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FREQUENCY DOMAIN SOLUTION TO THE DELAY-TYPE NEHARI PROBLEM [1]

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Abstract This paper presents a frequency domain solution to the delay-type Nehari problem. The solvability condition is formulated in terms of nonsingularities of three matrices. The optimal value γ_{opt} is the maximal value such that one of the three matrices becomes singular when γ decreases from $+\infty$ to 0. The all sub-optimal compensators are parameterized in a transparent structure with a modified Smith predictor. The J-spectral factorization of a general para-Hermitian matrix is also given in this paper as a requisite for proof. *Copyright ©2002 IFAC.*

Keywords: H_∞ control, Smith predictor, Nehari problem, time delay systems, Riccati equation, $\mathrm{L}_2[0,h]$-induced norm, J-spectral factorization

1. INTRODUCTION

The H_∞ control of processes with delay(s) has been an active research area since the mid 80's. There are mainly three kinds of methods: operator-theoretic methods (Foias *et al.*, 1996; Dym *et al.*, 1995; Zhou and Khargonekar, 1987), state-space methods (Nagpal and Ravi, 1997; Tadmor, 1997*a*; Tadmor, 2000; Başar and Bernhard, 1995) and frequency-domain methods (Mirkin, 2000; Meinsma and Zwart, 2000). It is known that a large class of H_∞ control problems, including the weighted sensitivity minimization problem, can be reduced to the Nehari problem (Francis, 1987). It is still true in the case with delay(s) and the simplified problem is a delay-type Nehari problem.

There are some papers calculating the infimum of the delay-type Nehari problem in stable case, see for example (Zhou and Khargonekar, 1987; Flamm and Mitter, 1987). It was shown in (Zhou and Khargonekar, 1987) that this problem (in stable case) is equivalent to calculate the $\mathrm{L}_2[0,h]$-induced norm. However, for unstable case, it is much more involved. Tadmor (1997*b*) presented a state space solution to this problem in unstable case, in which a differential/algebraic matrix Riccati equation-based method was used. The optimal value relies on the solution of a differential Riccati equation. The suboptimal solution, of which the structure is not transparent, is very complicated. A more transparent solution is demanded.

Motivated by the idea of Meinsma and Zwart (2000), this paper presents a frequency domain solution to the delay-type Nehari problem. The optimal value γ_{opt} is formulated in a clear way: it is the maximal value such that one of three matrices becomes singular when γ decreases from $+\infty$ to 0. Hence, one need no longer solve a differential Riccati equation any more. A prominent advantage is that the suboptimal solutions hold quite a transparent structure of modified Smith-predictor. With some man-machine interactive operation, it is very easy to find the optimal value using MATLAB.

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Notation

Assume

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is a rational transfer matrix $G(s) = D + C(sI - A)^{-1}B$. Two operators acting on rational transfer matrices, the *truncation* operator τ_h and the *completion* operator π_h, which depend on a parameter $h \geq 0$, are defined as:

$$\begin{aligned} \tau_h\{G\} &\doteq \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] - e^{-sh} \left[\begin{array}{c|c} A & e^{Ah}B \\ \hline C & 0 \end{array}\right] \\ &\doteq G(s) - e^{-sh}\tilde{G}(s), \end{aligned}$$

$$\begin{aligned} \pi_h\{G\} &\doteq \left[\begin{array}{c|c} A & B \\ \hline Ce^{-Ah} & 0 \end{array}\right] - e^{-sh} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \\ &\doteq \hat{G}(s) - e^{-sh}G(s). \end{aligned}$$

This follows (Mirkin, 2000), except for a small adjustment in notation. Note that these two operators map any rational transfer matrix G into an FIR block.

