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FORMULAE FOR DISCRETE $\mathcal{H}_\infty$ LOOP SHAPING DESIGN PROCEDURE CONTROLLERS

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Abstract: This paper briefly introduces the $\mathcal{H}_\infty$ loop shaping design procedure (LSDP) in the discrete-time case. Solution formulae are explicitly presented with the exposure of a relationship between the solutions to the three discrete-time, algebraic Riccati equations (DARE) required in the construction of an LSDP controller.

Keywords: Robust control systems design, Discrete-time $\mathcal{H}_\infty$ loop shaping design procedure, Normalized coprime factor perturbation

1. INTRODUCTION

The $\mathcal{H}_\infty$ loop shaping design procedure (LSDP) (MaFarlane and Glover, 1990; MaFarlane and Glover, 1992) is one of many $\mathcal{H}_\infty$ optimization design methods. The $\mathcal{H}_\infty$ LSDP method removes the restrictions on the number of right-half plane poles and produces no pole-zero cancellations between the nominal model and controller designed. Furthermore, the $\mathcal{H}_\infty$ loop shaping design procedure inherits classical loop shaping design ideas so that practising control engineers would feel more comfortable to use it.
The main purpose of this paper is to present the formulae for the discrete-time $\mathcal{H}_\infty$ LSDP controllers, which have not been explicitly available in the literature. Also, a theorem will be introduced which reveals a relation between the solutions to the three discrete-time, algebraic Riccati equations appearing in the solution procedure. This results implies that only two of such equations need be solved, which obviously is of great advantages in terms of computational effort and time in real designs and consequently increases the computational reliability of the resultant controller.

2. NORMALIZED COPRIME FACTORIZATION OF DISCRETE-TIME PLANT

Let $G(z)$ be a minimal realization, discrete-time model of a plant,

$$\begin{aligned} G(z) &= D + C(zI - A)^{-1}B \\ &\overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{aligned} \tag{2.1}$$

with $A : n \times n$, $B : n \times m$, $C : p \times n$, and $D : p \times m$. Matrices $(\tilde{M}(z),\ \tilde{N}(z)) \in \mathcal{H}_\infty^+$, where $\mathcal{H}_\infty^+$ denotes the space of functions with all poles in the open unit disc of the complex plane, constitute a left coprime factorization of $G(z) = \tilde{M}^{-1}\tilde{N}$ if and only if there exists $(\tilde{V},\ \tilde{U}) \in \mathcal{H}_\infty^+$ such that

$$\tilde{M}\tilde{V} \ + \ \tilde{N}\tilde{U} \ = \ I_p \tag{2.2}$$

A left coprime factorization of G is *normalized* if and only if

$$\tilde{N}(z)\tilde{N}^T(\frac{1}{z}) \; + \; \tilde{M}(z)\tilde{M}^T(\frac{1}{z}) \; = \; I_p. \quad (2.3)$$

The concept of right coprime factorization and normalized right coprime factorization can be introduced dually. However, the work presented in the paper will follow the (normalized) left coprime factorization, although all results concerning the (normalized) right coprime factorization can be derived similarly.

State-space constructions for the normalized coprime factorizations can be obtained in terms of the solutions to the following two discrete algebraic Riccati equations (DAREs),

$$\begin{aligned} \Phi^T P \Phi - P - \Phi^T P B Z_1 {Z_1}^T B^T P \Phi \\ + C^T {R_1}^{-1} C = 0 \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} \Phi Q \Phi^T - Q - \Phi Q C^T Z_2^T Z_2 C Q \Phi^T \\ + B {R_2}^{-1} B^T = 0 \end{aligned} \quad (2.5)$$

where $R_1 = I_p + DD^T$, $R_2 = I_m + D^T D$, $\Phi = A - B{R_2}^{-1}D^T C$, $Z_1 {Z_1}^T = (R_2 + B^T P B)^{-1}$, ${Z_2}^T Z_2 = (R_1 + CQC^T)^{-1}$. And, $P \geq 0$ and $Q \geq 0$ are the unique stabilizing solutions, respectively.

