15th Triennial World Congress, Barcelona, Spain

AN APPROACH TO STABLE CONTROLLER DESIGN OF PIECEWISE DISCRETE TIME LINEAR SYSTEMS

Gang Feng, Dong Sun and Shaosheng Zhou

Dept. of Manufacturing Engineering & Engineering Management
City University of Hong Kong
Tat Chee Ave., Kowloon
Hong Kong
Email: megfeng@cityu.edu.hk

Abstract*:* This paper presents a controller design method for piecewise discrete time linear systems based on a piecewise smooth Lyapunov function. It is shown that the resulting closed loop system is globally stable and the controller can be obtained by solving a set of linear matrix inequalities that is numerically feasible with commercially available software.

Key words: Control system design, Discrete time systems, Linear matrix inequality, Lyapunov functions, Piecewise linear controllers, Stability

1. INTRODUCTION

Piecewise linear systems have been a subject of research in the systems and control community for some time, see for example (Imura and van der Schaft, 2000; Johansson and Rantzer, 1998; Rantzer ans Johansson, 2000; Hassibi and Boyd, 1998; Wicks, et al., 1994; Pettit and Wellstead, 1995; Kantner, 1997; Yfoulis, et al., 1998; Banks and Khathur, 1989; Sontag, 1981; Chua and Deng, 1986; Chua and Ying, 1983). In fact, the piecewise linear systems constitute a special class of hybrid systems (Yfoulis, et al., 1998) and arise often in practical control systems when piecewise linear components are encountered. These components include dead-zone, saturation, relays, and hysteresis. In addition, many other classes of nonlinear systems can also be approximated by the piecewise linear systems. Thus the piecewise linear systems provide a powerful means of analysis and design for nonlinear control systems.

A number of significant results have been obtained on analysis and controller design of such piecewise continuous time linear systems during the last few years. For example, the authors in Imura and van der Schaft (2000) studied a basic issue, that is, the well-posedness of piecewise linear systems. Necessary and sufficient conditions for bimodal systems to be well-posed have been derived, and the extension to the multimodal case has also been discussed. The authors in Johansson and Rantzer (1998) and Rantzer ans Johansson (2000) presented results on stability and optimal performance analysis for piecewise linear systems based on a piecewise continuous Lyapunov function. It has been shown that lower bounds, as well as upper bounds, on the optimal control cost can be obtained by semidefinite programming, and the framework of piecewise linear systems can be used to analyze smooth nonlinear systems with arbitrary accuracy. The authors in Hassibi and Boyd (1998) discussed stability analysis and controller design of piecewise linear systems which may involve multiple equilibrium points based on a common quadratic Lyapunov function and a piecewise quadratic Lyapunov function. It has been shown that stability and performance analysis can be cast as convex optimization problems. A controller design method based on a common quadratic Lyapunov function and linear matrix inequalities has been proposed.

However, there is little result on stability analysis of piecewise discrete time systems using piecewise Lyapunov functions in the open literature. The difficulty seems to be due to the fact that the state of the discrete time system may never pass through the region boundaries; instead the state most likely jumps from one region to another. In such a case, the boundary information, like the matrices F's in Johansson and Rantzer (1998) and Rantzer ans

Johansson (2000), cannot be used to characterize the state transition from one region to another as dealt with in the case of continuous time systems. More specifically, it may not be helpful to construct a piecewise Lyapunov function that is continuous across boundaries for the discrete time systems to analyze stability of the system as in Johansson and Rantzer (1998) and Rantzer ans Johansson (2000) for the continuous time systems. Nevertheless, it may also be unnecessary to require the piecewise Lyapunov function to be continuous across boundaries for the discrete time piecewise linear systems since the state of such systems may never pass through the boundaries. In a recent paper (Feng, 2001), we propose a method for stability analysis of the piecewise discrete time linear systems by constructing a novel piecewise Lyapunov function. This function is guaranteed to be decreasing when the state of the system jumps from one region to another. It is shown that the piecewise Lyapunov function can be constructed by solving a set of linear matrix inequalities if it exists. The work presented in that paper can be considered as an extension of the work for the piecewise continuous time systems in Imura and van der Schaft (2000) and Johansson and Rantzer (1998) to their discrete time counterparts.

In this paper, we will use the stability results presented in Feng (2001) to synthesize a stable controller for the piecewise discrete time linear systems. It will be shown that the resulting closed loop system is globally stable and the controller can be obtained by solving a set of linear matrix inequalities.

The rest of the paper is organised as follows. Section 2 introduces the piecewise linear system model and the stability theorem. Section 3 presents a controller design method for such systems. Finally, conclusions are given in section 4.

