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BIFURCATION TAILORING OF NONLINEAR SYSTEMS

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Abstract: We discuss a novel approach to control bifurcations in nonlinear systems. The aim of bifurcation tailoring is to design an appropriate control law such that the controlled system has a desired bifurcation diagram. After describing two open-loop bifurcation tailoring techniques, this paper proposes two alternative modified bifurcation tailoring methods based on the use of the Newton-flow algorithm and the so-called Minimal Control Synthesis adaptive control strategy. The novel technique is applied to the Duffing system as an illustration example. *Copyright © 2002 IFAC*

Keywords: Bifurcation; Open-loop control; Adaptive control; Nonlinear system.

1. INTRODUCTION

In recent years, there has been rapidly growing interest in control of bifurcations in nonlinear dynamical systems (Chen, 1999; Chen *et al*., 2000). The goal of bifurcation control is typically achieved by delaying the onset of an inherent bifurcation and/or stabilizing an existing one. Some of the bifurcation control approaches to solve these problems presented in the literature include linear or nonlinear static feedback (Abed and Fu, 1986), washout filter-aided dynamic feedback (Wang & Abed, 1995), harmonic balance approximation (Berns *et al*., 1998) and normal forms-based feedback (Kang, 1998). In a broader sense, bifurcation control can be referred to as the task of designing a controller to modify the bifurcation properties of a given nonlinear system. Hence the goal becomes that of achieving a set of desirable asymptotic behaviours of the system as its parameters are varied.

Recently, motivated by the control of bifurcations in complex flight dynamics, Lowenberg *at al* proposed the concept of *bifurcation tailoring*. This novel method is aimed at changing the bifurcation diagram of a given system to a desired one by appropriately varying extra system parameters in addition to the bifurcation parameter (Lowenberg, 1998a, 1998b; Lowenberg and Richardson, 1999). The original bifurcation tailoring technique involves an 'inversion' of the bifurcation continuation method as used in software such as AUTO (Doedel & Wang, 1995), therefore, is open-loop in nature from a control point of view. In other words, it cannot guarantee the stability of the desired behavior (equilibrium points or limit cycles) at any given value of the bifurcation parameter. Therefore, in addition to the feedforward control, an effective feedback mechanism should be added to the original bifurcation tailoring technique to address disturbances and modeling errors, so as to guarantee the stability and robustness of the controlled system.

This paper is concerned with the development of such a feedback mechanism through the synthesis of novel bifurcation tailoring methods. These are based on the combined use of an on-line continuation technique (the Newton-flow algorithm) and a sophisticated adaptive control strategy, the Minimal Control Synthesis Algorithm or MCS (Stoten and Benchoubane, 1990a, 1990b).

The rest of the paper is outlined as follows. Definition of bifurcation tailoring is presented in Section 2. In Section 3, open-loop bifurcation tailoring techniques and their limitations are discussed. Section 4 proposes two open-loop plus close-loop bifurcation tailoring methods. An illustration example is presented in Section 5.

2. BIFURCATION TAILORING: STATEMENT OF THE PROBLEM

Consider a continuous-time dynamical system described by

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\ p, \mathbf{q}) \\ \mathbf{y} = \mathbf{h}(\mathbf{x}) \end{cases} \tag{1}$$

where $\mathbf{x} \in \Re^n$ and $\mathbf{y} \in \Re^m$ are the state and output of the system, respectively. Here, we assume that $p \in \Re$ is a slow-varying system parameter and $\mathbf{q} \in \Re^m$ is the vector of all other system parameters or external control inputs. The *bifurcation tailoring* problem is to design a control law $\mathbf{q}$ such that the output of the controlled system has a desired dynamical behavior as the parameter p varies from p_a to p_b, i.e.,

$$\mathbf{y}_d * (p) = \mathbf{g}(p, t) \qquad \text{for } p \in [p_a,\ p_b], \tag{2}$$

where for a given $p \in [p_a,\ p_b]$, $\mathbf{g}(p,t)$ could be a point, a limit cycle or even chaotic.