Let $N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$ be a 2×2 block transfer matrix. The following notations are introduced for two linear fractional transformations, which are called *homographic transformations* (*HMT*) in (Kimura, 1996; Delsarte *et al.*, 1979):

$$\mathcal{H}_r(N, Q) = (N_{11}Q + N_{12})(N_{21}Q + N_{22})^{-1},$$

$$\mathcal{H}_l(N, Q) = -(N_{11} - QN_{21})^{-1}(N_{12} - QN_{22}),$$

where the subscript l stands for *left* and r for *right.*

2. PROBLEM STATEMENT AND PRELIMINARY

Delay-type Nehari Problem(NP$_\mathbf{h}$): Given a minimal realization

$$G_\beta \doteq \left[\begin{array}{c|c} A & B \\ \hline -C & 0 \end{array}\right]$$

which is not necessarily stable, characterize the optimal value [3]

$$\gamma_{opt} = \inf\{\left\|G_\beta + e^{-sh}K\right\|_{L_\infty} : \ K(s) \in H_\infty\}$$

and, given $\gamma > \gamma_{opt}$, parameterize the suboptimal set of proper and causal $K(s) \in H_\infty$ such that

$$\left\|G_\beta + e^{-sh}K\right\|_{L_\infty} < \gamma. \qquad (1)$$

It is well-known (Gohberg *et al.*, 1993) that this problem is solvable iff

$$\gamma > \gamma_{opt} \doteq \left\|\Gamma_{e^{sh}G_\beta}\right\|,$$

[3] The argument of a transfer matrix, s, is omitted frequently hereafter for clarity.

where Γ denotes the Hankel operator. Inspecting the transfer matrix $e^{sh}G_\beta$, one can see that γ_{opt} is not less than the $\mathrm{L}_2[0,h]$-induced norm of G_β (Foias *et al.*, 1996; Zhou and Khargonekar, 1987; Gu *et al.*, 1996), *i.e.*,

$$\gamma_{opt} \geq \gamma_h \doteq \|G_\beta\|_{\mathrm{L}_2[0,h]}.$$

Under this condition, the matrix Σ_{22} is always nonsingular (Foias *et al.*, 1996; Zhou and Khargonekar, 1987; Gu *et al.*, 1996), where Σ_{22} is the $(2,2)$-block of

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \doteq e^{Hh}, \qquad (2)$$

which is an exponential function with regard to a Hamiltonian matrix

$$H = \begin{bmatrix} A & \gamma^{-2}BB^* \\ -C^*C & -A^* \end{bmatrix}.$$

As shown later, this exponential Hamiltonian matrix Σ plays quite an important role in the $H_\infty-$control of dead-time systems.

Various methods (Foias *et al.*, 1996; Zhou and Khargonekar, 1987; Gu *et al.*, 1996) have been proposed to compute γ_h. A simple representation is (Zhou and Khargonekar, 1987):

$$\gamma_h = \max\{\gamma : \ \det \Sigma_{22} = 0\}, \qquad (3)$$

i.e. the maximal γ that makes Σ_{22} singular or the maximal root of $\det \Sigma_{22} = 0$.

3. MAIN RESULT

Theorem 1. **(Delay-type Nehari problem)** Given strictly proper G_β has no $j\omega$-axis zero nor $j\omega$-axis pole, there always exist unique solutions $L_c \leq 0$ and $L_o \leq 0$, respectively, for the algebraic Riccati equations

$$\begin{bmatrix} -L_c & I \end{bmatrix} \begin{bmatrix} A & \gamma^{-2}BB^* \\ 0 & -A^* \end{bmatrix} \begin{bmatrix} I \\ L_c \end{bmatrix} = 0$$

$$\begin{bmatrix} I & -L_o \end{bmatrix} \begin{bmatrix} A & 0 \\ -C^*C & -A^* \end{bmatrix} \begin{bmatrix} L_o \\ I \end{bmatrix} = 0$$

such that $A + \gamma^{-2}BB^*L_c$ and $A + L_oC^*C$ are stable [4]. The optimal value γ_{opt} of the delay-type Nehari problem (1) is