2. PIECEWISE UNCERTAIN LINEAR SYSTEM MODEL AND STABILITY THEOREM

Consider autonomous piecewise uncertain discrete time linear systems of the form

$$x(t+1) = (A_l + \Delta A_l)x(t) + (B_l + \Delta B_l)u(t) + a_l + \Delta a_l$$
$$\text{for } x \in S_l,\ l = 1,2,...,m, \qquad (2.1)$$

where $\{S_l\}_{l\in L} \subseteq \Re^n$ denotes a partition of the state space into a number of closed polyhedral subspaces, L is the index set of subspaces, $x(t)\in \Re^n$ the system state variables, (A_l, B_l, a_l) the *l-th* local nominal model of the system, a_l the offset term, and $(\Delta A_l, \Delta B_l, \Delta a_l)$ are the uncertainties of the corresponding matrices. For the definition of state trajectory and solution to the piecewise linear system (2.1) please refer to Imura and van der Schaft (2000), Johansson and Rantzer (1998), and Rantzer ans Johansson (2000) for details. Here we assume that given any initial condition $x(0) = x_0$, the difference equation (2.1) has a unique solution for all $t > 0$. We also assume that when the state of the system transits from the region S_l to S_j at the time *t*, the dynamics of the system is governed by the dynamics of the local model of S_l at that time. For future use, we also define a set Ω that represents all possible transitions from one region to another, that is,

$$\Omega := \{l, j \mid x(t) \in S_l,\ x(t+1) \in S_j,\ j \neq l\}.$$

Remark 2.1: It is noted that the system models defined in (2.1) are in fact *affine* systems instead of linear systems. They include an additional offset term. In this paper, the notation of linear systems has been *abused* to represent the affine systems.

Define $L_0 \subseteq L$ as the set of indexes for subspaces that contain the origin and $L_1 \subseteq L$ the set of indexes for the subspaces that do not contain the origin. It is assumed that $a_l = 0$ for all $l \in L_0$.

For convenient notation, we introduce

$$\overline{A}_l = \begin{bmatrix} A_l & a_l \\ 0 & 1 \end{bmatrix},\ \overline{B}_l = \begin{bmatrix} B_l \\ 0 \end{bmatrix},\ \overline{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \qquad (2.2a)$$

$$\Delta\overline{A}_l = \begin{bmatrix} \Delta A_l & \Delta a_l \\ 0 & 0 \end{bmatrix},\ \Delta\overline{B}_l = \begin{bmatrix} \Delta B_l \\ 0 \end{bmatrix}. \qquad (2.2b)$$

Then using this notation, the system model (2.1) can be expressed as

$$\overline{x}(t+1) = (\overline{A}_l + \Delta\overline{A}_l)\overline{x}(t) + (\overline{B}_l + \Delta\overline{B}_l)u(t),$$
$$x(t) \in S_l \qquad (2.3)$$

For purpose of stability analysis and subsequent use, we introduce the following upper bounds for the uncertainty term of the system (2.1),

$$[\Delta A_l \quad \Delta a]^T[\Delta A_l \quad \Delta a] \le [E_{lA} \quad E_{la}]^T[E_{lA} \quad E_{la}],$$
$$[\Delta B_l]^T[\Delta B_l] \le E_{lB}^T E_{lB}. \qquad (2.4a)$$

Then

$$[\Delta\overline{A}_l]^T[\Delta\overline{A}_l] \le E_{l\overline{A}}^T E_{l\overline{A}} = [E_{lA} \quad E_{la}]^T[E_{lA} \quad E_{la}]$$
$$[\Delta\overline{B}_l]^T[\Delta\overline{B}_l] \le E_{l\overline{B}}^T E_{l\overline{B}} = E_{lB}^T E_{lB}. \qquad (2.4b)$$

As shown in Johansson and Rantzer (1998), we can define the following matrices $\overline{E}$'s for the so-called S-procedure. It is noted (Johansson and Rantzer, 1998; Rantzer ans Johansson, 2000) that these matrices $\overline{E}$'s can be constructed for each cell since they are polyhedra such that

$$\overline{E}_l \overline{x} \ge 0 \qquad (2.5)$$

where $\overline{E}_l = [E_l \quad e_l]$ with $e_l = 0$ for $l \in L_0$. It should be noted that the above vector inequality means that each entry of the vector is nonnegative.

Then we are ready to present the following stability result of the paper (Feng, 2001).

Lemma 2.1 (Feng, 2001): Consider the piecewise linear system without uncertainties (2.1) with $u \equiv 0$. If there exist symmetric positive definite matrices $P_l, l \in L_0, \overline{P}_l, l \in L_1$, symmetric matrices U_l, W_l and Q_{lj} such that U_l, W_l and Q_{lj} have nonnegative entries, and the following LMIs are satisfied,

$$0 < P_l - E_l^T U_l E_l, \quad l \in L_0, \tag{2.6}$$

$$A_l^T P_l A_l - P_l + E_l^T W_l E_l < 0, \; l \in L_0 \tag{2.7}$$

$$0 < \overline{P}_l - \overline{E}_l^T U_l \overline{E}_l, \quad l \in L_1, \tag{2.8}$$

