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POSITIVE REALNESS AND THE ANALYSIS OF A CLASS OF 2D LINEAR SYSTEMS

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Abstract: Repetitive processes are a distinct class of 2D linear systems with applications in areas ranging from long-wall coal cutting and metal rolling operations through to iterative learning control schemes. The main feature which makes them distinct from other classes of 2D linear systems is that information propagation in one of the two independent directions only occurs over a finite duration. This, in turn, means that a distinct systems theory must be developed for them, which can then be translated into efficient routinely applicable controller design algorithms for applications domains. In this paper, we give the first significant results on a positive realness based approach to the analysis of these processes.

Keywords: 2D linear systems, repetitive processes, positive realness.

1. INTRODUCTION

The essential unique characteristic of a repetitive, or multipass, process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass to pass direction.

To introduce a formal definition, let α be an integer and $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(p), 0 \le p \le \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(p), 0 \le p \le \alpha, k \ge 0$.

Physical examples of repetitive processes include long- wall coal cutting and metal rolling operations (Edwards 1974). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications of repetitive process theory include classes of iterative learning control schemes (Amann, Owens

& Rogers 1998) and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle (Roberts 2000).

Attempts to control these processes using standard (or 1D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass to pass and along a given pass, and the pass initial conditions are reset before the start of each new pass. In seeking a rigorous foundation on which to develop a control theory for these processes, it is natural to attempt to exploit structural links which exist between, in particular, the class of so-called discrete linear repetitive processes and 2D linear systems described by the extensively studied Roesser (Roesser 1975) or Fornasini Marchesini (Fornasini & Marchesini 1978) state space models. Discrete linear repetitive processes are distinct from such 2D linear systems in the sense that information propagation in one of the two independent directions (along the pass) only occurs over a finite duration.

In this paper, we develop the first significant results on a positive realness approach to the analysis of these processes. The starting point for this is the background results summarized in the next section.

2. BACKGROUND

Consider first a 1D discrete linear time-invariant system defined by the state space quadruple $\{A,B,C,D\}$ where the dimensions of the state, output and input vectors are n,m and l respectively. Suppose also that this system is controllable and observable with transfer function matrix $G(z)$. Let $x(i)$ be the state vector. Then this system is said to be asymptotically stable provided

$$\lim_{i\to\infty} ||x(i)|| = 0 \tag{1}$$

under zero input (i.e. $u(i)\equiv 0$). This property holds if, and only if, $\exists$ $n\times n$ symmetric positive definite matrix P such that the Lyapunov equation

$$A^TPA - P < 0 \tag{2}$$

(from this point onwards $>$ will (where relevant) be used to denote a positive definite matrix and <0 one which is negative definite). Suppose now that $m=l$. Then this 1D linear system is said to be positive real (PR) if
(i) its transfer function matrix $G(z)$ is analytic for $|z|>1$; and
(ii) $G(z)+G^*(z)\geq 0$, $|z|>1$, where $*$ denotes the complex conjugate transpose operation.
It is said to be strictly positive real (SPR) if (ii) above can be replaced by $G(z)+G^*(z)>0$, $|z|>1$.

The PR and SPR properties play a key role in various aspects of 1D linear systems theory and a natural question to ask is whether or not the same is true (to any meaningful extent) for 2D linear systems/discrete linear repetitive processes. In the former case, it is instructive to consider first the Roesser state space model, i.e.

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bu(i,j)$$
$$y(i,j) = C\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Du(i,j) \tag{3}$$

where $x^h(i,j)\in\mathbb{R}^{n_h}$ and $x^v(i,j)\in\mathbb{R}^{n_v}$ are the horizontal and vertical state vectors, respectively, $u(i,j)\in\mathbb{R}^q$ is the input vector, and $y(i,j)\in\mathbb{R}^s$ is the output vector for $(i,j)\geq 0$. The system boundary conditions are defined as follows

$$x_0 = \left[x^h(0,0)^T, x^h(0,1)^T, x^h(0,2)^T \ldots,\right.$$
$$\left. x^v(0,0)^T, x^v(1,0)^T, x^v(2,0)^T, \ldots\right]^T \tag{4}$$