Without loss of generality, we may assume that both Z_1 and Z_2 are square matrices, and $Z_1 = {Z_1}^T$, $Z_2 = {Z_2}^T$.

Further, define $H = -(AQC^T + BD^T){Z_2}^T Z_2$, and $F = -Z_1 {Z_1}^T (B^T P A + D^T C)$ then

$$\left[\, \tilde{N} \;\; \tilde{M} \,\right] \stackrel{s}{=} \left[\begin{array}{c|cc} A + HC & B + HD & H \\ \hline Z_2 C & Z_2 D & Z_2 \end{array}\right] \quad (2.6)$$

and

$$\begin{bmatrix} N \\ M \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} A + BF & BZ_1 \\ \hline C + DF & DZ_1 \\ F & Z_1 \end{array}\right] \quad (2.7)$$

are the normalizaed left, and right, coprime factorizations of G, correspondingly.

3. ROBUST CONTROLLER FORMULAE

Same as in the continuous-time case (MaFarlane and Glover, 1990; MaFarlane and Glover, 1992), the discrete-time $\mathcal{H}_\infty$ loop shaping design procedure is based on the construction of a robust stabilizing controller against the perturbations on the coprime factors, as depicted in Figure 1.

In the practical design using the $\mathcal{H}_\infty$ LSDP, the $(\tilde{M}, \tilde{N})$ in Figure 1 is the normalized left coprime facorization of an augmented system, the normal model with pre- and/or post- loop shaping weighting functions.

To maximize the robust stability margin of the closed-loop system given in Fig. 1, one must minimize

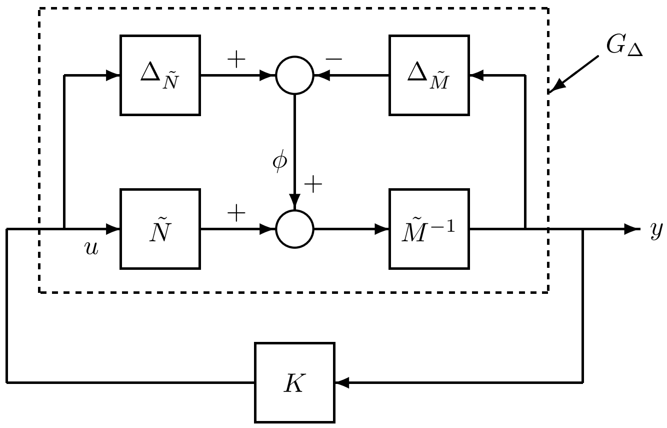

Fig. 1. Robust stabilization with regard to coprime factor uncertainty.

$$\gamma := \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \tilde{M}^{-1} \right\|_\infty$$

Thus, the lowest achievable value of γ for all stabilizing controllers K is

$$\gamma_o = \inf_{K \; stabilizing} \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \tilde{M}^{-1} \right\|_\infty \quad (3.1)$$

and is given in by

$$\gamma_o = (1 + \lambda_{max}(QP))^{1/2} \quad (3.2)$$

where Q and P are the solutions to (2.5) and (2.4), respectively.

For a given $\gamma > \gamma_o$, a sub-optimal $\mathcal{H}_\infty$ LSDP controller K is that K internally stabilizes the nominal system in Fig. 1 and achieves

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \tilde{M}^{-1} \right\|_\infty < \gamma$$

The synthesis of a sub-optimal $\mathcal{H}_\infty$ LSDP controller can be recast as a standard $\mathcal{H}_\infty$ sub-optimal control problem, which has been discussed in (MaFarlane and Glover, 1990; MaFarlane and Glover, 1992). The generalized (interconnected) system in this case is

$$\begin{aligned} \tilde{P}(z) &= \left[\begin{array}{c|c} 0 & I_m \\ \tilde{M}^{-1} & G \\ \hline \tilde{M}^{-1} & G \end{array}\right] \\ &= \left[\begin{array}{c|cc} A & -H{Z_2}^{-1} & B \\ \hline 0 & 0 & I_m \\ C & {Z_2}^{-1} & D \\ C & {Z_2}^{-1} & D \end{array}\right] \end{aligned} \quad (3.3)$$