$$\overline{A}_l^T \overline{P}_l \overline{A}_l - \overline{P}_l + \overline{E}_l^T W_l \overline{E}_l < 0, \; l \in L_1, \tag{2.9}$$

$$A_l^T P_j A_l - P_l + E_l^T Q_{lj} E_l < 0, \; l, j \in \Omega \cap L_0, \tag{2.10}$$

$$\overline{A}_l^T \overline{P}_j \overline{A}_l - \overline{P}_l + \overline{E}_l^T Q_{lj} \overline{E}_l < 0, l, j \in \Omega, l \in L_1, \tag{2.11}$$

$$A_l^T \widetilde{P}_j A_l - P_l + E_l^T Q_{lj} E_l < 0, l, j \in \Omega, l \in L_0, \tag{2.12}$$

where we define $\overline{P}_j = [I_{n\times n} \quad 0_{n\times 1}]^T P_j [I_{n\times n} \quad 0_{n\times 1}]$ for $j \in L_0$ in (2.11), and $\widetilde{P}_j = [I_{n\times n} \quad 0_{n\times 1}] \overline{P}_j [I_{n\times n} \quad 0_{n\times 1}]^T$ for $j \in L_1$ in (2.12), then the piecewise linear system is globally exponentially stable, that is, *x(t)* tends to the origin exponentially for every trajectory in the state space.

The above conditions are linear matrix inequalities in the variables $P_l, \overline{P}_l, U_l, W_l$, and Q_{lj}. A solution to those inequalities ensures there exists a Lyapunov function for the system. The LMI in (3.6) or (3.8) for each region guarantees that the function is positive and the LMI in (3.7) or (3.9) guarantees that the function decreases along all system trajectories in each region. The LMIs (3.10)-(3.12) guarantee that the function is decreasing when the state transits from one region to another. The terms involving $E_l, \overline{E}_l, U_l, W_l$, and Q_{lj} are related to the S-procedure to reduce the conservatism of those inequalities.

Remark 2.2: The stability test of the piecewise linear system in eqn. (3.6)-(3.12) can be easily facilitated by a commercially available software package Matlab LMI toolbox (Boyd et al., 1994; Gahinet et al., 1995).

Remark 2.3: The set Ω can be usually determined by all possible combinations of the adjacent or non-adjacent regions. If it is possible for the transitions happen between all regions, then $\Omega = L^2$, which is defined as a set of $\{l, j \mid l, j \in L, j \neq l\}$.

Then based on the lemma 2.1, we can have the following stability result.

Theorem 2.1: Consider the piecewise uncertain linear system (2.1) with $u \equiv 0$. If there exist a set of positive constants $\varepsilon_l, l = 1,2,\cdots,m$, a set of symmetric matrcies $P_l, l \in L_0, \overline{P}_l, l \in L_1$, symmetric matrices U_l, W_l and Q_{lj} such that U_l, W_l and Q_{lj} have nonnegative entries, and the following LMIs are satisfied,

$$0 < P_l - E_l^T U_l E_l, \; l \in L_0 \tag{2.13}$$

$$0 > \begin{bmatrix} A_l^T P_l A_l - P_l + \frac{1}{\varepsilon_l} E_{lA}^T E_{lA} + E_l^T W_l E_l & A_l^T P_l \\ P_l A_l & -(\frac{1}{\varepsilon_l} I - P_l) \end{bmatrix}$$

$$l \in L_0 \tag{2.14}$$

$$0 < \overline{P}_l - \overline{E}_l^T U_l \overline{E}_l, \quad l \in L_1 \tag{2.15}$$

$$0 > \begin{bmatrix} \overline{A}_l^T \overline{P}_l \overline{A}_l - \overline{P}_l + \frac{1}{\varepsilon_l} E_{\overline{l}A}^T E_{\overline{l}A} + \overline{E}_l^T W_l \overline{E}_l & \overline{A}_l^T \overline{P}_l \\ \overline{P}_l \overline{A}_l & -(\frac{1}{\varepsilon_l} I - \overline{P}_l) \end{bmatrix}$$

$$l \in L_1 \tag{2.16}$$

$$0 > \begin{bmatrix} A_l^T P_j A_l - P_l + \frac{1}{\varepsilon_l} E_{lA}^T E_{lA} + E_l^T Q_{lj} E_l & A_l^T P_j \\ P_j A_l & -(\frac{1}{\varepsilon_l} I - P_j) \end{bmatrix}$$

$$l, j \in \Omega \cap L_0 \tag{2.17}$$

$$0 > \begin{bmatrix} \overline{A}_l^T \overline{P}_j \overline{A}_l - \overline{P}_l + \frac{1}{\varepsilon_l} E_{\overline{l}A}^T E_{\overline{l}A} + \overline{E}_l^T Q_{lj} \overline{E}_l & \overline{A}_l^T \overline{P}_j \\ \overline{P}_j \overline{A}_l & -(\frac{1}{\varepsilon_l} I - \overline{P}_j) \end{bmatrix}$$

