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MPC FOR CONTINUOUS PIECEWISE-AFFINE SYSTEMS

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Abstract: First we show that continuous piecewise-affine systems are equivalent to max-min-plus-scaling systems (i.e., systems that can be modeled using maximization, minimization, addition and scalar multiplication). Next, we consider model predictive control for these systems. In general, this leads to nonlinear non-convex optimization problems. However, we present a method based on canonical forms for max-min-plus-scaling functions to solve these optimization problems in a more efficient way than by just applying nonlinear optimization as was done in previous research.

Keywords: generalized predictive control, control algorithms, model-based control, piecewise linear analysis, discrete event dynamic systems, hybrid systems

1. INTRODUCTION

In our previous work (De Schutter and van den Boom, 2001*b*) we have extended the model predictive control (MPC) framework to max-min-plus-scaling (MMPS) systems. MMPS systems are systems that can be modeled using maximization, minimization, addition and scalar multiplication. Typical examples of MMPS systems in a discrete event systems context are digital circuits, computer networks, telecommunication networks, and manufacturing plants. In (De Schutter and van den Boom, 2000) we have shown that this class encompasses several other classes of discrete event systems such as max-plus-linear systems, max-plus-bilinear systems, max-plus-polynomial systems, and max-min systems. So MMPS systems can be considered as a generalized framework for several classes of discrete event systems. Moreover, recently a link between *constrained* MMPS systems and hybrid systems — among which piecewise-affine (PWA) systems — has been established (Heemels *et al.*, 2001*a*; Heemels *et al.*, 2001*b*).

In this paper we will present a direct connection between *continuous* PWA systems and MMPS systems (without the need to introduce additional auxiliary variables or extra constraints as was done in (Heemels *et al.*, 2001*a*; Heemels *et al.*, 2001*b*)). Next, we use the link between PWA systems and MMPS systems to present a new approach to MPC for continuous PWA systems. In order to compute an MPC controller for a PWA system or for an MMPS system we have to solve a nonlinear non-convex optimization problem at each sample step. We propose an optimization algorithm that is based on canonical forms for MMPS functions and that is similar to the cutting-plane algorithm for convex optimization problems. The proposed algorithm consists in solving several linear programming problems and is more efficient than the algorithms used in (De Schutter and van den Boom, 2001*b*), which are based on multi-start nonlinear local optimization (sequential quadratic programming) or on the extended linear complementarity problem.

This paper is organized as follows. In Section 2 we present MMPS functions and systems, and PWA function and systems. We also discuss the connection between continuous PWA systems and (unconstrained) MMPS systems. Next, we consider canonical forms for MMPS functions in Section 3. Section 4 briefly recapitulates our previous results in connection with MPC for MMPS systems. Due to the link between PWA and MMPS systems, this approach can also be used for continuous PWA systems. Finally, we present an efficient algorithm to solve the MMPS-MPC and PWA-MPC optimization problems.

2. CONTINUOUS PWA SYSTEMS AND MMPS SYSTEMS

2.1 *MMPS functions and systems*

An MMPS function f of the variables $x_1,\ldots,x_n$ is defined by the recursive grammar [1]

$$f := x_i|\alpha|\max(f_k,f_l)|\min(f_k,f_l)|f_k+f_l|\beta f_k \ , \quad (1)$$

with $i\in\{1,\ldots,n\}$, $\alpha,\beta\in\mathbb{R}$, and where f_k and f_l are again MMPS functions.

Now we consider systems that can be described by state space equations of the following form:

$$\mathbf{x}(k) = \mathscr{M}_\mathbf{x}(\mathbf{x}(k-1),\mathbf{u}(k)) \quad (2)$$
$$\mathbf{y}(k) = \mathscr{M}_\mathbf{y}(\mathbf{x}(k),\mathbf{u}(k)) \ , \quad (3)$$

where $\mathscr{M}_\mathbf{x}$ and $\mathscr{M}_\mathbf{y}$ are vector-valued MMPS functions. Systems the behavior of which can be described by a model of the form (2)–(3) will be called *MMPS systems*.