$$\gamma_{opt} = \max\{\gamma_h, \ \gamma_1, \ \gamma_2\},$$

where

[4] In MATLAB, in order to obtain a solution $L_o \leq 0$ such that $A + L_oC^*C$ is stable, the second ARE should be equivalently changed as $\begin{bmatrix} -L_o & I \end{bmatrix} \begin{bmatrix} A^* & C^*C \\ 0 & -A \end{bmatrix} \begin{bmatrix} I \\ L_o \end{bmatrix} = 0.$

$$\gamma_h = \max\{\gamma : \det(\begin{bmatrix} 0 & I \end{bmatrix} \Sigma \begin{bmatrix} 0 \\ I \end{bmatrix}) = 0\},$$

$$\gamma_1 = \max\{\gamma : \det(\begin{bmatrix} 0 & I \end{bmatrix} \Sigma \begin{bmatrix} L_o \\ I \end{bmatrix}) = 0\},$$

$$\gamma_2 = \max\{\gamma : \det(\begin{bmatrix} -L_c & I \end{bmatrix} \Sigma \begin{bmatrix} L_o \\ I \end{bmatrix}) = 0\}.$$

Furthermore, if $\gamma > \gamma_{opt}$, then all $K(s) \in H_\infty$ satisfying (1) are parameterized as

$$K(s) = \mathcal{H}_r\left(\begin{bmatrix} I & 0 \\ \Delta(s) & I \end{bmatrix} W^{-1}, Q\right) \tag{4}$$

where [5]

$$\Delta(s) = -\pi_h\{\mathcal{F}_u(\begin{bmatrix} G_\beta & I \\ I & 0 \end{bmatrix}, \gamma^{-2} G_\beta^{\sim})\},$$

$$W^{-1}(s) = \left[\begin{array}{c|cc} A + \gamma^{-2} B B^* L_c & (I - L_{oh} L_c)^{-1} L_{oh} C^* & (I - L_{oh} L_c)^{-1}(L_{oh}\Sigma_{21} - \Sigma_{11})B \\ \hline -C & I & \\ \gamma^{-2} B^*(\Sigma_{21}^* - \Sigma_{11}^* L_c) & & I \end{array}\right]$$

with $L_{oh} = \mathcal{H}_r(\Sigma, L_o)$ and $\|Q(s)\|_{H_\infty} < \gamma$ is a free parameter.

Remark 2. It is clear that $\gamma > \gamma_h$ ensures the nonsingularity of Σ_{22}, that $\gamma > \gamma_1$ ensures the existence of L_{oh} and that $\gamma > \gamma_2$ ensures the existence of the J-spectral factorization and the stability of $K(s)$.

The structure of $K(s)$ is shown in Figure 1. It consists of an infinite-dimensional block $\Delta(s)$, which is an FIR block (modified Smith predictor), and a finite-dimensional block $W^{-1}(s)$. The *right*-upper tag means $W^{-1}(s)$ maps the *right* variables to the left variables while a *left* upper tag, if any, means the matrix maps the *left* variables to the right variables.

In order to prove this theorem, we need a result about the J-spectral (co-)factorization of a general para-Hermitian matrix.

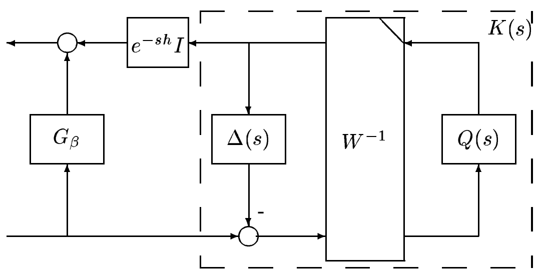

Figure 1. Structure of $K(s)$

4. J-SPECTRAL FACTORIZATION

Assume that the two signature matrices with appropriate dimensions defined as

$$J_\gamma = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \text{ and } J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

hold the same number of negative eigenvalues. Furthermore, assume a given rational transfer matrix

$$\Lambda \doteq \left[\begin{array}{cc|c} A & R & -B \\ -E & -A^* & C^* \\ \hline C & B^* & D^* J_\gamma D \end{array}\right] \tag{5}$$

satisfies the following conditions:

(i) $E = E^*$, $R = R^*$, which means $\Lambda^{\sim}(s) = \Lambda(s)$ (such a matrix is called para-Hermitian matrix (Kwakernaak, 2000));

(ii) Λ has no poles nor zeros on $j\omega$-axis;

(iii) D is nonsingular.