$$l \in L_1, l, j \in \Omega \tag{2.18}$$

$$0 > \begin{bmatrix} A_l^T \hat{P}_j A_l - P_l + \frac{1}{\varepsilon_l} E_{lA}^T E_{lA} + E_l^T Q_{lj} E_l & A_l^T \hat{P}_j \\ \hat{P}_j A_l & -(\frac{1}{\varepsilon_l} I - \hat{P}_j) \end{bmatrix}$$

$$l \in L_0, \; l, j \in \Omega \tag{2.19}$$

where we define $\overline{P}_j = [I_{n\times n} \quad 0_{n\times 1}]^T P_j [I_{n\times n} \quad 0_{n\times 1}]$ for $j \in L_0$ in (2.18), and $\hat{P}_j = [I_{n\times n} \quad 0_{n\times 1}] \overline{P}_j [I_{n\times n} \quad 0_{n\times 1}]^T$ for $j \in L_1$ in (2.19), then the piecewise uncertain linear system is globally exponentially stable, that is, *x(t)* tends to zero exponentially for every continuous piecewise trajectory in the state space.

Proof: Consider the following Lyapunov function candidate *V(t)*,

$$V(t) = \begin{cases} x^T P_l x, & x \in \overline{S}_l, l \in L_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \overline{P}_l \begin{bmatrix} x \\ 1 \end{bmatrix}, & x \in \overline{S}_l, l \in L_1 \end{cases} \tag{2.20a}$$

or in a more compact form,

$$V(x) = \overline{x}^T \overline{P}_l \overline{x}, \quad x \in \overline{S}_l, \quad l \in L. \tag{2.20b}$$

It is obvious from (2.20) that in an open neighborhood of the origin there exists a constant $\beta > 0$ such that

$$V(t) \le \beta \| x \|^2,$$

since the affine term does not appear in this case. Moreover, (2.13) and (2.15) imply that there exists a constant $\alpha > 0$ such that

$$\alpha \| \bar{x} \|^2 \le \bar{x}^T (\bar{P}_l - \bar{E}_l^T U_l \bar{E}_l)\bar{x} \le \bar{x}^T \bar{P}_l \bar{x}$$

for $x \in \bar{S}_l$. Thus we have,

$$\alpha \| x \|^2 \le V(t) \le \beta \| x \|^2 . \quad (2.21)$$

Using the lemma A.1, we have

$$(\bar{A}_l + \Delta\bar{A}_l)^T \bar{P}_j (\bar{A}_l + \Delta\bar{A}_l) - \bar{P}_l + \bar{E}_l^T W_l \bar{E}_l$$
$$= \bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l + \bar{A}_l^T \bar{P}_j \Delta\bar{A}_l + \Delta\bar{A}_l^T \bar{P}_j \bar{A}_l$$
$$+ \Delta\bar{A}_l^T \bar{P}_j \Delta\bar{A}_l + \bar{E}_l^T W_l \bar{E}_l$$
$$\le \bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l + \bar{A}_l^T \bar{P}_j (\frac{1}{\varepsilon_l} I - \bar{P}_j)^{-1} \bar{P}_j \bar{A}_l$$
$$+ \frac{1}{\varepsilon_l} \Delta\bar{A}_l^T \Delta\bar{A}_l + \bar{E}_l^T W_l \bar{E}_l$$
$$\le \bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l + \bar{A}_l^T \bar{P}_j (\frac{1}{\varepsilon_l} I - \bar{P}_j)^{-1} \bar{P}_j \bar{A}_l$$
$$+ \frac{1}{\varepsilon_l} E_{l\bar{A}}^T E_{l\bar{A}} + \bar{E}_l^T W_l \bar{E}_l \quad (2.22)$$

Then by the Schur complement formula, it follows that (2.14) and (2.16)-(2.19) imply

$$\bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l + \bar{A}_l^T \bar{P}_j (\frac{1}{\varepsilon_l} I - \bar{P}_j)^{-1} \bar{P}_j \bar{A}_l$$
$$+ \frac{1}{\varepsilon_l} E_{l\bar{A}}^T E_{l\bar{A}} + \bar{E}_l^T W_l \bar{E}_l < 0 \quad (2.23)$$

Thus it follows from (2.22) and (2.23) that

$$(\bar{A}_l + \Delta\bar{A}_l)^T \bar{P}_j (\bar{A}_l + \Delta\bar{A}_l) - \bar{P}_l + \bar{E}_l^T W_l \bar{E}_l < 0,$$

which in turn implies that there exists a constant $\rho > 0$ such that

$$(\bar{A}_l + \Delta\bar{A}_l)^T \bar{P}_l (\bar{A}_l + \Delta\bar{A}_l) - \bar{P}_l + \rho I < 0 . \quad (2.24)$$