Following the solution procedure given in (Green and Limebeer, 1995), one more DARE need be solved in order to compute the required controller. In general $\mathcal{H}_\infty$ sub-optimal problems, two more algebraic Riccati equations are to be solved. Here, however, due to the structure of $\tilde{P}(z)$ in (3.3), it can be shown that the solution to one of the

DARE is always zero. The third DARE is the following

$$A^T X_\infty A - X_\infty - \tilde{F}^T (R + \begin{bmatrix} -Z_2{}^{-1} H^T \\ R_2{}^{-1/2} B^T \end{bmatrix} X_\infty \begin{bmatrix} -H Z_2{}^{-1} & B R_2{}^{-1/2} \end{bmatrix}) \tilde{F} + C^T C = 0 \quad (3.4)$$

where

$$\tilde{F} = -(R + \begin{bmatrix} -Z_2{}^{-1} H^T \\ R_2{}^{-1/2} B^T \end{bmatrix} X_\infty \begin{bmatrix} -H Z_2{}^{-1} & B R_2{}^{-1/2} \end{bmatrix})^{-1} (\begin{bmatrix} -Z_2{}^{-1} C \\ D^T R_1{}^{-1/2} C \end{bmatrix} + \begin{bmatrix} -Z_2{}^{-1} H^T \\ R_2{}^{-1/2} B^T \end{bmatrix} X_\infty A)$$

and

$$R = \begin{bmatrix} Z_2{}^{-2} - \gamma^2 I_p & Z_2{}^{-1} R_1{}^{-1/2} D \\ D^T R_1{}^{-1/2} Z_2{}^{-1} & I_m \end{bmatrix}$$

Further, by defining $\tilde{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$, where, $F_1 : p \times n$ and $F_2 : m \times n$, the sub-optimal $\mathcal{H}_\infty$ discrete-time LSDP controller K can be constructed as

$$K(z) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

where

$$\begin{aligned} A_K &= \hat{A}_K - \hat{B}_K D (I + \hat{D}_K D)^{-1} \hat{C}_K \\ B_K &= \hat{B}_K (I + D \hat{D}_K)^{-1} \\ C_K &= (I + \hat{D}_K D)^{-1} \hat{C}_K \\ D_K &= \hat{D}_K (I + D \hat{D}_K)^{-1} \end{aligned} \quad (3.5)$$

with

$$\begin{aligned} \hat{D}_K &= -(R_2 + B^T X_\infty B)^{-1} (D^T - B^T X_\infty H) \\ \hat{B}_K &= -H + B \hat{D}_K \\ \hat{C}_K &= R_2^{-1/2} F_2 - \hat{D}_K (C + Z_2{}^{-1} F_1) \\ \hat{A}_K &= A + HC + B \hat{C}_K \end{aligned} \quad (3.6)$$

4. THE STRICTLY PROPER CASE

It may be appropriate to say that most plants considered in the practical, discrete-time control systems design are strictly proper, i.e. $D = 0$. When the plant under consideration is strictly proper, all the computations and formulae described above will be significantly simpler. The two DAREs (2.4) and (2.5) become

$$A^T P A - P - A^T P B Z_1 Z_1{}^T B^T P A + C^T C = 0 \quad (4.1)$$

and

$$A Q A^T - Q - A Q C^T Z_2^T Z_2 C Q A^T + B B^T = 0 \quad (4.2)$$

where $Z_1 Z_1{}^T = (I_m + B^T P B)^{-1}$, and $Z_2{}^T Z_2 = (I_p + C Q C^T)^{-1}$.

