimate similar to (25) when substituted $u_0 \to u_*$. This implies that time-independent feedback control $u_*(p,q)$ can be used as a nearly-optimal control for the original non-stationary problem.

Note that solutions (24), (26) have a simple physical sense. These controls maximize the mean velocity of the drift into the safe region at each moment t.

If $M^u(\tau)$ is a Gaussian process, maximization of m is equivalent to minimizatiom of the capture probability

3.1. Example

An example of the controlled Duffing system is intended only to be illustrative. We demonstrate the connection between limiting control performance and the probability of capture.

Consider a standard Duffing equation in the form

$$\frac{dq}{dt} = p$$

$$\frac{dp}{dt} = -V'(q) + \varepsilon[u + \sigma\xi(t+\tau)] \tag{27}$$

with the potential $V(q) = q^4/4 - q^2/2$ Let $|u| \le N$. We obtain from (24), (26)

$$u_*(p,q) = -N\text{sign } p, \quad u_0(t) = -N\text{sign } p_0(t) \tag{28}$$

where $p_0(t) = -(2)^{1/2} \sinh t/\cosh^2 t$ (Wiggins, 1990).

Define *limiting control performance*. Let $\xi(t)$ be Gaussian white noise with unit variance. In this case we find from (18), (28)

$$m(u_*) = m_N = -(2)^{3/2} N \tag{29}$$

Thus the maximum probability of capture is (Gardiner, 1990)

$$P_N = \frac{1}{2}[1 + \Phi(-\frac{m_N}{d})] \tag{30}$$

where $\Phi(z)$ is the Gauss probability integral, the variance $d^2 = 4\sigma^2/3$.

Let the system be reliable if $P_N > 0{,}972$. This value of P_N can be achieved if (Gardiner, 1990)

$$|m_N|/d = 6^{1/2} N/\sigma \ge 2, \quad N \ge (2/3)^{1/2}\sigma \tag{31}$$

If $N < (2/3)^{1/2}\sigma$, the desired quality of the system cannot be achieved. Condition (31) demonstrates the direct connection between control constraints and the requisite probability of capture.

Let the control goal be to drive the system from the outer domain B into the inner domain A_+ during one encirclement between the vertices b and a_1 (Fig. 1). Since $h^* > 0$, but $Eh^{**} = h^* + m(u) < 0$, the basin of attraction is defined as

$$0 < h^* < h_A = -m_N = (2)^{3/2} N \tag{32}$$

(terms of higher orders are neglected). If $h^* > h_A$, then $Eh^{**} > 0$, and the probability of capture during one encirclement becomes relatively small. It is easy to find that $Eh^{**} > 0$ if $N < N^* = h^*/(2)^{3/2}$. This allows estimation of limiting control performance in the form $N > N^* = h^*/(2)^{3/2}$.

Other types of criteria associated with the change of energy can be considered as for deterministic systems (Kovaleva, 1998b).

4. CONCLUSIONS

We develop an asymptotic method for constructing a nearly-optimal control against escape from a potential well. Motion near the separatrix is presented as a sequence of encirclements between two consecutive vertices. This allows extention of the stochastic Melnikov method to optimal control problems. It is shown that the problem of avoiding escape can be reduced to minimization of the mean value of the Melnikov integral $M(\tau)$. The asymptotic solution is constructed as time-independent feedback, which is proved to be a nearly-optimal control for the original nonstationary problem.

The perturbed system has a chaotic near-separatrix layer of width $d \sim O(\varepsilon)$. The chaos arises due to crossings of stable and unstable manifolds of the perturbed system (Wiggins, 1990). This implies that the capture into the safe region can be considered if the internal vertex lies within the safe region but beyond the chaotic layer.

As known, the system can escape from a region of bounded oscillations either to a domain of infinite motion (rotation), or to another domain of oscillatory or chaotic motions. The energetic approach explores, in principle, non-chaotic motion. On the other hand, the Melnikov integral $M(\tau)$ is a measure of a distance between stable and unstable manifolds in a perturbed system (Wiggins, 1990). If $M(\tau)$ has simple zeros, the stable and unstable manifolds intersects and the system exhibits irregular escape associated with chaotic motion. This implies that the probability of capture $P\{M(\tau) < 0\}$ is identical to the probability of non-occurrence of chaos. This allows extension of the results to the field of controlling chaos.

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