Then along trajectories of the system, we have

$$\Delta V(t) = \bar{x}^T ((\bar{A}_l + \Delta\bar{A}_l)^T \bar{P}_l (\bar{A}_l + \Delta\bar{A}_l) - \bar{P}_l)\bar{x}$$
$$\le \bar{x}^T (-\rho I)\bar{x} \quad (2.25)$$
$$\le -\rho \| x \|^2$$

Therefore, the desired result follows directly from (2.21) and (2.25) based on the standard Lyapunov theory. $\nabla\nabla$

3. CONTROLLER SYNTHESIS OF PIECEWISE LINEAR SYSTEMS

In this section, we will address the controller synthesis problem for the discrete time piecewise linear systems introduced in the last section. Consider the piecewise system model (2.1) on every subspace,

$$x(t+1) = (A_l + \Delta A_l)x(t) + (B_l + \Delta B_l)u(t) + a_l + \Delta a_l,$$
$$x(t) \in S_l \quad (3.1)$$

or in more compact form,

$$\bar{x}(t+1) = (\bar{A}_l + \Delta\bar{A}_l)\bar{x}(t) + (\bar{B}_l + \Delta\bar{B}_l)u(t),\ x(t) \in S_l \quad (3.2)$$

For the stabilization of the system (3.1) or equivalently (3.2), we consider the following piecewise continuous controller as

$$u(t) = \begin{cases} K_l x(t) & x(t) \in S_l,\ l \in L_0 \\ K_l \bar{x}(t) & x(t) \in S_l \quad l \in L_1 \end{cases} \quad (3.3)$$

With the control law (3.3), the global closed loop system is obtained by combining the system (3.2) and the controller (3.3), and can be described by the following equation,

$$\bar{x}(t+1) = A_{cl}\bar{x}(t), \quad x(t) \in S_l \quad (3.4)$$

where

$$A_{cl} = \bar{A}_l + \Delta\bar{A}_l + (\bar{B}_l + \Delta\bar{B}_l)K_l . \quad (3.5)$$

Then we have the following result.

Theorem 3.1: The system (3.4) is globally stable, if there exist a set of positive constants $\varepsilon_l, l = 1,2,\cdots,m$, a set of symmetric matrcies $P_l, l \in L_0, \bar{P}_l, l \in L_1$, symmetric matrices U_l, W_l and Q_{lj} such that U_l, W_l and Q_{lj} have nonnegeative entries, and the following LMIs are satisfied,

$$0 < P_l - E_l^T U_l E_l, \quad l \in L_0 \quad (3.6)$$

$$0 > \begin{bmatrix} A_l^T P_l A_l - P_l + \frac{2}{\varepsilon_l} E_{lA}^T E_{lA} + E_l^T W_l E_l & A_l^T P_l \tilde{B}_l \\ \tilde{B}_l^T P_l A_l & -(R_l - \tilde{B}_l^T P_l \tilde{B}_l) \end{bmatrix}$$
$$l \in L_0 \quad (3.7)$$

$$0 < P_l - E_l^T U_l E_l, \quad l \in L_1 \quad (3.8)$$

$$0 > \begin{bmatrix} \bar{A}_l^T \bar{P}_l \bar{A}_l - \bar{P}_l + \frac{2}{\varepsilon_l} E_{l\bar{A}}^T E_{l\bar{A}} + \bar{E}_l^T W_l \bar{E}_l & \bar{A}_l^T \bar{P}_l \tilde{\bar{B}}_l \\ \tilde{\bar{B}}_l^T \bar{P}_l \bar{A}_l & -(\bar{R}_l - \tilde{\bar{B}}_l^T \bar{P}_l \tilde{\bar{B}}_l) \end{bmatrix}$$
$$l \in L_1 \quad (3.9)$$

$$0 > \begin{bmatrix} A_l^T P_j A_l - P_l + \frac{2}{\varepsilon_l} E_{lA}^T E_{lA} + E_l^T Q_{lj} E_l & A_l^T P_j \tilde{B}_l \\ \tilde{B}_l^T P_j A_l & -(R_l - \tilde{B}_l^T P_j \tilde{B}_l) \end{bmatrix}$$
$$l, j \in \Omega \cap L_0 \quad (3.10)$$

$$0 > \begin{bmatrix} \bar{A}_l^T \bar{P}_j \bar{A}_l - \bar{P}_l + \frac{2}{\varepsilon_l} E_{l\bar{A}}^T E_{l\bar{A}} + \bar{E}_l^T Q_{lj} \bar{E}_l & \bar{A}_l^T \bar{P}_j \tilde{\bar{B}}_l \\ \tilde{\bar{B}}_l^T \bar{P}_j \bar{A}_l & -(\bar{R}_l - \tilde{\bar{B}}_l^T \bar{P}_j \tilde{\bar{B}}_l) \end{bmatrix}$$
$$l \in L_1, l, j \in \Omega \quad (3.11)$$