This Roesser model is said to be asymptotically stable provided

$$\lim_{i,j\to\infty} ||x(i,j)|| = 0 \tag{5}$$

under zero input (i.e. $u(i,j)\equiv 0$) and the boundary conditions are such that $\sup_j ||x^h(0,j)|| < \infty$ and $\sup_i ||x^v(i,0)|| < \infty$, where $x(i,j) = \left[x^h(i,j)^T, x^v(i,j)^T\right]^T$. A sufficient condition for this property is that there exists a block-diagonal matrix $P=\mathrm{diag}(P_h,P_v)>0$ with $P_h\in\mathbb{R}^{n_h\times n_h}$ and $P_v\in\mathbb{R}^{n_v\times n_v}$ such that

$$A^TPA - P < 0 \tag{6}$$

With the same number of inputs and outputs, the 2D transfer function matrix of the state space model (3) is given by

$$G(z_1,z_2) = C\left(I(z_1,z_2)-A\right)^{-1}B + D \tag{7}$$

where $I(z_1,z_2)=\mathrm{diag}(z_1I_{n_h}, z_2I_{n_v})$.
This model is said to be PR if
(i) $G(z_1,z_2)$ is analytic in $|z_1|>1$, $|z_2|>1$; and
(ii) $G(z_1,z_2)+G^*(z_1,z_2)\geq 0$ for $|z_1|>1$, $|z_2|>1$. It is said to be SPR if
(i) $G(z_1,z_2)$ is analytic in $|z_1|\geq 1$, $|z_2|\geq 1$; and
(ii) $G(e^{\imath\theta_1},e^{\imath\theta_2})+G^*(e^{\imath\theta_1},e^{\imath\theta_2})>0$ for $\theta_1,\theta_2\in[0,2\pi)$.
Finally, it is said to be extended strictly positive real (ESPR) if it is SPR and $G(\infty,\infty)+G^T(\infty,\infty)>0$.

The following result gives a sufficient condition for the system (4) to be asymptotically stable and have the ESPR property (Xu, Lam, Galkowski, Rogers & Owens 2002).

Theorem 1. The system (3) is asymptotically stable and ESPR if $\exists$ a block-diagonal matrix $P = diag(P_h,P_v)>0$ with $P_h\in\mathbb{R}^{n_h\times n_h}$ and $P_v\in\mathbb{R}^{n_v\times n_v}$ such that the following LMI holds.

$$\begin{bmatrix} A^TPA-P & C^T-A^TPB \\ C-B^TPA & -\left(D+D^T-B^TPB\right)\end{bmatrix} < 0 \tag{8}$$

By Schur complements, it follows that if (8) holds then

$$A^T PA - P < 0 \tag{9}$$

and

$$D + D^T - B^T PB > 0 \tag{10}$$

Also it follows immediately from (10) that a necessary condition for positive realness is that

$$D + D^T > 0 \tag{11}$$

In the next section we will immediately see that the condition corresponding to (11) does not hold for discrete linear repetitive processes. Hence the use of positive realness theory for these processes cannot proceed by simply interpreting that already existing for systems described by (3) (or alternatives).

3. ANALYSIS

The state space model of a discrete linear repetitive process has the following form over $0 \le p \le \alpha$, $k \ge 0$,

$$\begin{aligned} x_{k+1}(p+1) &= \widehat{A}x_{k+1}(p) + \widehat{B}u_{k+1}(p) \\ &\quad + \widehat{B}_0 y_k(p), \\ y_{k+1}(p) &= \widehat{C}x_{k+1}(p) + \widehat{D}u_{k+1}(p) \\ &\quad + \widehat{D}_0 y_k(p) \end{aligned} \tag{12}$$

Here on pass $x_k(p) \in \mathbb{R}^n$ is the current pass state vector, $y_k(p) \in \mathbb{R}^m$ is the current pass profile vector, and $u_k(p) \in \mathbb{R}^m$ is the vector of current pass inputs. To complete the process description, it is necessary to specify the initial, or boundary, conditions, i.e. the state initial vector on each pass $x_{k+1}(0)$, $k \ge 0$, and the initial pass profile $y_0(p)$. Here these are taken to be of the simplest possible form, i.e. $x_{k+1}(0) = d_{k+1}$, $k \ge 0$, and $y_0(p) = y(p)$, $0 \le p \le \alpha$, where d_{k+1} is an $n \times 1$ vector with constant entries and the entries in the $m \times 1$ vector $y(p)$ are known functions of p. Note, however, that the structure of the boundary conditions alone can cause instability in these processes. (See (Owens & Rogers 1999) where this fact is established for the differential counterparts of the processes considered here using a pass state initial vector sequence which is an explicit function of points on the previous pass profile.)