Without loss of generality, we could assume that the outputs are the first *m* states of the system which we label with $\mathbf{x}_I$, i.e.,

$$\mathbf{y} = \mathbf{x}_I = [x_1, \cdots x_m]^T. \tag{3}$$

3. OPEN-LOOP BIFURCATION TAILORING

3.1 Bifurcation Tailoring of Equilibria

In what follows we focus our attention on the problem where the desired objective is for the controlled system to exhibit a branch of equilibria such that, as the parameter *p* is varied,

$$\mathbf{x}_I(p) = \mathbf{g}(p), \quad \text{for } p \in [p_a, \quad p_b] \tag{4}$$

where $\mathbf{g}$ is an *m*-dimensional smooth function of p. This means that, for a given $p \in [p_a,\ p_b]$, the desired output is a point and this point varies smoothly as p varies from p_a to p_b.

Partition the state vector $\mathbf{x}$ as $\mathbf{x} = [\mathbf{x}_I^T\ \mathbf{x}_{II}^T]^T$, where $\mathbf{x}_{II} = [x_{m+1}, \cdots x_n]^T$ and define the auxiliary vector $\mathbf{z}$ as:

$$\mathbf{z} = \begin{bmatrix} \mathbf{x}_{II} \\ \mathbf{q} \end{bmatrix}. \tag{5}$$

For any given $p \in [p_a,\ p_b]$, $\mathbf{z}$ should satisfy the equation:

$$\mathbf{f}(\mathbf{g}(p), \mathbf{x}_{II}, p, \mathbf{q}) \equiv \tilde{\mathbf{f}}(\mathbf{z}, p) = 0. \tag{6}$$

In fact, if (6) is satisfied the system will exhibit a branch of equilibria with the desired shape over the parameter range $p \in [p_a,\ p_b]$.

The Implicit Function Theorem (IFT) states that if the Jacobian of **f** with respect to $\mathbf{z}$ is invertible, then Eq. (6) implicitly defines **z** as a function of *p*, i.e.,

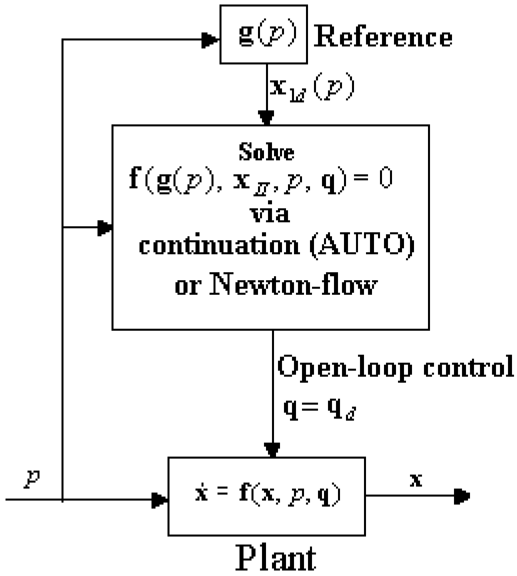

Fig. 1. Structure of the open-loop bifurcation tailoring.

$$\mathbf{z} = \mathbf{z}_d * (p) = \begin{bmatrix} \mathbf{x}_{IId} * (p) \\ \mathbf{q}_d * (p) \end{bmatrix}, \tag{7}$$

which means that

$$\mathbf{x}_d * = \begin{bmatrix} \mathbf{x}_{Id} * \\ \mathbf{x}_{IId} * \end{bmatrix} \tag{8}$$

is an equilibrium point of the open-loop system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\ p, \mathbf{q}_d * (p)). \tag{9}$$

Therefore, if $\mathbf{x}_d *$ is an asymptotically stable equilibrium point of (9) for any $p \in [p_a,\ p_b]$, then one may use the open-loop control input $\mathbf{q} = \mathbf{q}_d * (p)$ to achieve the goal of bifurcation tailoring.

Thus, as shown in Fig. 1, the implementation of bifurcation tailoring requires the on-line solution of equation (6), i.e. the so-called continuation of the system solution. This can be achieved in two different ways.