$$0 > \begin{bmatrix} A_l^T \hat{P}_j A_l - P_l + \frac{2}{\varepsilon_l} E_{lA}^T E_{lA} + E_l^T Q_{lj} E_l & A_l^T \hat{P}_j \tilde{B}_l \\ \tilde{B}_l^T \hat{P}_j A_l & -(R_l - \tilde{B}_l^T \hat{P}_j \tilde{B}_l) \end{bmatrix}$$
$$l \in L_0,\ l, j \in \Omega \quad (3.12)$$

where we define $\overline{P}_j = [I_{n\times n} \quad 0_{n\times 1}]^T P_j [I_{n\times n} \quad 0_{n\times 1}]$ for $j \in L_0$ in (3.11), and $\hat{P}_j = [I_{n\times n} \quad 0_{n\times 1}]\overline{P}_j [I_{n\times n} \quad 0_{n\times 1}]^T$ for $j \in L_1$ in (3.12),

$$R_l = \begin{bmatrix} \alpha I + \frac{2}{\varepsilon_l} E_{lB}^T E_{lB} & 0 \\ 0 & (1/\varepsilon_l) I \end{bmatrix}, \ \widetilde{B}_l = [B_l \quad I],$$

$$\overline{R}_l = \begin{bmatrix} \alpha I + \frac{2}{\varepsilon_l} E_{l\overline{B}}^T E_{l\overline{B}} & 0 \\ 0 & (1/\varepsilon_l) I \end{bmatrix}, \ \widetilde{\overline{B}}_l = [\overline{B}_l \quad I],$$

and $\alpha > 0$ is any constant large enough such that the matrices $R_l\,(\overline{R}_l)$ are positive definte.

Moreover, the controller gain for each local subsystem is given by

$$K_l = \begin{cases} -[\alpha I + \frac{2}{\varepsilon_l} E_{lB}^T E_{lB} + B_l^T \widetilde{P}_l B_l]^{-1} B_l^T \widetilde{P}_l A_l, & l \in L_0 \\ -[\alpha I + \frac{2}{\varepsilon_l} E_{l\overline{B}}^T E_{l\overline{B}} + \overline{B}_l^T \widetilde{\overline{P}}_l \overline{B}_l]^{-1} \overline{B}_l^T \widetilde{\overline{P}}_l \overline{A}_l, & l \in L_1 \end{cases} \tag{3.13}$$

and

$$K_l = \begin{cases} -[\alpha I + \frac{2}{\varepsilon_l} E_{lB}^T E_{lB} + B_l^T \widetilde{P}_j B_l]^{-1} B_l^T \widetilde{P}_j A_l, & l, j \in \Omega, l \in L_0 \\ -[\alpha I + \frac{2}{\varepsilon_l} E_{l\overline{B}}^T E_{l\overline{B}} + \overline{B}_l^T \widetilde{\overline{P}}_j \overline{B}_l]^{-1} \overline{B}_l^T \widetilde{\overline{P}}_j \overline{A}_l, & l, j \in \Omega, l \in L_1 \end{cases} \tag{3.14}$$

where $\widetilde{P}_l = (P_l^{-1} - \varepsilon_l I)^{-1}$, and $\widetilde{\overline{P}}_l = (\overline{P}_l^{-1} - \varepsilon_l I)^{-1}$.

Proof: Based on the result in Theorem 2.2, we learn that the system (3.4) is globally stable if there exist a set of symmetric matrices $P_l, l \in L_0, \overline{P}_l, l \in L_1$, and symmetric matrices U_l, W_l and Q_{lj} such that U_l, W_l and Q_{lj} have nonnegeative entries, P_l and $\overline{P}_l$ satisfy the inequality (3.6) and (3.8) respectively, and the following inequalities are satisfied,

$$A_{cl}^T P_l A_{cl} - P_l + E_l^T W_l E_l < 0, \ l \in L_0 \tag{3.15}$$

$$\overline{A}_{cl}^T \overline{P}_l \overline{A}_{cl} - \overline{P}_l + \overline{E}_l^T W_l \overline{E}_l < 0, l \in L_1 \tag{3.16}$$

$$A_{cl}^T P_j A_{cl} - P_l + E_l^T Q_{lj} E_l < 0, l, j \in \Omega \cap L_0, \tag{3.17}$$

$$\overline{A}_{cl}^T \overline{P}_j \overline{A}_{cl} - \overline{P}_l + \overline{E}_l^T Q_{lj} \overline{E}_l < 0, l, j \in \Omega, l \in L_1, \tag{3.18}$$

$$A_{cl}^T \hat{P}_j A_{cl} - P_l + E_l^T Q_{lj} E_l < 0, l, j \in \Omega, l \in L_0. \tag{3.19}$$