In Roesser model terms, the pass profile vector here $y_k(p)$ plays the role of the vertical state vector and the pass state vector $x_k(p)$ plays the role of the horizontal state vector. Also the pass profile vector is simultaneously the output vector in Roesser model terms and hence we can write for $k \ge 1$

$$\begin{aligned} y_k(p) &= \widehat{C}\begin{bmatrix} x_k(p) \\ y_k(p) \end{bmatrix} + \widehat{D}u_k(p) \\ &= \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} x_k(p) \\ y_k(p) \end{bmatrix} + 0u_k(p) \end{aligned} \tag{13}$$

The corresponding 2D z transfer function matrix is

$$G(z_1, z_2) = \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} I - z_1\widehat{A} & -\widehat{B}_0 \\ -\widehat{C} & I - z_2\widehat{D}_0 \end{bmatrix}^{-1}\begin{bmatrix} \widehat{B} \\ \widehat{D} \end{bmatrix} \tag{14}$$

Hence, it follows immediately that no discrete linear repetitive process of the form considered here can ever be asymptotically stable and ESPR since $\widehat{D} = 0$ and hence the equivalent of (11), i.e. $\widehat{D} + \widehat{D}^T > 0$, can never hold.

To apply PR theory to discrete linear repetitive processes, we propose a route via the 1D equivalent state space model description of the underlying dynamics. This 1D equivalent model has been developed in, for example, (Galkowski, Rogers & Owens 1998) and here we need only give the final construction.

The starting point for this is to make the substitutions $l = k + 1$ and $y_{k-1}(p) = \upsilon_k(p)$, $0 \le p \le \alpha - 1$, $l = 1, 2, \ldots$. Now define the so-called global pass profile, state and input vectors respectively for (12) as

$$\begin{aligned} Y(l) &:= \begin{bmatrix} \upsilon_l^T(0), & \upsilon_l^T(1), & \ldots, & \upsilon_l^T(\alpha-1) \end{bmatrix}^T \\ X(l) &:= \begin{bmatrix} x_l^T(1), & x_l^T(2), & \ldots, & x_l^T(\alpha) \end{bmatrix}^T \\ U(l) &:= \begin{bmatrix} u_l^T(0), & u_l^T(1), & \ldots, & u_l^T(\alpha-1) \end{bmatrix}^T \end{aligned} \tag{15}$$

Then, assuming without loss of generality that the state initial vector on each pass is zero, i.e. $d_{k+1} = 0$, $k \ge 0$, the 1D equivalent state space model of the dynamics of (12) has the form

$$\begin{aligned} Y(l+1) &= \Phi Y(l) + \Delta U(l) \\ X(l) &= \Gamma Y(l) + \Sigma U(l) \end{aligned} \tag{16}$$

where

$$\Phi = \begin{bmatrix} \widehat{D}_0 & 0 & \ldots & 0 \\ \widehat{C}\widehat{B}_0 & \widehat{D}_0 & \ldots & 0 \\ \widehat{C}\widehat{A}\widehat{B}_0 & \widehat{C}\widehat{B}_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{C}\widehat{A}^{\alpha-2}\widehat{B}_0 & \widehat{C}\widehat{A}^{\alpha-3}\widehat{B}_0 & \ldots & \widehat{D}_0 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \widehat{D} & 0 & 0 & \cdots & 0 \\ \widehat{C}\widehat{B} & D & 0 & \cdots & 0 \\ \widehat{C}\widehat{A}\widehat{B} & \widehat{C}\widehat{B} & \widehat{D} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{C}\widehat{A}^{\alpha-2}\widehat{B} & \widehat{C}\widehat{A}^{\alpha-3}\widehat{B} & \widehat{C}\widehat{A}^{\alpha-4}\widehat{B} & \cdots & \widehat{D} \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} \widehat{B}_0 & 0 & 0 & \cdots & 0 \\ \widehat{A}\widehat{B}_0 & \widehat{B}_0 & 0 & \cdots & 0 \\ \widehat{A}^2\widehat{B}_0 & \widehat{A}\widehat{B}_0 & \widehat{B}_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{A}^{\alpha-1}\widehat{B}_0 & \widehat{A}^{\alpha-2}\widehat{B}_0 & \widehat{A}^{\alpha-3}\widehat{B}_0 & \cdots & \widehat{B}_0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \widehat{B} & 0 & 0 & \cdots & 0 \\ \widehat{A}\widehat{B} & \widehat{B} & 0 & \cdots & 0 \\ \widehat{A}^2\widehat{B} & \widehat{A}\widehat{B} & \widehat{B} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{A}^{\alpha-1}\widehat{B} & \widehat{A}^{\alpha-2}\widehat{B} & \widehat{A}^{\alpha-3}\widehat{B} & \cdots & \widehat{B} \end{bmatrix} \tag{17}$$

