15th Triennial World Congress, Barcelona, Spain

ON-LINE BIFURCATION TAILORING: AN APPLICATION TO A NONLINEAR AIRCRAFT MODEL

G.A.Charles [*,**,***] **D.P.Stoten** [***] **X.Wang** [*,**,***]
M.Di Bernardo [**] **M.H.Lowenberg** [*]

[*] *Aerospace Engineering, University of Bristol, BS8 1TR*
[**] *Engineering Maths, University of Bristol, BS8 1TR*
[***] *Mechanical Engineering, University of Bristol, BS8 1TR*

Abstract: Bifurcation tailoring is a novel control technique aimed at changing the entire bifurcation diagram of a given nonlinear system to some desired one. Bifurcation tailoring was successfully carried out on a second order nonlinear highly manoeuvrable aircraft model so as to control the angle of attack to an arbitrary prescribed bifurcation diagram under the variation of elevator. On-line feedforward scheduling was carried out using a Newton Flow method, and feedback stabilisation was provided by an adaptive control strategy known as the Minimal Control Synthesis (MCS).

Keywords: Nonlinear aircraft model; Scheduled feedforward control; Adaptive feedback control

1. INTRODUCTION

The widespread growth in the use of nonlinear dynamics and bifurcation theory (Strogatz, 1994) has led to a considerable amount of research into the nonlinear analysis of complex flight dynamics. Particularly dynamics at the edge of the flight envelope, such as high angles of attack, spins and departures, have received significant amounts of attention (Thompson *et al.*, 1998; Goman *et al.*, 2001). The use of nonlinear stabilisation and control methods have also received a good deal of interest within aerospace research (Abed and Lee, 1990; Jahnke and Chen, 1995).

Bifurcation tailoring is a novel technique that allows the aircraft dynamicist to control the aircraft throughout its flight regime by altering the system's entire bifurcation diagram. This is achieved by changing the system bifurcation diagram to some given desired one. Bifurcation tailoring has been successfully applied to flight models in an *open loop* sense, i.e. in an entirely scheduled feedforward control guise, where the feedforward signal was created in an off-line continuation program (Lowenberg, 1998). However, in this feedforward only configuration the stability or uniqueness of solution cannot be guaranteed. (Lowenberg and Richardson, 2001) proposed using the bifurcation tailoring to schedule the gains in a feedback controller throughout the flight regime: an improvement over the standard approach of interpolating between several calculated gain values at individual points in the flight regime.

This paper aims to present a process of online bifurcation tailoring where scheduled feedforward control is combined with an adaptive model reference feedback controller known as the Minimal Control Synthesis (MCS) (Stoten and Benchoubane, 1990*a*; Stoten and Benchoubane, 1990*b*). This will ensure the stability and uniqueness *throughout* the desired bifurcation diagram, *and* provide the control designer with an opportunity to control the dynamic response of the aircraft through the eigenvalues of the reference model chosen in the MCS controller.

Section 2 covers some background information; an overview of the bifurcation tailoring theory, an on-line approach to creating the feedforward signal, the MCS controller algorithms, and the Hypothetical High angle of Incidence Research Model (HHIRM) aircraft model used as the application. Section 3 contains the results and discussion for the bifurcation tailoring, and section 4 contains the conclusions and future work.

2. BACKGROUND

A complete background can be found in (Wang *et al.*, 2001; Charles *et al.*, 2001).

2.1 *Bifurcation Tailoring*

Consider a continuous time dynamical system described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, p, \mathbf{q}) \tag{1}$$

where $\mathbf{x} \in \Re^n$ is the state of the system, $p \in \Re$ we assume to be a slow varying system parameter (bifurcation parameter) and $\mathbf{q} \in \Re^m$ is the vector of all the other system parameters. The *bifurcation tailoring* problem is to design a control law $\mathbf{q}$ such that the controlled system has the desired dynamical behaviour as the parameter p varies from p_a *to* p_b.