3.2Open-loop Bifurcation Tailoring via Continuation

Partitioning the interval $[p_a,\ p_b]$ into *l-1* sufficiently small subintervals, the endpoints of the subintervals are as follows:

$$p_a = p_0 < p_1 < \cdots < p_{l-1} < p_l = p_b. \tag{10}$$

Starting from

$$\mathbf{z}_d * (p_a) = \begin{bmatrix} \mathbf{x}_{II}(p_a) \\ \mathbf{q}(p_a) \end{bmatrix}, \tag{11}$$

one can use numerical continuation technique (such as AUTO) to find consecutive points of a solution branch to equation (6).

3.3 *Open-loop Bifurcation Tailoring via Newton Flow*

One may also use the following continuous Newton algorithm:

$$\begin{bmatrix} \dot{\mathbf{x}}_{IId} \\ \dot{\mathbf{q}}_d \end{bmatrix} = -\begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial \mathbf{x}_{IId}} & \dfrac{\partial \mathbf{f}}{\partial \mathbf{q}_d} \end{bmatrix}^{-1} \cdot \mathbf{f}(\mathbf{g}(p(t)), \mathbf{x}_{IId}, p(t), \mathbf{q}_d) \qquad (12)$$

to solve equation (6) as $p = p(t)$ varies slowly from $p(0) = p_a$ to $p(t_f) = p_b$ at some time t_f.

The initial values are taken as

$$\mathbf{x}_{IId}(0) = \mathbf{x}_{II}(p_a), \quad \mathbf{q}_d(0) = \mathbf{q}(p_a) \quad . \qquad (13)$$

For fixed value of p, the right hand side of (12) is commonly referred to as (gradient) Newton flow. The stability of Newton flows has been studied extensively (Jongen *et al.*, 1986; Zufiria and Guttalu, 1990). If $p = p(t)$ varies sufficiently slowly, then

$$\mathbf{q}_d(t) \approx \mathbf{q}_d{}^*(p(t)), \ \mathbf{x}_{IId}(t) \approx \mathbf{x}_{IId}{}^*(p(t)) .$$

3.4 Limitation of Open-Loop Bifurcation Tailoring

As anticipated in the introduction, the open-loop bifurcation tailoring technique presented above presents some limitations and disadvantages.

I. It's possible that for some $p \in [p_a, p_b]$, $\mathbf{x}_d{}^*$ is not an asymptotically stable equilibrium point of (9). In this case, the open-loop input $\mathbf{q} = \mathbf{q}_d{}^*(p)$ cannot achieve the goal of control.

II. The basic assumption needed for the successful implementation of the open-loop bifurcation tailoring is the 'almost complete knowledge' of the system under investigation. For realistic engineering applications such knowledge cannot be assumed.

The limitations of open-loop bifurcation tailoring suggest that some sort of feedback mechanism should be necessary to achieve the goal of bifurcation tailoring. The ability to achieve a prescribed goal in the presence of uncertainties is the main reason for feedback. In other words, we want to design an open-loop plus close-loop control law of the form,

$$\mathbf{q}(p) = \mathbf{q}_d{}^*(p) + \delta\mathbf{q}(p), \qquad (14)$$

so that $\mathbf{x}_d{}^*(p)$ is an asymptotically stable equilibrium point of the controlled system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, p, \mathbf{q}(p)) . \qquad (15)$$

for $p \in [p_a, p_b]$ (Fig. 2).

4. OPEN-LOOP PLUS CLOSE-LOOP BIFURCATION TAILORING

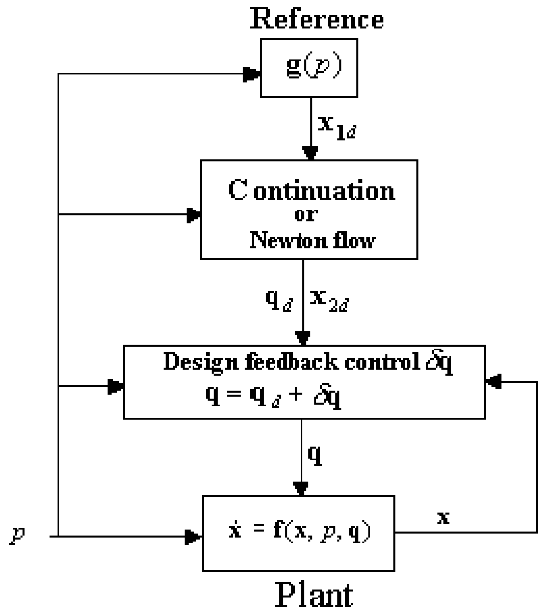

Fig.2. Structure of open-loop plus close-loop bifurcation tailoring.