Given this 1D equivalent model, we can now establish the main result in this paper which requires the additional assumption that the dimension of $x_k(p)$ is equal to that of $u_k(p)$. This assumption arises from the fact that in the 1D equivalent model the pass profile, which in the 2D linear systems interpretation of the dynamics of these processes, is the subject of dynamic updating and the pass profile vector (horizontally transmitted information in the 2D setting) is embedded in a static (or purely algebraic) equation.

Theorem 2. Discrete repetitive processes described by the 1D equivalent state space model of (16) and (17) are asymptotically stable and ESPR if and only if $\exists$ an $m\alpha \times m\alpha$ symmetric matrix $P > 0$ such that the following LMI is satisfied

$$\begin{bmatrix} \Phi^T P\Phi - P & \Gamma^T - \Phi^T P\Delta \\ \Gamma - \Delta^T P\Phi & -\left(\Sigma + \Sigma^T - \Delta^T P\Delta\right) \end{bmatrix} < 0 \qquad (18)$$

The proof of this result is immediate from the known result for 1D systems (Sun, Khargonekar & Shim 1994) and the structure of the 1D equivalent state space model.

The only major difficulty with Theorem 2 is that the (potentially) large dimension of the matrix P may cause numerical difficulties. In what follows, we develop a feasible way of avoiding such problems by assuming that P has a block diagonal form, i.e.

$$P = diag\left(P_1, P_2, \ldots, P_\alpha\right) \qquad (19)$$

Under the assumption of (19), the block sub-matrices of (18) can be expressed as

$$\Phi^T P\Phi - P = \left[\Omega^1_{ij}\right]_{\alpha\times\alpha} \qquad (20)$$

where

$$\begin{aligned} \Omega^1_{ii} &= \widehat{D}_0^T P_i \widehat{D}_0 \\ &\quad + \sum_{k=0}^{\alpha-1-i} \widehat{B}_0^T \widehat{A}^{kT} \widehat{C}^T P_{k+i+1} \widehat{C}\widehat{A}^k \widehat{B}_0 - P_i \\ \Omega^1_{i+q,i} &= \widehat{D}_0^T P_{i+q} \widehat{C}\widehat{A}^{q-1}\widehat{B}_0 \\ &\quad + \sum_{k=q}^{\alpha-1-i} \widehat{B}_0^T \widehat{A}^{k-q,T} \widehat{C}^T P_{k+i+1} \widehat{C}\widehat{A}^k \widehat{B}_0, \\ \Omega^1_{i,i+q} &= \Omega^{1T}_{i+q,i}, \end{aligned} \qquad (21)$$

$i = 1, 2, \ldots, \alpha$; $q = 1, 2, \ldots, \alpha - i$,

$$\Gamma^T - \Phi^T P\Delta = \left[\Omega^2_{ij}\right]_{\alpha\times\alpha} \qquad (22)$$

with

$$\begin{aligned} \Omega^2_{ii} &= \widehat{B}_0^T - \widehat{D}_0^T P_i \widehat{D} \\ &\quad - \sum_{k=0}^{\alpha-1-i} \widehat{B}_0^T \widehat{A}^{kT} \widehat{C}^T P_{k+i+1} \widehat{C}\widehat{A}^k \widehat{B} \\ \Omega^2_{i+q,i} &= -\widehat{D}_0^T P_{i+q} \widehat{C}\widehat{A}^{q-1}\widehat{B} \\ &\quad - \sum_{k=q}^{\alpha-1-i} \widehat{B}_0^T \widehat{A}^{k-q,T} \widehat{C}^T P_{k+i+1} \widehat{C}\widehat{A}^k \widehat{B} \\ \Omega^2_{i,i+q} &= \widehat{B}_0^T \widehat{A}^{qT} - \widehat{B}_0^T \widehat{A}^{q-1,T} \widehat{C}^T P_{i+q} \widehat{D} \\ &\quad - \sum_{k=q}^{\alpha-1-i} \widehat{B}_0^T \widehat{A}^{kT} \widehat{C}^T P_{k+i+1} \widehat{C}\widehat{A}^{k-q} \widehat{B} \end{aligned} \qquad (23)$$