Since, in general, $R \neq 0$ and/or $E \neq C^* J_\gamma C$, it is impossible to directly write Λ in the form $G^{\sim}(s) J_\gamma G(s)$. This makes the J-spectral factorization much more complex. Many researchers have already studied the J-spectral factorization. However, by the knowledge of the authors, they started with $G^{\sim}(s) J_\gamma G(s)$ with certain stable $G(s)$ and only considered the case $\Lambda(s)$ which can be explicitly written in the form of $G^{\sim}(s) J_\gamma G(s)$. In this case, one needs three steps to find the J-spectral factor of a matrix (Meinsma, 1995): firstly, to find the modal factorization; secondly, to construct a stable $G_+(s)$ such that the original matrix is equivalent to $G_+^{\sim}(s) J_\gamma G_+(s)$; thirdly, to derive the J-spectral factor.

Here we show an alternative way to find the J-spectral factor for a general para-Hermitian matrix in one step without modal factorization. Hence, the A-matrix is not split.

Lemma 3. **($J-$spectral factorization)** Assume $\Lambda(s)$ satisfies the above conditions and (E, A) is detectable, then $\Lambda(s)$ has a $J-$spectral factorization if and only if the following two conditions hold:

(i) Two algebraic Riccati equations

$$\begin{bmatrix} I & -\lambda_o \end{bmatrix} \begin{bmatrix} A & R \\ -E & -A^* \end{bmatrix} \begin{bmatrix} \lambda_o \\ I \end{bmatrix} = 0$$

and

$$\begin{bmatrix} -\lambda_c & I \end{bmatrix} (\begin{bmatrix} A & R \\ -E & -A^* \end{bmatrix} - \begin{bmatrix} -B \\ C^* \end{bmatrix} D^{-1} J_\gamma^{-1} D^{-*} \begin{bmatrix} C & B^* \end{bmatrix}) \begin{bmatrix} I \\ \lambda_c \end{bmatrix} = 0$$

have unique symmetric solutions λ_o and λ_c, respectively, such that

$$\begin{bmatrix} I & -\lambda_o \end{bmatrix} \begin{bmatrix} A & R \\ -E & -A^* \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} I & 0 \end{bmatrix} (\begin{bmatrix} A & R \\ -E & -A^* \end{bmatrix} - \begin{bmatrix} -B \\ C^* \end{bmatrix} D^{-1} J_\gamma^{-1} D^{-*} \begin{bmatrix} C & B^* \end{bmatrix}) \begin{bmatrix} I \\ \lambda_c \end{bmatrix}$$

are stable;

(ii) $\det(I - \lambda_c \lambda_o) \neq 0$.

If these conditions hold, then one $J-$spectral factor is represented as

[5] The "0" elements in some matrices hereafter are omitted quite frequently for clarity.

$$W(s) = \begin{bmatrix} I & 0 \\ 0 & \gamma I \end{bmatrix} \left[\begin{array}{c|c} A + \lambda_o E & B + \lambda_o C^* \\ \hline -J_\gamma^{-1} D^{-*}(B^* \lambda_c + C)(I - \lambda_o \lambda_c)^{-1} & D \end{array} \right].$$

Dually, the following lemma holds:

Lemma 4. **($J-$spectral co-factorization)** Assume $\Lambda(s)$ satisfies the above three conditions and (A, R) is stabilizable, then $\Lambda(s)$ has a $J-$spectral co-factorization if and only if the following two conditions hold:

(i) Two algebraic Riccati equations

$$\begin{bmatrix} -\lambda_c & I \end{bmatrix} \begin{bmatrix} A & R \\ -E & -A^* \end{bmatrix} \begin{bmatrix} I \\ \lambda_c \end{bmatrix} = 0$$

and

$$\begin{bmatrix} I & -\lambda_o \end{bmatrix} \left(\begin{bmatrix} A & R \\ -E & -A^* \end{bmatrix} - \begin{bmatrix} -B \\ C^* \end{bmatrix} D^{-1} J_\gamma^{-1} D^{-*} \begin{bmatrix} C & B^* \end{bmatrix} \right) \begin{bmatrix} \lambda_o \\ I \end{bmatrix} = 0$$

have unique symmetric stabilizing solutions λ_c and λ_o, respectively, such that

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A & R \\ -E & -A^* \end{bmatrix} \begin{bmatrix} I \\ \lambda_c \end{bmatrix} = A + R\lambda_c$$

and

$$\begin{bmatrix} I & -\lambda_o \end{bmatrix} \left(\begin{bmatrix} A & R \\ -E & -A^* \end{bmatrix} - \begin{bmatrix} -B \\ C^* \end{bmatrix} D^{-1} J_\gamma^{-1} D^{-*} \begin{bmatrix} C & B^* \end{bmatrix} \right) \begin{bmatrix} I \\ 0 \end{bmatrix}$$

are stable;

(ii) $\det(I - \lambda_o \lambda_c) \neq 0$.

If these conditions hold, then one $J-$spectral co-factor is represented as

$$W(s) = \left[\begin{array}{c|c} A + R\lambda_c & -(I - \lambda_o \lambda_c)^{-1}(B + \lambda_o C^*) D^{-1} J_\gamma^{-1} \\ \hline B^* \lambda_c + C & D^* \end{array} \right] \begin{bmatrix} I & 0 \\ 0 & \gamma I \end{bmatrix}.$$

PROOF. Omitted because of page limitation.

5. THE PROOF OF THEOREM 1

Associate the $\mathbf{NP_h}$ problem (1) with the following system in input-output representation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} G_\beta & e^{-sh} I \\ I & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$u_2 = K z_2$$

or, equivalently, in chain-scattering representation

$$\begin{bmatrix} z_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} e^{-sh} I & G_\beta \\ 0 & I \end{bmatrix} \begin{bmatrix} u_2 \\ z_2 \end{bmatrix} \doteq G(s) \begin{bmatrix} u_2 \\ z_2 \end{bmatrix},$$

then the closed-loop transfer matrix can be re-written as

$$T_{z_1 u_1} = \mathcal{H}_r(G, K) = G_\beta + e^{-sh} K.$$

PROOF. (of Theorem 1) It has already been well known (Kimura, 1996; Meinsma and Zwart, 2000; Green *et al.*, 1990) that the H_∞ control problem

$$\|\mathcal{H}_r(G, K)\|_\infty < \gamma$$

is equivalent to that $G^\sim J_\gamma G$ has a J-spectral factor $V(s)$ such that the $(2, 2)$-block of GV^{-1} is bistable. Hence, in this proof, we characterize the conditions to meet these requirements.

The main idea underlying is to find a unimodular matrix to equivalently rationalize the system and then to find the J-spectral factorization of the rationalized system. This idea was used in (Meinsma and Zwart, 2000) where a 2-block problem was considered but the result was for a stable case and cannot be directly used here because $G(s)$ is not necessarily stable. We borrowed some ideas from there but we use a very basic tool, similarity transformation, to find the realization of the rationalized system. Here, we prefer to keep the A-matrix in the original form and not to split it by modal factorization.

The proof is divided into three steps:

(i) Find a predictor $\Delta(s)$ to equivalently rationalize the system;

(ii) Find the realization of the rationalized system;

(iii) Find the J-spectral factorization of the rationalized system.