4.1 Close-Loop Bifurcation Tailoring via Linear Feedback

Denote $\delta\mathbf{x}_p = \mathbf{x} - \mathbf{x}_d{}^*(p)$, $\delta\mathbf{q}_p = \mathbf{q} - \mathbf{q}^*(p)$. The linearization of (15) at the equilibrium point $\mathbf{x}_d{}^*(p)$ is given by

$$\delta\dot{\mathbf{x}}_p = \mathbf{A}_p \delta\mathbf{x}_p + \mathbf{b}_p \delta\mathbf{q}_p , \qquad (16)$$

where

$$\mathbf{A}_p = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_d{}^*(p) \\ q=q^*(p)}}, \quad \mathbf{b}_p = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\right|_{\substack{\mathbf{x}=\mathbf{x}_d{}^*(p) \\ \mathbf{q}=\mathbf{q}^*(p)}} . \qquad (17)$$

Suppose that the open-loop control law is derived using the continuation method of Section 3.2. At $p = p_i$, (16) becomes

$$\delta\dot{\mathbf{x}}_{p_i} = \mathbf{A}_{p_i} \delta\mathbf{x}_{p_i} + \mathbf{b}_{p_i} \delta\mathbf{q}_{p_i} . \qquad (18)$$

The feedback control law is taken as

$$\delta\mathbf{q}_{p_i} = -\mathbf{k}_{p_i}{}^T \delta\mathbf{x}_{p_i} , \qquad (19)$$

where the gain vector $\mathbf{k}_{p_i}$ is chosen so that the real parts of the eigenvalues of the matrix

$$\mathbf{J}(p_i) = \mathbf{A}_{p_i} - \mathbf{b}_{p_i} \mathbf{k}_{p_i}{}^T \qquad (20)$$

are all negative. Therefore, $\mathbf{x}_d{}^*(p_i)$ is a locally asymptotically stable equilibrium point of the controlled system,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, p_i, \mathbf{q}(p_i)), \quad i = 1, 2, \cdots, l , \qquad (21)$$

where

$$\mathbf{q}(p_i) = \mathbf{q}^*(p_i) + \delta\mathbf{q}(p_i) . \qquad (22)$$

The whole control law is a piece-wise linear control law of the form (Fig. 3)

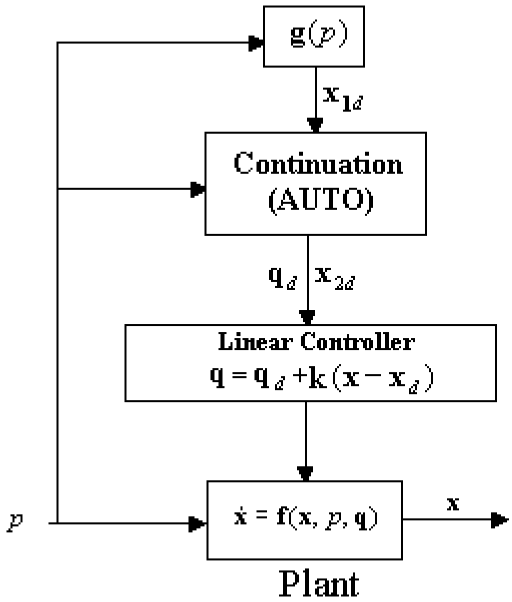

Fig. 3. Structure of open-loop plus linear feedback bifurcation tailoring.

$$\mathbf{q}(p) = \mathbf{q}^*(p_i) - \mathbf{k}_{p_i}^{\ T}\ \delta\mathbf{x}_{p_i} \tag{23}$$

for $p \in [p_i,\ p_{i+1})$, $i = 1, 2, \cdots l-1$.