and

$$-\left(\Sigma + \Sigma^T - \Delta^T P\Delta\right) = \left[\Omega^3_{ij}\right]_{\alpha\times\alpha} \qquad (24)$$

with

$$\begin{aligned} \Omega^3_{ii} &= -\widehat{B} - \widehat{B}^T + \widehat{D}^T P_i \widehat{D} \\ &\quad + \sum_{k=0}^{\alpha-1-i} \widehat{B}^T \widehat{A}^{kT} \widehat{C}^T P_{k+i+1} \widehat{C}\widehat{A}^k \widehat{B} \\ \Omega^3_{i+q,i} &= -A^q \widehat{B} + \widehat{D}^T P_{i+q} \widehat{C}\widehat{A}^{q-1}\widehat{B} \\ &\quad + \sum_{k=q}^{\alpha-1-i} \widehat{B}^T \widehat{A}^{k-q,T} \widehat{C}^T P_{k+i+1} \widehat{C}\widehat{A}^k \widehat{B} \\ \Omega^3_{i,i+q} &= \Omega^{3T}_{i+q,i} \end{aligned} \qquad (25)$$

and $i = 1, 2, \ldots, \alpha$; $q = 1, 2, \ldots, \alpha - i$.

Hence, all blocks in (18) are of the form

$$K_0 + \sum_{i=1}^{\alpha} K_i P_i L_i \qquad (26)$$

where the matrices K_i and L_i have constant entries, which are defined by the matrices in the original process state space model, and the positive definite P_i, $1 \le i \le \alpha$, are the problem solution matrices to be searched for in the LMI computation. Note also that the underlying assumption here, i.e. that P has a block diagonal structure, will make the stability condition more conservative. Also this would be increased further if it were to be assumed that $P_j = P$, $j = 1, 2, \ldots, \alpha$.

4. SYNTHESIS

Consider the following repetitive process

$$\begin{aligned} x_{k+1}(p+1) &= \widehat{A}x_{k+1}(p) + \widehat{B}u_{k+1}(p) \\ &\quad + \widehat{B}_0 y_k(p) + \widehat{E}w_{k+1}(p), \\ y_{k+1}(p) &= \widehat{C}x_{k+1}(p) + \widehat{D}u_{k+1}(p) \\ &\quad + \widehat{D}_0 y_k(p) + \widehat{R}w_{k+1}(p) \end{aligned} \qquad (27)$$

where $w_{k+1}(p)$ is an exogenous input. Then the 1D equivalent state space model of the dynamics of (27) (with the pass state initial vector sequence set equal to zero) has the form

$$\begin{aligned} Y(l+1) &= \Phi Y(l) + \Delta U(l) + \Phi W(l) \\ X(l) &= \Gamma Y(l) + \Sigma U(l) + \Upsilon W(l) \end{aligned} \qquad (28)$$

where Φ, Δ, Γ and Σ are given in (17),

$$W(l) := \begin{bmatrix} w_l(0) \\ w_l(1) \\ \vdots \\ w_l(\alpha-1) \end{bmatrix} \qquad (29)$$

and

$$\Pi = \begin{bmatrix} \widehat{R} & 0 & 0 & \cdots & 0 \\ \widehat{C}\widehat{E} & \widehat{R} & 0 & \cdots & 0 \\ \widehat{C}\widehat{A}\widehat{E} & \widehat{C}\widehat{E} & \widehat{R} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{C}\widehat{A}^{\alpha-2}\widehat{E} & \widehat{C}\widehat{A}^{\alpha-3}\widehat{E} & \widehat{C}\widehat{A}^{\alpha-4}\widehat{E} & \cdots & \widehat{R} \end{bmatrix},$$

$$\Upsilon = \begin{bmatrix} \widehat{E} & 0 & 0 & \cdots & 0 \\ \widehat{A}\widehat{E} & \widehat{E} & 0 & \cdots & 0 \\ \widehat{A}^2\widehat{E} & \widehat{A}\widehat{E} & \widehat{E} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{A}^{\alpha-1}\widehat{E} & \widehat{A}^{\alpha-2}\widehat{E} & \widehat{A}^{\alpha-3}\widehat{E} & \cdots & \widehat{E} \end{bmatrix}$$

Then we have the following synthesis result whose proof is immediate from Theorem 2 and a simple application of the Schur complement's formula.