Consider the bifurcation tailoring problem where the desired objective for the controlled system is to exhibit a branch of equilibria such that, as the parameter p is varied,

$$\mathbf{x}_I = \mathbf{g}(p) \tag{2}$$

where $\mathbf{x} = \begin{bmatrix} \mathbf{x}_I \\ \mathbf{x}_{II} \end{bmatrix}$, $\mathbf{x}_I = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, $\mathbf{x}_{II} = \begin{bmatrix} x_{m+1} \\ \vdots \\ x_n \end{bmatrix}$

Defining the auxiliary vector as:

$$\mathbf{z} = \begin{bmatrix} \mathbf{x}_{II} \\ \mathbf{q} \end{bmatrix} \tag{3}$$

we have that for any given $p \in [p_a, p_b]$ the system must satisfy the equation:

$$\mathbf{f}(\mathbf{g}(p), \mathbf{x}_{II}, p, \mathbf{q}) \equiv \tilde{\mathbf{f}}(\mathbf{g}(p), p, \mathbf{z}) = 0 \tag{4}$$

The Implicit Function Theorem (Glendinning, 1994) states that if the Jacobian of $\mathbf{f}$ w.r.t. $\mathbf{z}$ is invertible, then (4) implicitly defines $\mathbf{z}$ as a function of p i.e.

$$\mathbf{z} = \mathbf{z}_d^*(p) = \begin{bmatrix} \mathbf{x}_{IId}^*(p) \\ \mathbf{q}_d^*(p) \end{bmatrix} \tag{5}$$

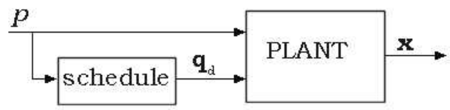


Fig. 1. Block diagram of the feedforward scheduled bifurcation tailoring.

which means that

$$\mathbf{x} = \mathbf{x}_d^*(p) = \begin{bmatrix} \mathbf{x}_{Id}^* \\ \mathbf{x}_{IId}^* \end{bmatrix} \tag{6}$$

is the desired equilibrium point of the feedforward open loop system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, p, \mathbf{q}_d(p)) \tag{7}$$

Hence the desired equilibria defined by (2) is a set of equilibria in the feedforward system (7). Figure 1 shows the block diagram for this technique.

2.2 *Newton Flow*

Typically the schedule, $\mathbf{q}_d$, may not be found analytically, but can be found off-line by 'inverting' the numerical continuation routines in bifurcation analysis programs such as AUTO (Doedel and Wang, 1995). In this paper an on-line Newton Algorithm method was also used:

$$\dot{\mathbf{z}} = -\left[\frac{\partial \mathbf{f}}{\partial \mathbf{z}}\right]^{-1} \tilde{\mathbf{f}}(\mathbf{g}(p(t)), p(t), \mathbf{z}(t)) \tag{8}$$

to solve for $\mathbf{z}$. If $p = p(t)$ varies sufficiently slowly, then $\mathbf{z}(t)$ can be found on-line using (8), i.e. $\mathbf{z}(t) = \mathbf{z}_d^*$.

2.3 *Feedforward Bifurcation Tailoring Limitations*

(1) The Newton Flow equation (8) requires a *full, accurate* mathematical model to create the correct $\mathbf{q}_d^*$. Note: this is also true of the off-line continuation methods (AUTO).
(2) Undesired equilibria may be created in addition to the desired equilibria.
(3) The stability of the equilibria is not assured.
(4) The equilibrium $\mathbf{x}_d^*$ may not exist for some value of $p = p_e$, in which case there is no solution for $\mathbf{q}_d^*(p_e)$.

Problems 2 and 3 suggest that some sort of feedback mechanism would be beneficial. This would also overcome some inaccuracy in the mathematical model used to create the feedforward signal (problem 1). Problem 4 of course is a problem for *any* controller using the same control input, $\mathbf{q}$, and may be seen as, for example, control actuator saturation. The technique of bifurcation tailoring can, in fact, bring these limitations to the attention of the control designer very early in the design process.

2.4 *Feedback Stabilisation*

For the bifurcation tailoring applications in this paper an adaptive model reference controller known as Minimal Control Synthesis (MCS) was used. This is an appropriate solution as the controller automatically tunes the gains as the plant changes throughout the range of p. The control input is given by:

$$\mathbf{q}(t) = \mathbf{q}_d(t) + \Delta\mathbf{q}(t) \tag{9}$$

where $\mathbf{q}_d(t)$ is the schedule from the Newton Flow equation, $\Delta\mathbf{q}(t)$ is the stabilisation control from the MCS equations:

$$\begin{aligned}
\Delta\mathbf{q}_d(t) &= K(t)\mathbf{x}(t) + K_R(t)\mathbf{x}_d(t) \\
K(t) &= \alpha_M \int \mathbf{y}_e(\tau)\mathbf{x}^T(\tau)d\tau + \beta_M \mathbf{y}_e(t)\mathbf{x}^T(t) \\
K_R(t) &= \alpha_M \int \mathbf{y}_e(\tau)\mathbf{x}_d^T(\tau)d\tau + \beta_M \mathbf{y}_e(t)\mathbf{x}_d^T(t) \\
\mathbf{y}_e(t) &= C_e\mathbf{x}_e(t) = C_e(\mathbf{x}_m - \mathbf{x})
\end{aligned}$$

where $\alpha_M \in \Re$, $\beta_M \in \Re$ and $C_e \in \Re^{m,n}$ are constants,

and where the linear reference model is:

$$\dot{\mathbf{x}}_m = A_m\mathbf{x}_m + B_m\mathbf{x}_d \tag{10}$$

The MCS controller is so useful in this application as not only can the stability of the desired solution be assured (Stoten and Benchoubane, 1990*a*; Stoten and Benchoubane, 1990*b*), but the dynamic response of the aircraft can be controlled in the region around the desired solution through the linear reference model (Landau, 1979). If we set $B_m = -A_m$ we ensure that $x_m = x_d$ at equilibrium. Bearing this in mind, inspection of the MCS equations reveals that, providing the eigenvalues of A_m have negative real parts, at equilibrium $x = x_d$. The MCS controller will ensure that the system tracks the reference model in the region around the equilibrium value, allowing the control designer to place the eigenvalues of the system throughout the range of p. Figure 2 shows the block diagram for the feedforward plus MCS feedback stabilised control.

2.5 *Aircraft Model*

The aircraft model used in this report is a highly manoeuvrable non-linear model called the Hypothetical High angle of Incidence Research Model

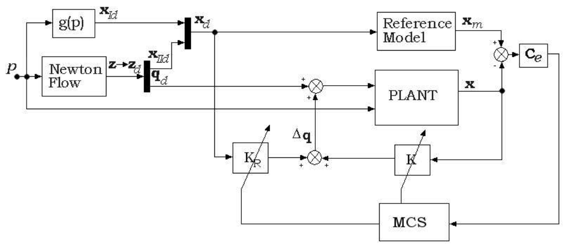


Fig. 2. Block diagram for the feedforward, plus MCS feedback stabilised, scheduled HHIRM model.

(HHIRM) (Goman *et al.*, 1995), provided by QinetiQ. For this paper the model is used in a second order form that describes the fast dynamics of the aircraft (Etkin and Reid, 1996):

$$\begin{cases} \dot{\alpha} = q + \dfrac{Z - X}{mV_T} + \dfrac{g\cos(\alpha)}{V_T} \\ \dot{q} = \dfrac{M}{I_y} \end{cases} \tag{11}$$

where α is the angle of attack, q is the pitch rate, $Z = f_Z(\alpha, \delta_{el}, \delta_{tp})$ is the force in the z direction, $X = f_X(\alpha, \delta_{el}, \delta_{tp})$ is the thrust in the x direction, $M = f_M(\alpha, \delta_{el}, \delta_{tp})$ is the pitching moment, δ_{el} is the elevator angle, δ_{tp} is the thrust vectoring angle in the pitching sense; m is the mass (constant), V_T is the airspeed (constant), g is the acceleration due to gravity (constant), I_y is the moment of inertia about the pitching axis (constant). We can write (11) more generally as

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \mathbf{f}(\alpha, q, \delta_{el}, \delta_{tp}) \tag{12}$$

This paper contains the results for bifurcation tailoring applied to the HHIRM model using δ_{el} as the bifurcation parameter and δ_{tp} as the control input. Since there is only one control input ($\mathbf{q} = \delta_{tp}$), i.e. $m = 1$, there can only be one pre-defined desired state in $\mathbf{x}_I$ in equation (2).

3. BIFURCATION TAILORING APPLIED TO THE HHIRM MODEL

Figure 3 shows the original bifurcation plot for α vs δ_{el} for the HHIRM model. Figure 4 shows the HHIRM response under the same conditions ($\delta_{tp} = 0$). When a gradual decrease in the elevator angle from zero to $-25°$ was applied to the HHIRM model simulation the angle of attack gradually increased up to $-19.5°$ ($\alpha = 0.59\,\text{rad} = 33.8°$) where there was a catastrophic 'drop' from the edge of the fold in figure 3 up to the higher α branch.