The third DARE (3.4) is now the following

$$A^T X_\infty A - X_\infty - \tilde{F}^T (R + \begin{bmatrix} -Z_2{}^{-1} H^T \\ B^T \end{bmatrix} X_\infty \begin{bmatrix} -H Z_2{}^{-1} & B \end{bmatrix}) \tilde{F} + C^T C = 0 \quad (4.3)$$

where

$$\tilde{F} = -(R + \begin{bmatrix} -Z_2{}^{-1} H^T \\ B^T \end{bmatrix} X_\infty \begin{bmatrix} -H Z_2{}^{-1} & B \end{bmatrix})^{-1} (\begin{bmatrix} -Z_2{}^{-1} C \\ 0 \end{bmatrix} + \begin{bmatrix} -Z_2{}^{-1} H^T \\ B^T \end{bmatrix} X_\infty A)$$

and

$$\begin{aligned} R &= \begin{bmatrix} Z_2{}^{-2} - \gamma^2 I_p & 0 \\ 0 & I_m \end{bmatrix} \\ H &= -A Q C^T Z_2{}^T Z_2. \end{aligned}$$

Further, by defining $\tilde{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$, where, $F_1 : p \times n$ and $F_2 : m \times n$, the sub-optimal $\mathcal{H}_\infty$ discrete-time LSDP controller K in the case of a strictly proper G can be constructed as

$$K(z) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

where

$$\begin{aligned} D_K &= (I_m + B^T X_\infty B)^{-1} B^T X_\infty H \\ B_K &= -H + B D_K \\ C_K &= F_2 - D_K (C + Z_2{}^{-1} F_1) \\ A_K &= A + HC + B C_K \end{aligned} \quad (4.4)$$

5. ON THE THREE DARE SOLUTIONS

As discussed above, the discrete-time $\mathcal{H}_\infty$ LSDP sub-optimal controller formulae require the solutions to the three discrete-time algebraic Riccati equations, (2.4), (2.5) and (3.4), or (4.1), (4.2) and (4.3) in the strictly proper case. In this section, we will reveal that there is a relation between those three solutions, namely the solution X_∞ to the third DARE can be calculated directly from the first two solutions P and Q. This fact is important and useful, especially in the numerical implementation of the discrete-time LSDP routines.

We start with a general DARE, hence the notations are not related to those defined earlier in the paper,

$$F^T X F - X - F^T X G_1 (G_2 + G_1^T X G_1)^{-1} G_1^T X F + C^T C = 0 \quad (5.1)$$

where $F,\ H,\ X \in \mathcal{R}^{n \times n}$, $G_1 \in \mathcal{R}^{n \times m}$, $G_2 \in \mathcal{R}^{m \times m}$, and $G_2 = G_2^T > 0$. We assume that $(F,\ G_1)$ is a stabilizable pair and that $(F,\ C)$ a detectable pair. We also define $G = G_1 G_2^{-1} G_1^T$.

It is well known that, see (Pappas *et al.*, 1980) or others, solutions to DARE (5.1) are closely linked with a matrix pencil pair $(M,\ L)$, where

$$\begin{aligned} M &= \begin{bmatrix} F & 0 \\ -H & I \end{bmatrix} \\ L &= \begin{bmatrix} I & G \\ 0 & F^T \end{bmatrix} \end{aligned} \tag{5.2}$$

It also can be shown that if there exist $n \times n$ matrices S, U_1 and U_2, with U_1 invertible, such that

$$M \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = L \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} S \tag{5.3}$$

then, $X = U_2 U_1^{-1}$ is a solution to (5.1). Further, the matrix $F - G_1(G_2 + G_1^T X G_1)^{-1} G_1^T X F$ shares the same spectrum as S. Hence, if S is stable, i.e. all the eigenvalues are within the open unit disc, $F - G_1(G_2 + G_1^T X G_1)^{-1} G_1^T X F$ is also stable. Such an X is non-negative definite and unique, and is called the stabilizing solution to (5.1).