Theorem 3. Consider the discrete repetitive processes described by the 1D equivalent state space model of (28). Then if $\exists$ an $m\alpha \times m\alpha$ symmetric matrix $P > 0$ and a matrix Z such that the following LMI is satisfied

$$\begin{bmatrix} -P & 0 & (\Phi P + \Delta Z)^T \\ 0 & -\left(\Sigma + \Sigma^T\right) & -\Pi^T \\ \Phi P + \Delta Z & -\Pi & -P \end{bmatrix} < 0 \quad (30)$$

the state feedback control law

$$U(l) = KY(l) \quad (31)$$

where

$$K = ZP^{-1} \quad (32)$$

will be such that the resulting closed-loop system formed by (28) and (31) is asymptotically stable and ESPR.

5. EXAMPLES

Consider first the case when the matrices in the state space model (27) are

$$\widehat{A} = [-0.4],\ \widehat{B} = [0.2],\ \widehat{B}_0 = [0.1],$$
$$\widehat{C} = [0.1],\ \widehat{D} = [0.2],\ \widehat{D}_0 = [0.9]$$

Then the LMI of (18) is feasible, i.e. $\exists$ a positive definite 10×10 matrix P satisfying (18). Hence Theorem 2 holds and therefore this process is asymptotically stable and ESPR. Note also the LMI of (19) with block diagonal P is also feasible in this case with

$$\begin{array}{lll} P_1 = 0.8314, & P_2 = 0.8299, & P_3 = 0.8297, \\ P_4 = 0.8296, & P_5 = 0.8296, & P_6 = 0.8296, \\ P_7 = 0.8296, & P_8 = 0.8296, & P_9 = 0.8296, \\ & P_{10} = 0.8297. & \end{array}$$

Also the LMI of (18) is feasible in this case with $\tilde{P} = 0.8298$. Finally, these conclusions are easily seen to hold in simulation studies where, by way of illustration, Fig.1 shows the current pass state vector sequence generated over 10 passes with $\alpha = 10$ and Fig.2 the pass profile vector sequence over the same number of passes and pass length. In both cases, a zero control input sequence was applied and the initial pass profile was a unit step signal applied at $p = 0$.

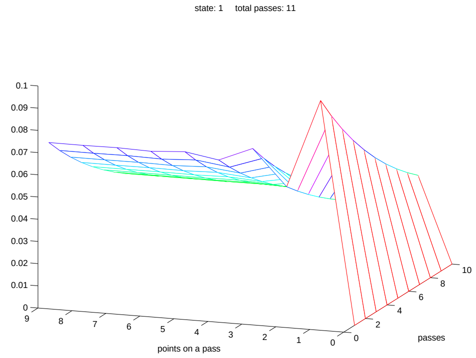

Fig. 1. Current pass state vector response sequence

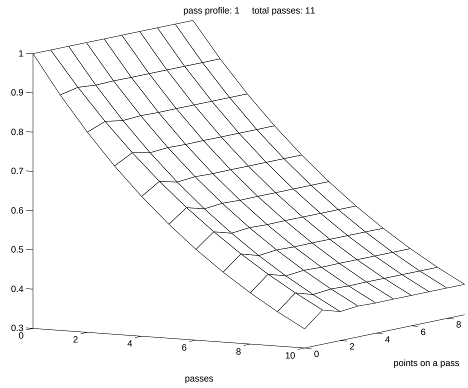

Fig. 2. Pass profile vector sequence

To illustrate the application of Theorem 3, consider the case when the matrices in the state space model (27) are

$$\widehat{A} = [0.6],\ \widehat{B} = [0.2],\ \widehat{B}_0 = [0.1],$$
$$\widehat{C} = [0.1],\ \widehat{D} = [0.2],\ \widehat{D}_0 = [0.99],$$
$$\widehat{R} = [0.5],\ \widehat{E} = [0.3]$$