To perform an asymptotical error analysis of the controlled system, for $p \in (q_i,\ q_{i+1})$, we have

$$\begin{aligned}\delta\dot{\mathbf{x}}_p &= \mathbf{A}_p\ \delta\mathbf{x}_p + \mathbf{b}_p \delta\mathbf{q}_{p_i} \\ &= [\mathbf{A}_p - \mathbf{b}_p \mathbf{k}_{p_i}^T]\delta\mathbf{x}_p + \mathbf{b}_p \mathbf{k}_{p_i}^T(\mathbf{x}_{p_i} - \mathbf{x}_p)\end{aligned} \tag{24}$$

According to the continuity of the solutions of equations and the eigenvalues of matrices, if the subinterval $(p_i,\ p_{i+1})$ is sufficiently small, then $\|\mathbf{x}_{p_i} - \mathbf{x}_p\|$ is also sufficiently small and $\mathbf{A}_p - \mathbf{b}_p \mathbf{k}_{p_i}^T$ is still a stable matrix for $p \in (p_i,\ p_{i+1})$. Moreover,

$$\begin{aligned}\delta\mathbf{y}_p(t) &= \mathbf{y}_p(t) - \mathbf{y}_d^*(p) \\ &= \mathbf{c}\, e^{[\mathbf{A}_p - \mathbf{b}_p \mathbf{k}_{q_i}^T]t}\mathbf{x}_o \\ &\quad + \int_0^t \mathbf{c}\, e^{[\mathbf{A}_p - \mathbf{b}_p \mathbf{k}_{p_i}^T]\tau}\mathbf{b}_p \mathbf{k}_{p_i}^T(\mathbf{x}_{p_i} - \mathbf{x}_p)\, d\tau\end{aligned} \tag{25}$$

where $\mathbf{c} = [1, \cdots, 1, 0, \cdots 0]$, which means that

$$\|\mathbf{y} - \mathbf{y}_d^*\| \le M\ \|\mathbf{x}_{p_i} - \mathbf{x}_p\| \quad \text{as } t \to \infty, \tag{26}$$

where M is a positive constant. Better performance may be obtained by PI control.

4.2 Close-Loop Bifurcation Tailoring via Adaptive MCS Control

Now suppose that the open-loop bifurcation tailoring is implemented via Newton-flow of Section 3.3. We want to design an open-loop plus close-loop control law of the form

$$\mathbf{q}(t) = \mathbf{q}_d(t) + \delta\mathbf{q}(t), \tag{27}$$

so that $\mathbf{x}_1(t) \approx \mathbf{x}_{1d}^*(p(t))$ for $t \le t_f$.

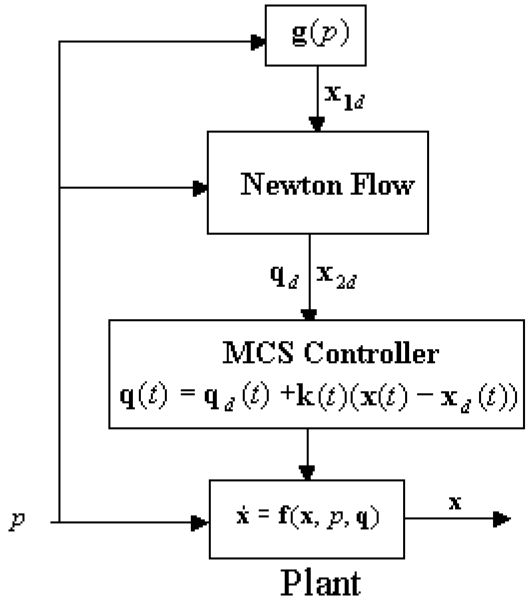

Fig. 4. Structure of open-loop plus MCS adaptive bifurcation tailoring.