We will first show that the inequality (3.7) implies (3.15). Using Lemma A.1, the left hand side of inequality (3.15) can be expressed as,

$$\begin{aligned} LH &:= A_{cl}^T P_l A_{cl} - P_l + E_l^T W_l E_l \\ &= [A_l + \Delta A_l + (B_l + \Delta B_l) K_l]^T P_l [A_l + \Delta A_l + (B_l + \Delta B_l) K_l] \\ &\quad - P_l + E_l^T W_l E_l \\ &= (A_l + B_l K_l)^T P_l (A_l + B_l K_l) \\ &\quad + (A_l + B_l K_l)^T P_l (\Delta A_l + \Delta B_l K_l) \\ &\quad + (\Delta A_l + \Delta B_l K_l)^T P_l (A_l + B_l K_l) - P_l + E_l^T W_l E_l \\ &\le (A_l + B_l K_l)^T P_l (A_l + B_l K_l) \\ &\quad + (A_l + B_l K_l)^T P_l (\frac{1}{\varepsilon_l} I - P_l)^{-1} P_l (A_l + B_l K_l) \\ &\quad + \frac{1}{\varepsilon} (\Delta A_l + \Delta B_l K_l)^T (\Delta A_l + \Delta B_l K_l) - P_l + E_l^T W_l E_l \\ &\le (A_l + B_l K_l)^T (P_l + P_l (\frac{1}{\varepsilon_l} I - P_l)^{-1} P_l)(A_l + B_l K_l) \\ &\quad + \frac{2}{\varepsilon_l} \Delta A_l^T \Delta A_l + \frac{2}{\varepsilon_l} K_l^T \Delta B_l^T \Delta B_l K_l - P_l + E_l^T W_l E_l \\ &\le (A_l + B_l K_l)^T (P_l^{-1} - \varepsilon_l I)^{-1} (A_l + B_l K_l) \\ &\quad + \frac{2}{\varepsilon_l} E_{lA}^T E_{lA} + \frac{2}{\varepsilon_l} K_l^T E_{lB}^T E_{lB} K_l - P_l + E_l^T W_l E_l \\ &\le (A_l + B_l K_l)^T (P_l^{-1} - \varepsilon_l I)^{-1} (A_l + B_l K_l) \\ &\quad + \frac{2}{\varepsilon_l} E_{lA}^T E_{lA} + K_l^T (\alpha I + \frac{2}{\varepsilon_l} E_{lB}^T E_{lB}) K_l - P_l + E_l^T W_l E_l \end{aligned} \tag{3.20}$$

Let

$$K_l = -[\alpha I + \frac{2}{\varepsilon_l} E_{lB}^T E_{lB} + B_l^T \widetilde{P}_l B_l]^{-1} B_l^T \widetilde{P}_l A_l. \tag{3.21}$$

Substituting (3.21) into (3.20) and using the matrix inversion lemma, one has

$$\begin{aligned} LH &\le A_l^T \widetilde{P}_l A_l + A_l^T \widetilde{P}_l B_l (\alpha I + \frac{2}{\varepsilon_l} E_{lB}^T E_{lB} + B_l^T \widetilde{P}_l B_l)^{-1} B_l^T \widetilde{P}_l A_l \\ &\quad + \frac{2}{\varepsilon_l} E_{lA}^T E_{lA} - P_l + E_l^T W_l E_l \\ &= A_l^T [\widetilde{P}_l^{-1} - B_l (\alpha I + \frac{2}{\varepsilon_l} E_{lB}^T E_{lB})^{-1} B_l^T]^{-1} A_l \\ &\quad + \frac{2}{\varepsilon_l} E_{lA}^T E_{lA} - P_l + E_l^T W_l E_l \\ &= A_l^T [P_l^{-1} - \varepsilon_l I - B_l (\alpha I + \frac{2}{\varepsilon_l} E_{lB}^T E_{lB})^{-1} B_l^T]^{-1} A_l \\ &\quad + \frac{2}{\varepsilon_l} E_{lA}^T E_{lA} - P_l + E_l^T W_l E_l \\ &= A_l^T [P_l^{-1} - \widetilde{B}_l R_l^{-1} \widetilde{B}_l^T]^{-1} A_l \\ &\quad + \frac{2}{\varepsilon_l} E_{lA}^T E_{lA} - P_l + E_l^T W_l E_l \\ &= A_l^T P_l A_l + A_l^T P_l \widetilde{B}_l (R_l - \widetilde{B}_l^T P_l \widetilde{B}_l)^{-1} \widetilde{B}_l^T P_l A_l \\ &\quad + \frac{2}{\varepsilon_l} E_{lA}^T E_{lA} - P_l + E_l^T W_l E_l \end{aligned} \tag{3.22}$$

It then can be easily seen that the following inequality,

$$\begin{aligned} &A_l^T P_l A_l + A_l^T P_l \widetilde{B}_l (R_l - \widetilde{B}_l^T P_l \widetilde{B}_l)^{-1} \widetilde{B}_l^T P_l A_l \\ &+ \frac{2}{\varepsilon_l} E_{lA}^T E_{lA} - P_l + E_l^T W_l E_l < 0 \end{aligned} \tag{3.23}$$

implies (3.15). Using Schur complement formulas, it is easily shown that the inequality (3.23) is in turn equivalent to the linear matrix inequality (3.7). Thus, we have shown that the inequality (3.7) implies (3.15). Following the above procedure, it can also be shown that the inequality (3.9)-(3.12) implies the inequality (3.16)-(3.19) respectively. Therefore, it can be concluded from the Theorem 2.2 that the closed loop control system is globally stable and thus the proof is completed. ∇∇

Based on the above theorem, the following algorithm can be developed.