This process is not ESPR stable since (18) does not hold. The LMI of (30) is, however, feasible and one solution is the positive definite 10×10 matrix $P = 1.6877 I_{10}$, where I_{10} is the 10×10 identity matrix and

$$Z = \begin{bmatrix} -8.3542 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7510 & -8.3542 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3755 & 0.7510 & -8.3542 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1878 & 0.3755 & 0.7510 & -8.3542 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0939 & 0.1878 & 0.3755 & 0.7510 & -8.3542 & 0 & 0 & 0 & 0 & 0 \\ 0.0469 & 0.0939 & 0.1878 & 0.3755 & 0.7510 & -8.3542 & 0 & 0 & 0 & 0 \\ 0.0235 & 0.0469 & 0.0939 & 0.1878 & 0.3755 & 0.7510 & -8.3542 & 0 & 0 & 0 \\ 0.0117 & 0.0235 & 0.0469 & 0.0939 & 0.1878 & 0.3755 & 0.7510 & -8.3542 & 0 & 0 \\ 0.0059 & 0.0117 & 0.0235 & 0.0469 & 0.0939 & 0.1878 & 0.3755 & 0.7510 & -8.3542 & 0 \\ 0.0029 & 0.0059 & 0.0117 & 0.0235 & 0.0469 & 0.0939 & 0.1878 & 0.3755 & 0.7510 & -8.3542 \end{bmatrix}$$

Hence state feedback control law (31) with

$$K = \begin{bmatrix} -4.9500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4450 & -4.9500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2225 & 0.4450 & -4.9500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1112 & 0.2225 & 0.4450 & -4.9500 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0556 & 0.1113 & 0.2225 & 0.4450 & -4.9500 & 0 & 0 & 0 & 0 & 0 \\ 0.0278 & 0.0556 & 0.1112 & 0.2225 & 0.4450 & -4.9500 & 0 & 0 & 0 & 0 \\ 0.0139 & 0.0278 & 0.0556 & 0.1113 & 0.2225 & 0.4450 & -4.9500 & 0 & 0 & 0 \\ 0.0070 & 0.0139 & 0.0278 & 0.0556 & 0.1112 & 0.2225 & 0.4450 & -4.9500 & 0 & 0 \\ 0.0035 & 0.0070 & 0.0139 & 0.0278 & 0.0556 & 0.1113 & 0.2225 & 0.4450 & -4.9500 & 0 \\ 0.0017 & 0.0035 & 0.0070 & 0.0139 & 0.0278 & 0.0556 & 0.1112 & 0.2225 & 0.4450 & -4.9500 \end{bmatrix}$$

will ensure that the resulting closed loop system is asymptotically stable and ESPR.

6. CONCLUSIONS

This paper has produced the first significant results on a positive realness based theory for the control related analysis of discrete linear repetitive processes. The major results are an LMI based characterization of this property and the use of this same setting to design a control law to guarantee that this property holds closed loop. Currently in depth development/extension of these results is in progress and will be reported on in due course.

7. REFERENCES

Amann, N., Owens, D. & Rogers, E. (1998). Predictive optimal iterative learning control, *International Journal of Control* **69**: 203–226.

Edwards, J. B. (1974). Stability problems in the control of multipass processes, *Proceedings of The Institution of Electrical Engineers* **12**: 1425–1431.

Fornasini, E. & Marchesini, G. (1978). Doubly-indexed dynamical systems: state space models and structural properties, *Mathematical Systems Theory* **12**: 59–72.

Galkowski, K., Rogers, E. & Owens, D. (1998). Matrix rank based conditions for reachability / controllability of discrete linear repetitive processes, *Linear Algebra and its Applications* **275-276**: 201–224.

Owens, D. H. & Rogers, E. (1999). Stability analysis for a class of 2D continuous-discrete linear systems with dynamic boundary conditions, *Systems and Control Letters* **37**: 55–60.

Roberts, P. D. (2000). Numerical investigations of a stability theorem arising from 2-dimensional analysis of an iterative optimal control algorithm, *Multidimensional Systems and Signal Processing* **11 (1/2)**: 109–124.

Roesser, R. P. (1975). A discrete state space model for linear image processing, *IEEE Transactions on Automatic Control* **AC-20 (1)**: 1 – 10.

Sun, W., Khargonekar, P. & Shim, D. (1994). Solution to the positive real control problem for linear time-invariant systems, *International Journal of Control* **39**: 2034–2046.

Xu, S., Lam, J., Galkowski, K., Rogers, E. & Owens, D. H. (2002). Positive real control of classes of 2D discrete linear systems, *International Journal of Control* . (to appear).