Denote $\delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}_d(t)$ with

$$\mathbf{x}_d(t) = \begin{bmatrix}\mathbf{x}_{1d}^*(p(t)) \\ \mathbf{x}_{2d}(t)\end{bmatrix}. \tag{28}$$

Linearization of the controlled system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\ p,\ \mathbf{q}(t)) \tag{29}$$

at $(\mathbf{x}_d, \mathbf{q}_d)$ is given by

$$\delta\dot{\mathbf{x}} = \mathbf{A}(t)\,\delta\mathbf{x} + \mathbf{b}(t)\delta\mathbf{q}, \tag{30}$$

where

$$\mathbf{A}(t) = \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_d(t)\\ \mathbf{q}=\mathbf{q}_d(t)}},\quad \mathbf{b}(t) = \left.\frac{\partial\mathbf{f}}{\partial\mathbf{q}}\right|_{\substack{\mathbf{x}=\mathbf{x}_d(t)\\ \mathbf{q}=\mathbf{q}_d(t)}}. \tag{31}$$

In order to guarantee the stability of the origin of the time-varying system (30), we choose a particularly attractive adaptive control technique. Namely, we consider the so-called minimal control synthesis (MCS) adaptive control algorithm first presented in (Stoten and Benchoubane, 1990a, 1990b; Stoten and Di Bernardo, 1996). According to the MCS algorithm, the control signal $\delta\mathbf{q}$ is chosen as (Fig. 4):

$$\delta\mathbf{q}(t) = \mathbf{k}(t)\ \delta\mathbf{x}(t), \tag{32}$$

with

$$\mathbf{k}(t) = \alpha\int_0^t \mathbf{y}_e(\tau)\delta\mathbf{x}^T(\tau)d\tau + \beta\mathbf{y}_e(t)\delta\mathbf{x}^T(t),$$

where α and β are positive scalar adaptation weights and the initial conditions $\mathbf{k}(0)$ are usually set to zero. The output error is obtained as

$$\mathbf{y}_e(t) = -\mathbf{c}_e\delta\mathbf{x}(t), \tag{33}$$

where the output error matrix $\mathbf{c}_e$ is determined from the positive-definite solution of the Lyapunov equation

$$\mathbf{PA}_m + \mathbf{A}_m^T\mathbf{P} = -\mathbf{Q}, \quad \mathbf{Q} > \mathbf{0}, \tag{34}$$

as

$$\mathbf{c}_e = \mathbf{b}_e^T\mathbf{P}, \quad \mathbf{b}_e^T = [0 \quad \cdots \quad 0 \quad 1], \tag{35}$$

where we take $\mathbf{Q} = \mathbf{I}$ and

$$\mathbf{A}_m = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{m1} & -a_{m2} & -a_{m3} & \cdots & -a_{mn} \end{bmatrix}$$

is a Hurwitz matrix.

5. SIMULATION RESULTS

As an illustrative example, we consider a second-order Duffing system of the form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = px_1 - x_1^3 - x_2 + q \end{cases} \tag{36}$$

This system is often used as a testbed for bifurcation and chaos control techniques and can be used to model nonlinear mechanical oscillators (Chen 1999). For fixed value of $q = 0$, this system exhibits a pitchfork bifurcation at $p = 0$. It has one equilibrium point $(0, 0)$ for $p \le 0$, and three equilibrium points $(0, 0)$, $(\pm\sqrt{p}, 0)$ for $p > 0$. Fig. 5 shows that the stable node $(0, 0)$ for $p < 0$ bifurcates into a saddle $(0, 0)$ and two stable nodes $(\pm\sqrt{p}, 0)$.

Now assume that $p = p(t)$ is a slow varying parameter. We want to design a control law $q(t)$ so that the controlled system has the following desired behavior:

$$x_{1d} * (p) = 0.5p(t), \tag{37}$$

as p increases from $p(0) = 0$ to $p(t_f) = 5$. The corresponding Newton flow becomes

$$\begin{cases} \dot{x}_{2d}(t) = -x_{2d}(t) \\ \dot{q}_d(t) = -0.5p^2(t) + 0.5^3 p^3(t) - q_d(t) \end{cases} \tag{38}$$

Fig. 6 shows the trajectory of $q_d(t)$ which is used as the open-loop input to system (36). However, as shown in Fig. 7, the open-loop input alone cannot create the desired system behavior. Therefore, we add a close-loop input $\delta q(t)$ computed by the MCS algorithm with $\alpha = 0.2$ and $\beta = 0.5$. As shown in Fig. 8, under the hybrid open-loop and close-loop control input $q(t) = q_d(t) + \delta q(t)$, the controlled system (36) has the desired behavior (37).