The first two steps are also used in (Zhong and Mirkin, 2001) to prove the result of the extended Nehari problem with a delay, where the stability condition of $K(s)$ is not needed (but the stability of $G_\beta + e^{-sh} K$ is required). The predictor was obtained as

$$\Delta(s) = -\pi_h \{ \mathcal{F}_u (\begin{bmatrix} G_\beta & I \\ I & 0 \end{bmatrix}, \gamma^{-2} G_\beta^\sim) \} \quad (6)$$

and the realization of the rationalized matrix

$$\Theta \doteq \begin{bmatrix} I & \Delta(s)^\sim \\ 0 & I \end{bmatrix} G^\sim J_\gamma G \begin{bmatrix} I & 0 \\ \Delta(s) & I \end{bmatrix}$$

is

$$\Theta = \left[\begin{array}{cc|cc} A & 0 & -\Sigma_{12}^* C^* & B \\ -C^* C & -A^* & \Sigma_{11}^* C^* & 0 \\ \hline -C\Sigma_{11} & -C\Sigma_{12} & I & \\ 0 & B^* & & -\gamma^2 I \end{array} \right].$$

Its inverse is

$$\Theta^{-1} = \left[\begin{array}{cc|cc} A & \gamma^{-2} B B^* & 0 & \gamma^{-2} \Sigma_{11} B \\ & -A^* & -C^* & \gamma^{-2} \Sigma_{21} B \\ \hline -C & & I & \\ \gamma^{-2} B^* \Sigma_{21}^* & -\gamma^{-2} B^* \Sigma_{11}^* & & -\gamma^{-2} I \end{array} \right].$$

Obviously, Θ^{-1} is in the form of (5). Directly applying the result obtained in Lemma 4, $\Theta^{-1}(s)$ has a $J-$spectral co-factorization if and only if two Riccati equations

$$\begin{bmatrix} -L_c & I \end{bmatrix} \begin{bmatrix} A & \gamma^{-2} B B^* \\ 0 & -A^* \end{bmatrix} \begin{bmatrix} I \\ L_c \end{bmatrix} = 0$$

$$\begin{bmatrix} I & -L_{oh} \end{bmatrix} \Sigma \begin{bmatrix} A & 0 \\ -C^* C & -A^* \end{bmatrix} \Sigma^{-1} \begin{bmatrix} L_{oh} \\ I \end{bmatrix} = 0 \quad (7)$$

exist unique symmetric stabilizing solutions $L_c \leq 0$ and L_{oh}, respectively, and $\det(I - L_{oh}L_c) \neq 0$.

Since the second Hamiltonian matrix in (7) is similar to $\begin{bmatrix} A & 0 \\ -C^*C & -A^* \end{bmatrix}$, the unique stabilizing solution L_{oh} can also be obtained as

$$L_{oh} = (\Sigma_{11}L_o + \Sigma_{12})(\Sigma_{21}L_o + \Sigma_{22})^{-1} = \mathcal{H}_r(\Sigma, L_o)$$

(if $\Sigma_{21}L_o + \Sigma_{22}$ is nonsingular), where $L_o \leq 0$ is the unique stabilizing solution of

$$\begin{bmatrix} I & -L_o \end{bmatrix} \begin{bmatrix} A & 0 \\ -C^*C & -A^* \end{bmatrix} \begin{bmatrix} L_o \\ I \end{bmatrix} = 0.$$

The J-spectral co-factor of Θ^{-1}, $W_0^{-1}(s)$, can then be simplified as

$$W_0^{-1}(s) = \left[\begin{array}{c|cc} A + \gamma^{-2}BB^*L_c & (I - L_{oh}L_c)^{-1}L_{oh}C^* & (I - L_{oh}L_c)^{-1}(L_{oh}\Sigma_{21} - \Sigma_{11})\gamma^{-1}B \\ \hline -C & I & \\ \gamma^{-2}B^*(\Sigma_{21}^* - \Sigma_{11}^*L_c) & & \gamma^{-1}I \end{array}\right].$$