Figure 5 shows the desired bifurcation diagram for α. This arbitrary shape was chosen because of its simplicity and for its smooth qualities, hence it

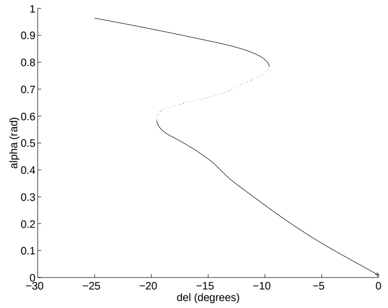


Fig. 3. Bifurcation plot for the HHIRM with $\delta_{tp} = 0$ (created in AUTO). Solid lines = stable, dotted lines = unstable.

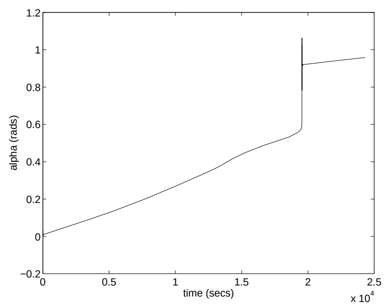


Fig. 4. Simulation of the HHIRM model with $\delta_{tp} = 0$ and $\delta_{el} = -0.001°/\text{sec}$

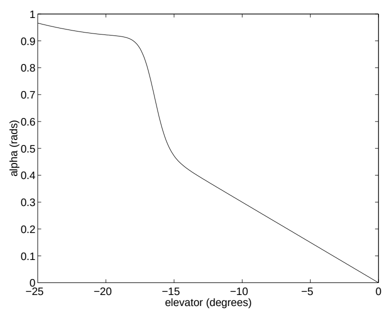


Fig. 5. Desired bifurcation function for the HHIRM example.

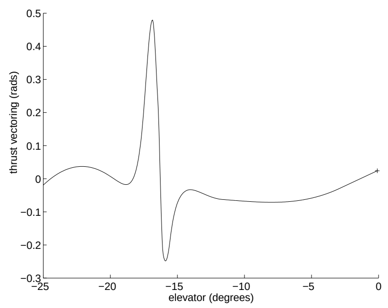


Fig. 6. The ideal schedule that when applied to the HHIRM model results in the desired equilibria (figure 5)

can easily be applied to a bifurcation continuation program or numerical simulation. Figure 6 shows the ideal schedule for δ_{tp} created via AUTO, that when applied to the HHIRM model ensured that the desired equilibria existed.

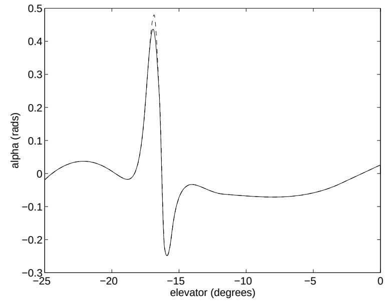


Fig. 7. Schedule produced by the Newton Flow algorithm. Dashed shows the ideal schedule created by AUTO. $\delta_{el} = -0.001°/\text{sec}$

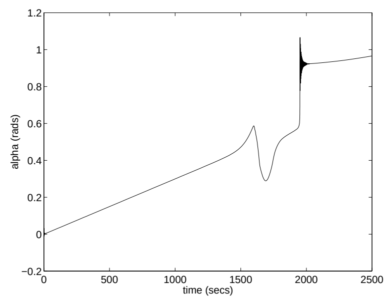


Fig. 8. Simulation of the HHIRM model using the Newton Flow feedforward schedule (alone). $\delta_{el} = -0.001°/\text{sec}$

3.1 *Feedforward scheduling using the Newton Flow method*

The Newton Flow (NF) algorithm as laid out in equation 8 was used to supply the feed forward thrust vectoring to the HHIRM model. Figure 7 shows the schedule that was produced by the NF equations. It can be noted that the NF schedule is not quite the same as the (dashed) ideal schedule that was produced using AUTO. The difference can be accounted for by the time varying nature of the NF method. The NF equations solve for the schedule in real time whereas the off line numerical method solves for the precise equilibrium value in parameter space. The NF equations therefore require a finite settling time to reach a steady state solution. In parts of the schedule where the thrust vectoring is varying quickly over a relatively short change in the elevator the NF will not converge to the ideal schedule quickly enough.