6. CONCLUSIONS

This work investigated open-loop plus close-loop bifurcation tailoring techniques. Using a representative example, we showed that the methodology presented in this paper can be successfully applied to control the bifurcation diagram of a nonlinear system. A more realistic application to the control of aircraft dynamics is currently under investigation (Charles, *et al.*, 2001).

Generally, from a control theory point of view, the bifurcation tailoring problem can be viewed as a particular type of output tracking problem which consists of designing a control law able to achieve tracking of a prescribed reference signal. The difficulty of bifurcation tailoring is that the reference signal is a function of a slow-varying system parameter p and this function may be discontinuous at some bifurcation point. We should guarantee the tracking of the reference signal as p varies from p_a to p_b.

The problem of controlling the output of a system so as to achieve asymptotic tracking of prescribed trajectories and asymptotic rejection of undesired disturbances is one of the most fundamental problems in control theory (Isidori and Byrnes, 1990). An investigation on the applicability of output tracking techniques to bifurcation tailoring should be pursued in future work.

ACKNOWLEDGEMENTS

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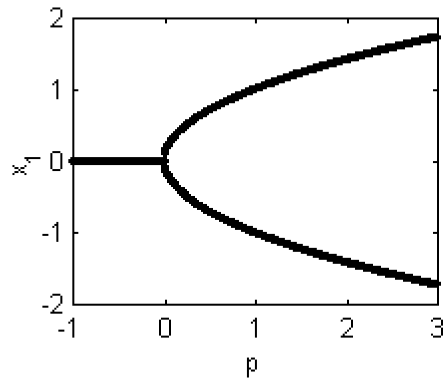

Fig. 5. Bifurcation diagram of the uncontrolled system (36).

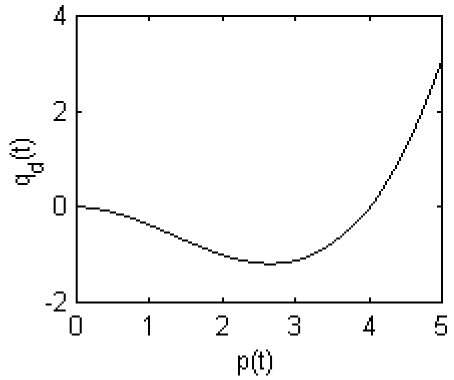

Fig. 6. The nominal feedforward input $q_d(t)$.

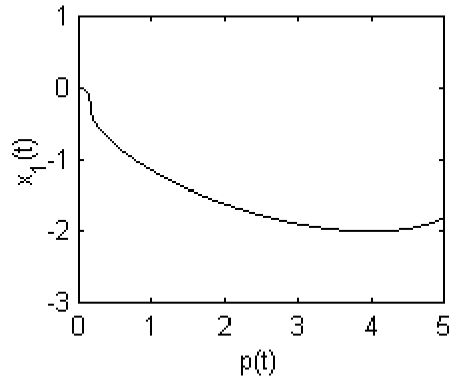

Fig. 7. Bifurcation diagram of the controlled system (36) with feedforward control input only.

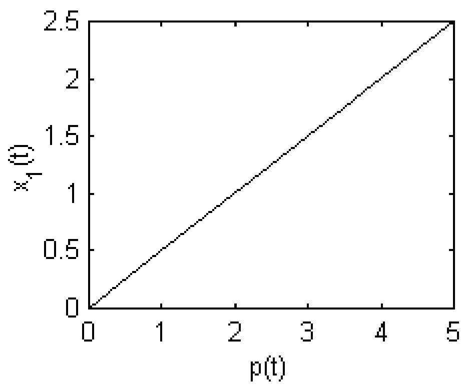

Fig. 8. Bifurcation diagram of the controlled system (36) with hybrid control input.