Under the above assumptions on (5.1), it was shown in (Pappas *et al.*, 1980) that none of the generalized eigenvalues of the pair $(M,\ L)$ lies on the unit circle, and if $\lambda \neq 0$ is a generalized eigenvalue of the pair, then $1/\lambda$ is also a generalized eigenvalue of the same multiplicity. In other words, the stable spectrum, consisting of n generalized eigenvalues lying in the open unit disc, is unique. Therefore, if there exists another triple $(V_1,\ V_2,\ T)$ satisfying (5.3), with V_1 being invertible and T stable, then there must exist an invertible R such that $T = R^{-1} S R$. Consequently,

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} R^{-1} \tag{5.4}$$

The solution of course remains the same, since $X = V_2 V_1^{-1} = (U_2 R)(U_1 R)^{-1} = U_2 U_1^{-1}$.

In our present study, we can accordingly define the three matrix pencils as

$$\begin{aligned} M_P &= \begin{bmatrix} \Phi & 0 \\ -C^T R_1^{-1} C & I \end{bmatrix} \\ L_P &= \begin{bmatrix} I & B R_2^{-1} B^T \\ 0 & \Phi^T \end{bmatrix} \end{aligned} \tag{5.5}$$

$$\begin{aligned} M_Q &= \begin{bmatrix} \Phi^T & 0 \\ -B R_2^{-1} B^T & I \end{bmatrix} \\ L_Q &= \begin{bmatrix} I & C^T R_1^{-1} C \\ 0 & \Phi \end{bmatrix} \end{aligned} \tag{5.6}$$

$$M_X = \begin{bmatrix} A - \begin{bmatrix} -H Z_2^{-1} & B R_2^{-1/2} \end{bmatrix} R^{-1} \begin{bmatrix} Z_2^{-1} C \\ D^T R_1^{-1/2} C \end{bmatrix} & 0 \\ -C^T C + \begin{bmatrix} C^T Z_2^{-1} & C^T R_1^{-1/2} D \end{bmatrix} R^{-1} \begin{bmatrix} Z_2^{-1} C \\ D^T R_1^{-1/2} C \end{bmatrix} & I \end{bmatrix}$$

$$L_X = \begin{bmatrix} I & \begin{bmatrix} -H Z_2^{-1} & B R_2^{-1/2} \end{bmatrix} R^{-1} \begin{bmatrix} -Z_2^{-1} H^T \\ R_2^{-1/2} B^T \end{bmatrix} \\ 0 & A^T - \begin{bmatrix} C^T Z_2^{-1} & C^T R_1^{-1/2} D \end{bmatrix} R^{-1} \begin{bmatrix} -Z_2^{-1} H^T \\ R_2^{-1/2} B^T \end{bmatrix} \end{bmatrix} \tag{5.7}$$

With all the above properties of the DAREs and the notations, we are ready to prove the following theorem.

Theorem 1 *Let P, Q and X_∞ be the stabilizing solutions to the DAREs (2.4), (2.5) and (3.4), (or, (4.1), (4.2) and (4.3) when G is strictly proper), respectively, the following identity holds*

$$\begin{aligned} X_\infty &= P\left[\left(1 - \gamma^{-2}\right) I_n - \gamma^{-2} Q P\right]^{-1} \\ &= \gamma^2 P \left[\gamma^2 I_n - (I_n + QP)\right]^{-1} \end{aligned} \tag{5.8}$$

■

Proof: (See Appendix A.)

The above result is useful with regard to numerical calculations. The DAREs (2.4) and (2.5) are directly based on the original data while the DARE (3.4) is more complex and involved with results from previous computations, and is thus more vulnerable to numerical inaccuracy.

Similar work was reported in (Walker, 1990), where the relationship between three discrete-time algebraic Riccati equations arising in the general $\mathcal{H}_\infty$ sub-optimal design was discussed. The result revealed here is explicitly concerning the discrete-time loop shaping design procedure, with three different DAREs and a slightly different conclusion.

6. CONCLUSIONS

The discrete-time $\mathcal{H}_\infty$ loop shaping design procedure has been summarised in this paper. The solution formulae included those for the general case as well as those for the case of a strictly proper plant model. A result on the relationship between the three discrete-time algebraic Riccati equation solutions required was presented, which would be useful in making the computation more efficient and more reliable. The formulae have been recently implemented in the control software package *SLICOT* (http://www.win.tue.nl/niconet/NIC2/slicot.html) with focus particularly on numerical efficiency and reliability.