Now, we have obtained the following identity:

$$G^{\sim}J_{\gamma}G = \begin{bmatrix} I & -\Delta(s)^{\sim} \\ 0 & I \end{bmatrix} W^{\sim}J_{\gamma}W \begin{bmatrix} I & 0 \\ -\Delta(s) & I \end{bmatrix},$$

where

$$W^{-1}(s) = \left[\begin{array}{c|cc} A + \gamma^{-2}BB^*L_c & (I - L_{oh}L_c)^{-1}L_{oh}C^* & (I - L_{oh}L_c)^{-1}(L_{oh}\Sigma_{21} - \Sigma_{11})B \\ \hline -C & I & \\ \gamma^{-2}B^*(\Sigma_{21}^* - \Sigma_{11}^*L_c) & & I \end{array}\right].$$

As we have shown, $W(s)$ and $\begin{bmatrix} I & 0 \\ -\Delta(s) & I \end{bmatrix}$ are all bistable and, hence, $W(s)\begin{bmatrix} I & 0 \\ -\Delta(s) & I \end{bmatrix}$ is a J-spectral factor of $G^{\sim}J_{\gamma}G$. This means that any $K(s)$ in the form

$$K(s) = \mathcal{H}_r\left(\begin{bmatrix} I & 0 \\ \Delta(s) & I \end{bmatrix} W^{-1}, Q\right)$$

(where $\|Q(s)\|_{H_\infty} < \gamma$ is a free parameter) satisfies

$$\left\|G_\beta + e^{-sh}K\right\|_{L_\infty} < \gamma.$$

Furthermore, in order to make $K(s) \in H_\infty$, the bistability of the $(2,2)$-block of the matrix

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \doteq G \begin{bmatrix} I & 0 \\ \Delta(s) & I \end{bmatrix} W^{-1}$$

is required.

With similar argument of Meinsma and Zwart (2000), the bistability of Π_{22} is equivalent to the existence of L_{oh} and the nonsingularity of $I - L_{oh}L_c$ (or equivalently the nonsingularity of $I - L_cL_{oh}$) not only for γ but also for any number larger than γ. Since $\gamma > \gamma_h$ is a necessary condition and, under this condition, Σ_{22} is always nonsingular (Zhou and Khargonekar, 1987), the existence of L_{oh} is equivalent to the nonsingularity of $\Sigma_{21}L_o + \Sigma_{22}$. Hence, the solvability condition can be summarized as follows:

(i) There exists a $\gamma_0 > 0$ such that $\Sigma_{22} = \begin{bmatrix} 0 & I \end{bmatrix} \Sigma \begin{bmatrix} 0 \\ I \end{bmatrix}$ is always nonsingular for $\gamma > \gamma_0$. This means $\gamma_0 = \gamma_h$;

(ii) There exists a $\gamma_1 > 0$ such that $\Sigma_{21}L_o + \Sigma_{22} = \begin{bmatrix} 0 & I \end{bmatrix} \Sigma \begin{bmatrix} L_o \\ I \end{bmatrix}$ is always nonsingular for $\gamma > \gamma_1$;

(iii) There exists a $\gamma_2 > 0$ such that $I - L_cL_{oh}$ is always nonsingular for $\gamma > \gamma_2$. When the condition (ii) is satisfied, the nonsingularity of $I - L_cL_{oh}$ is equivalent to that of $(I - L_cL_{oh})(\Sigma_{21}L_o + \Sigma_{22}) = \Sigma_{21}L_o + \Sigma_{22} - L_c(\Sigma_{11}L_o + \Sigma_{12}) = \begin{bmatrix} -L_c & I \end{bmatrix} \Sigma \begin{bmatrix} L_o \\ I \end{bmatrix}$.

The minimal γ which satisfies the above three conditions is the optimal value γ_{opt}:

$$\gamma_{opt} = \max\{\gamma_h,\ \gamma_1,\ \gamma_2\},$$

where

$$\gamma_h = \max\{\gamma : \det(\begin{bmatrix} 0 & I \end{bmatrix} \Sigma \begin{bmatrix} 0 \\ I \end{bmatrix}) = 0\},$$
$$\gamma_1 = \max\{\gamma : \det(\begin{bmatrix} 0 & I \end{bmatrix} \Sigma \begin{bmatrix} L_o \\ I \end{bmatrix}) = 0\},$$
$$\gamma_2 = \max\{\gamma : \det(\begin{bmatrix} -L_c & I \end{bmatrix} \Sigma \begin{bmatrix} L_o \\ I \end{bmatrix}) = 0\}.$$

This completes the proof.