Figure 8 shows the feedforward response of the HHIRM model when the NF schedule (figure 7) was used for the thrust vectoring. Figure 9 shows the AUTO bifurcation plot for the feedforward scheduled HHIRM using the NF equations. Note that because AUTO performs continuation on the equilibria, the bifurcation plot shown in figure 9 is that with the equilibrium values of the NF schedule, i.e no different to the ideal schedule. Inspection of figures 8 and 9 shows that the HHIRM response follows that implied by the bi-

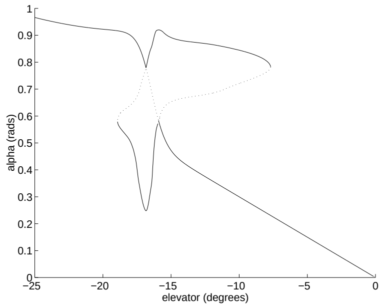


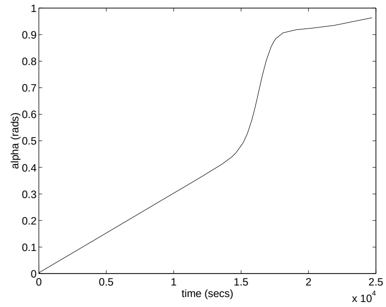
Fig. 9. Bifurcation plot created using AUTO for the feed forward scheduled HHIRM model.

furcation diagram. The stable branch is followed until around $-19.5°$ where the system 'dropped' onto the high α branch. The bifurcation diagram indicates that for the equilibrium state the NF equations provide the correct schedule to create the desired equilibria. However, as stated in section 2.3, the desired equilibria are not necessarily stable or unique throughout the prescribed range of δ_{el}.

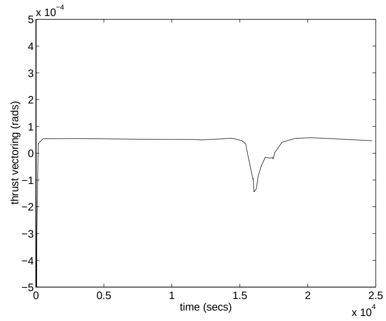
3.2 *Feedback Stabilisation*

The feedback stabilisation was provided by the MCS algorithm (see section 2.4) in order to stabilise the desired α vs δ_{el} equilibria in the $\delta_{el} = -16°$ to $\delta_{el} = -17°$ region (see figures 8 and 9). Figure 10a shows the response of the HHIRM model after the addition of the feedback stabilisation. The small control effort required by the MCS controller can be seen in figure 10b. The entire range of elevator is now stable and produces a unique (desired) equilibrium. As the δ_{el} lowers (the aircraft moves in a nose up sense), the response is now to move in a smooth path from the 'lower' branch to the 'upper' branch, instead of the abrupt change in α seen in figures 4 or 8. Moreover, it can be noted that the reference model in the MCS algorithm (equation 10) allows the control designer to control some aspects of the response of the aircraft away from the equilibria. In effect the eigenvalues of the controlled system are set via the reference model over the desired bifurcation diagram equilibria.

The purpose of the MCS stabilisation is also to ensure the correct bifurcation branch with the addition of unknowns in the system. This could be in the form of noise on the output signals, but in this case was achieved by including some variation in the pitching moment coefficient over the range of α. Figure 11a shows that the desired response is still achieved under these conditions using the feedforward plus MCS control. It must be noted that the feedforward control alone would not achieve the desired equilibria in this case, since the feedforward schedule was created using the model with no variations. This is in contrast to

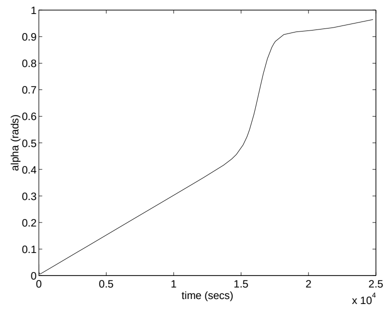


(a) α

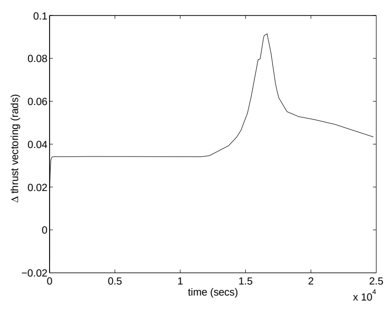


(b) $\Delta \mathbf{q}$

Fig. 10. Simulation of the HHIRM with feedforward scheduling and MCS feedback stabilisation. $\delta_{el} = -0.001°/\text{sec}$



(a) α



(b) $\Delta \mathbf{q}$

Fig. 11. Simulation of the HHIRM (including variations from the 'normal' model) with feedforward scheduling and MCS feedback stabilisation. $\delta_{el} = -0.001°/\text{sec}$

section 3.1 (where the model has no variation) in which the desired equilibria is achieved, but the stability is not addressed. The difference between these two situations is indicated in figure 11b by the additional effort that the MCS controller has to put in, throughout the range of δ_{el}, in order to counter the variations in the model.