Algorithm 1:

Step 1. Set $\varepsilon_l, l = 1,2,...,m$ to small constants, say $\varepsilon_l = 1, l = 1,2,...,m$.
Step 2. Solve the linear matrix inequalities (3.6)-(3.12) for a set of positive definite matrices $P_l, l \in L_0, \overline{P}_l, l \in L_1$. This can be facilitated by using the Matlab *LMI toolbox* (Gahinet et al., 1995).
Step 3. If the solutions are found, the controller parameters can be obtained by (3.13) and (3.14), and then stop. Otherwise, set $\varepsilon_l = \varepsilon_l / 2$, for those inequalities having no solution, and check whether $\varepsilon_l, l = 1,2,...,m$, are greater than some given threshold. If it is the case, then go back to step 2. Otherwise, claim the present controller design fails.

4. CONCLUSIONS

In this paper, a new stability result has been developed for piecewise uncertain discrete time linear systems based on a piecewise Lyapunov function, and then a stabilization controller design method has also been developed. It has been shown that the stability test can be accomplished by checking a set of LMIs and the controller gains can also be determined by solving a set of LMIs. Finally a constructive algorithm for the controller design is also given.

APPENDIX

Lemma A.1(Garcia et al., 1994): Let A and E be matrices of appropriate dimensions, and P be a positive-definite symmetric matrix satisfying

$$\frac{1}{\varepsilon} I - P > 0, \quad \varepsilon > 0,$$

then

$$A^T PE + E^T PA + E^T PE \le A^T P(\frac{1}{\varepsilon} I - P)^{-1} PA + \frac{1}{\varepsilon} E^T E .$$

REFERENCES

Banks, S.P. and S.A. Khathur (1989). Structure and control of piecewise-linear systems. *Int. J. Contr.*, **50**, 667-686.

Body, S., L. Ghaoui, E. Feron, and V. Balakrishnan (1994). *Linear matrix inequalities in systems and control theory.* PA: SIAM, Philadelphia.

Chua, L. and A. Deng (1986). Canonical piecewise-linear modelling. *IEEE Trans. Circuits and Systems,* **33**, 511-525.

Chua, L. and R. Ying (1983). Canonical piecewise-linear analysis. *IEEE Trans. Circuits and Systems,* **30**, 125-140.

Feng, G. (2001). Stability analysis of piecewise discrete time linear systems. Submitted to *IEEE Trans. Automat. Contr.*.

Gahinet, P., A. Nemirovski, A. Laub, and M. Chilali (1995). *The LMI Control Toolbox*. The Mathworks, Inc..

Garcia, G., J. Bernussou, and D. Arzelier (1994). Robust stabilization of discrete time linear systems with norm bounded time-varying uncertainty. *Syste. Contr. Lett.*, **22**, 327-339.

Hassibi, A. and S. Boyd (1998). Qradratic stabilization and control of piecewise-linear systems. in *Proc. American Control Conf.*, Philadelphia, 3659-3664.

Imura, J.I. and A. van der Schaft (2000). Characterization of well-posedness of piecewise-linear systems. *IEEE Trans. Automat. Contr.*, **45**, 1600-1619.

Johansson, M. and A. Rantzer (1998). Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Trans. Automat. Contr.*, **43**, 555-559.

Kantner, M. (1997). Robust stability of piecewise linear discrete time systems. in *Proc. American Control Conf.*, Albuquerque, 1241-1245.

Pettit, N.B.O. and P.E. Wellstead (1995). Analyzing piecewise linear dynamical systems. *IEEE Control Systems Magazine,* **15**, 43-50.

Rantzer, A. and M. Johansson (2000). Piecewise linear quadratic optimal control. *IEEE Trans. Automat. Contr.*, **45**, 629-637.

Wicks, M.A., P. Peleties, and R.A. DeCarlo (1994). Construction of piecewise Lyapunov functions for stabilizing switched systems. in *Proc. 33rd IEEE Conf. on Decision and Control*, Lake Buena Vista, FL, 3492-3497.

Yfoulis, C.A., A. Muir, N.B.O.L. Pettit, P.E. Wellstead (1998). Stabilization of orthogonal piecewise linear systems using piecewise Lyapunov-like functions. in *Proc. 37th IEEE Conf. on Decision and Control*, Tampa, 1476-1481.

Sontag, E.D. (1981). Nonlinear regulation: the piecewise linear approach. *IEEE Trans. Automat. Contr.,* **26**, 346-357.