Three special cases are outlined in the following corollaries:

Case 1: If A is stable, then $L_o = 0$, $L_c = 0$ and $L_{oh} = \Sigma_{12}\Sigma_{22}^{-1}$. Condition (iii) is always satisfied and condition (ii) becomes the same as condition (i), i.e., only the nonsingularity of Σ_{22} is required. In this case, $\gamma_{opt} = \gamma_h$.

Corollary 5. Given strictly proper stable G_β has no $j\omega$-axis zero nor pole, the delay-type Nehari problem (1) is solvable iff $\gamma > \gamma_h$, or equivalently, Σ_{22} is nonsingular not only for γ but also for any number larger than γ. Furthermore, if this condition holds, then $K(s)$ is parameterized as (4) where

$$W^{-1}(s) = \left[\begin{array}{c|cc} A & \Sigma_{12}\Sigma_{22}^{-1}C^* & -\Sigma_{22}^{-*}B \\ \hline -C & I & \\ \gamma^{-2}B^*\Sigma_{21}^* & & I \end{array}\right].$$

Case 2: If delay $h = 0$, then $\Sigma = I$ and $L_{oh} = L_o$. The conditions (i) and (ii) are always satisfied and L_{oh} always exists. Hence, the conditions are reduced to the nonsingularity of $I - L_oL_c$ for any $\gamma > \gamma_2$. In this case, $\gamma_{opt} = \gamma_2 = \max\{\gamma : \det(I - L_oL_c) = 0\}$.

Corollary 6. Given strictly proper G_β has no $j\omega$-axis zero nor pole, the delay-free Nehari problem (to find $K(s) \in H_\infty$ such that $\|G_\beta(s) + K(s)\|_{L_\infty} < \gamma$) is solvable iff $\gamma > \max\{\gamma : \det(I - L_oL_c) = 0\}$. Furthermore, if this condition holds, then $K(s)$ is parameterized as

$$K(s) = \mathcal{H}_r\left(W^{-1}, Q\right),$$

where

$$W^{-1}(s) = \left[\begin{array}{c|cc} A + \gamma^{-2}BB^*L_c & (I - L_oL_c)^{-1}L_oC^* & -(I - L_oL_c)^{-1}B \\ \hline -C & I & \\ -\gamma^{-2}B^*L_c & & I \end{array}\right]$$

and $\|Q(s)\|_{H_\infty} < \gamma$ is a free parameter.

Remark 7. This is an alternative solution to the well-known Nehari problem which has been addressed extensively, e.g. in (Francis, 1987; Green *et al.*, 1990). The A-matrix A is not split here. In common situation, it was handled by modal decomposition, see, e.g. (Green *et al.*, 1990), and the A-matrix A is split into two parts, a stable part and an anti-stable part. Actually, it can be shown that, in this case, $\gamma_{opt} = \left\|\Gamma_{G_\beta}\right\| = \max\{\gamma : \det(I - L_oL_c) = 0\}$.

Case 3: If delay $h = 0$ and A is stable, then $L_o = 0$, $L_c = 0$, $L_{oh} = 0$, and $\Sigma = I$. The conditions are always satisfied for $\gamma > 0$. $K(s)$ is parameterized as (4) where $\Delta(s) = 0$ and

$$W^{-1}(s) = \left[\begin{array}{c|cc} A & 0 & B \\ \hline C & I & \\ 0 & & I \end{array}\right] = \begin{bmatrix} I & -G_\beta \\ 0 & I \end{bmatrix}.$$

This is obvious. For stable delay-free Nehari problem, the solution is definitely $K = -G_\beta + Q(s)$ for any $\gamma > 0$, where $\|Q(s)\|_{H_\infty} < \gamma$ is a free parameter. In this case, $\gamma_{opt} = 0$.

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