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APPENDIX A. THE PROOF OF THEOREM 1

Assume that $P = P_2 P_1^{-1}$ and $X_\infty = X_2 X_1^{-1}$, satisfying

$$M_P \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = L_P \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} S_P \tag{6.1}$$

and

$$M_X \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = L_X \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} S_X \tag{6.2}$$

with S_P and S_X being stable.

The main idea in the proof is to find an invertible matrix W such that

$$M_X \begin{bmatrix} -\gamma^{-2}(1-\gamma^2)I_n & -\gamma^{-2}Q \\ 0 & I \end{bmatrix} = W M_P \tag{6.3}$$

and

$$L_X \begin{bmatrix} -\gamma^{-2}(1-\gamma^2)I_n & -\gamma^{-2}Q \\ 0 & I \end{bmatrix} = W L_P \tag{6.4}$$

If such a W exists, then by pre-multiplying W on the both side of (6.1), we have

$$\begin{aligned} & M_X \begin{bmatrix} -\gamma^{-2}(1-\gamma^2)I_n & -\gamma^{-2}Q \\ 0 & I \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \\ &= L_X \begin{bmatrix} -\gamma^{-2}(1-\gamma^2)I_n & -\gamma^{-2}Q \\ 0 & I \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} S_P \end{aligned} \tag{6.5}$$

By comparing (6.5) with (6.2), and from the properties of the stabilizing solution to a DARE discussed in Section 5, it can be deducted that

$$\begin{aligned} X_\infty &= \left(\begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \right) \left(\begin{bmatrix} -\gamma^{-2}(1-\gamma^2)I_n & -\gamma^{-2}Q \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \right)^{-1} \\ &= P_2 \left[-\gamma^{-2}(1-\gamma^2)P_1 - \gamma^{-2}QP_2 \right]^{-1} \\ &= P_2 P_1^{-1} \left[(1-\gamma^{-2})I_n - \gamma^{-2}QP_2P_1^{-1} \right]^{-1} \\ &= P \left[(1-\gamma^{-2})I_n - \gamma^{-2}QP \right]^{-1} \\ &= \gamma^2 P \left[\gamma^2 I_n - (I_n + QP) \right]^{-1} \end{aligned} \tag{6.6}$$

It can be shown that

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{6.7}$$

where

$$\begin{aligned} W_{11} &= -\gamma^{-2}(1-\gamma^2)I_n \\ W_{21} &= 0 \\ W_{12} &= -\gamma^{-2}(1-\gamma^2)(A - BD^T R_1^{-1} C)Q \\ &\quad \left[(1-\gamma^2)I_n + C^T R_1^{-1} CQ \right]^{-1} \\ W_{22} &= (1-\gamma^2) \left[(1-\gamma^2)I_n + C^T R_1^{-1} CQ \right]^{-1} \end{aligned}$$

satisfies (6.3) and (6.4), by routine matrix manipulations and noticing that

$$\begin{aligned} R^{-1} = & \begin{bmatrix} I_p & 0 \\ -D^T R_1^{-1/2} Z_2^{-1} & I_m \end{bmatrix} \begin{bmatrix} (Z_2^{-1} R_1^{-1} Z_2^{-1} - \gamma^2 I_p) & 0 \\ 0 & I_m \end{bmatrix} \\ & \begin{bmatrix} I_p & -Z_2^{-1} R_1^{-1/2} D \\ 0 & I_m \end{bmatrix} \end{aligned}$$

In the strictly proper case, W is simply

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{6.8}$$

with

$$\begin{aligned} W_{11} &= -\gamma^{-2}(1-\gamma^2)I_n \\ W_{21} &= 0 \\ W_{12} &= -\gamma^{-2}(1-\gamma^2)AQ \left[(1-\gamma^2)I_n + C^T CQ \right]^{-1} \\ W_{22} &= (1-\gamma^2) \left[(1-\gamma^2)I_n + C^T CQ \right]^{-1} \end{aligned}$$

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