4. CONCLUSIONS

The practical uses of bifurcation analysis and nonlinear control in the area of aerospace is already well known and increasing in popularity. The novel method of control of nonlinear systems given by the bifurcation tailoring technique allows the control designer to entirely change the bifurcation diagram of the controlled system. Using the Newton Flow method avoids the need to use the cumbersome continuation packages, although some understanding of the underlying dynamics of the system is recommended in order to choose the desired bifurcation diagram. The addition of the adaptive feedback controller in the guise of the MCS completes the strategy in terms of stability and uniqueness of solution and gives the designer the chance to control the response of the system around the desired equilibria by selecting the reference model in the MCS equations.

In the particular highly manoeuvrable aircraft model used in this paper we have successfully removed the two folds and unstable region in the bifurcation diagram and replaced it with a smooth unique set of equilibria by applying bifurcation tailoring as described above. Although a purely arbitrary bifurcation diagram was selected to demonstrate this process, the power of bifurcation tailoring is obvious. The authors hope to explore further the applications of bifurcation tailoring to the HHIRM aircraft model. Further work will in particular concentrate on practical considerations such as noise rejection, actuator dynamics or computational requirements and the use of higher order asymmetric incarnations of the model.

ACKNOWLEDGEMENTS

The authors would like to thank the EPSRC (grant number GR/M97923) for their support. The support of QinetiQ in providing the HHIRM flight model is also gratefully acknowledged.

5. REFERENCES

Abed, E.H. and H.C. Lee (1990). Nonlinear stabilisation of high angle-of-attack flight dynamics using bifurcation control. number TR 90-32. American Control Conference.

Charles, G., M. di Bernardo, M.H. Lowenberg, D.P. Stoten and X.F. Wang (2001). Bifurcation tailoring of equilibria: a feedback control approach. *Latin American Applied Research* **31**(3), 199–210.

Doedel, E.J and X.J. Wang (1995). Auto94: Software for continuation and bifurcation problems in ordinary differential equations. technical report. Technical Report CRPC-95-2. California Institute of Technology.

Etkin, B. and L.D. Reid (1996). *Dynamics of Flight: Stability and Control.* John Wiley and Sons, Inc.

Glendinning, Paul (1994). *Stability,instability and chaos: an introduction to the theory of nonlinear differential equations.* Cambridge University press.

Goman, M., A. Khramtsovsky and M. Shapiro (2001). Aerodynamics modeling and dynamics simulation at high angles of attack. Technical report. TsAGI.

Goman, M.G., A.V. Khramtsovsky and S. Usoltev (1995). High incidence aerodynamics model for hypothetical aircraft. Technical Report 15/5dra. TsAGI.

Jahnke, C.C. and G. Chen (1995). Nonlinear stability analysis of aircraft steady states. number AIAA-953449. AIAA.

Landau, Y.D. (1979). *Adaptive Control: The Model Reference Approach.* Marcel Dekker.

Lowenberg, M.H. (1998). Development of control strategies to modify spin behaviour. number AIAA-98-4267. AIAA Atmospheric Flight Mechanics Conference.

Lowenberg, M.H. and T.S. Richardson (2001). The continuation design framework for nonlinear aircraft control. number AIAA-2001-4100. AIAA Guidance, Navigation and Control Conference.

Stoten, D.P. and H. Benchoubane (1990*a*). Empirical studies of an mrac algorithm with minimal control synthesis. *International Journal of Control* (51), 823–849.

Stoten, D.P. and H. Benchoubane (1990*b*). Robustness of a minimal control synthesis algorithm. *International Journal of Control* (51), 851–861.

Strogatz, S.H. (1994). *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry and Engineering.* Addison-Wesley.

Thompson, J.M.T et al. (1998). Nonlinear flight dynamcis of high-performance aircraft. *Philosophical Transcriptions of the Royal Society of London* **356**, 2163–2333. A theme compiled and edited by J.M.T. Thompson and F.B.J. Macmillen.

Wang, X.F., M. di Bernardo, M.H. Lowenberg, D.P. Stoten and G.A.Charles (2001). Bifurcation tailoring of nonlinear sytems. submitted to IFAC 2002.