2.2 *PWA functions and systems*

A function $f:\mathbb{R}^n\to\mathbb{R}$ is said to be a continuous PWA function if and only if the following conditions hold (Chua and Deng, 1988):

(1) The domain space $\mathbb{R}^n$ is divided into a finite number of polyhedral regions $R_{(1)},\ldots,R_{(N)}$.
(2) For each $i\in\{1,\ldots,N\}$, f can be expressed as
$$f(\mathbf{x}) = \boldsymbol{\alpha}_{(i)}^T\mathbf{x}+\beta_{(i)} \quad (4)$$
for any $\mathbf{x}\in R_{(i)}$ with $\boldsymbol{\alpha}_{(i)}\in\mathbb{R}^n$ and $\beta_{(i)}\in\mathbb{R}$.
(3) f is continuous on any boundary between two regions.

For more information on PWA functions we refer to (Chua and Deng, 1988; Leenaerts and van Bokhoven, 1998) and the references therein.

A PWA system is a system of the form

$$\mathbf{x}(k) = \mathscr{P}_\mathbf{x}(\mathbf{x}(k-1),\mathbf{u}(k)) \quad (5)$$
$$\mathbf{y}(k) = \mathscr{P}_\mathbf{y}(\mathbf{x}(k),\mathbf{u}(k)) \ , \quad (6)$$

where $\mathscr{P}_\mathbf{x}$ and $\mathscr{P}_\mathbf{y}$ are vector-valued PWA functions. A model of the form (5)–(6) is called a PWA model. If $\mathscr{P}_\mathbf{x}$ and $\mathscr{P}_\mathbf{y}$ are continuous, then we say that the model is a continuous PWA model.

Note that continuous PWA models can also be used as approximations of more general state space models of the form

$$\mathbf{x}(k) = \mathbf{f}(\mathbf{x}(k-1),\mathbf{u}(k))$$
$$\mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k),\mathbf{u}(k)) \ ,$$

with **f** and **g** continuous functions.

[1] The symbol | stands for "or".

2.3 *Equivalence of PWA and MMPS systems*

Theorem 1. If f is a continuous PWA function of the form (4), then there exist index sets $I_1,\ldots,I_\ell\subseteq\{1,\ldots,N\}$ such that

$$f = \max_{j=1,\ldots,\ell}\ \min_{i\in I_j}\ (\boldsymbol{\alpha}_{(i)}^T\mathbf{x}+\beta_{(i)}) \ .$$

PROOF. See (Gorokhovik and Zorko, 1994; Ovchinnikov, 2001). □

From the definition of MMPS functions it follows that (see also (Gorokhovik and Zorko, 1994; Ovchinnikov, 2001)):

Lemma 2. Any MMPS function is also a continuous PWA function.

From Theorem 1 and Lemma 2 it follows that continuous PWA systems and MMPS systems are equivalent, i.e., for a given continuous PWA model there exists an MMPS model (and vice versa) such that the input-output behavior of both models coincide.

Corollary 3. Continuous PWA models and MMPS models are equivalent.

Note that this is an extension of the results of (Heemels *et al.*, 2001*a*; Heemels *et al.*, 2001*b*), which already prove an equivalence between (not necessarily continuous) PWA models and MMPS models, but there some extra auxiliary variables and some additional algebraic MMPS constraints between the states, the inputs and the auxiliary variables were required to transform the PWA model into an MMPS model.

3. CANONICAL FORMS OF MMPS FUNCTIONS

Let $\alpha,\beta,\gamma,\delta\in\mathbb{R}$. Now we consider some easily verifiable properties of the max and min operators that will be used in the proof of the main theorem of this section.

- Minimization is distributive [2] w.r.t. maximization, i.e., $\min\big(\alpha,\max(\beta,\gamma)\big)=\max\big(\min(\alpha,\beta),\min(\alpha,\gamma)\big)$, which results in:
$$\min\big(\max(\alpha,\beta),\max(\gamma,\delta)\big) = \max\big(\min(\alpha,\gamma),\min(\alpha,\delta),\min(\beta,\gamma),\min(\beta,\delta)\big) \ . \quad (7)$$
- The max operation is distributive w.r.t. min. Hence,

[2] If we use the operator symbols $\vee$ and $\wedge$ to denote max and min respectively, this distributivity property can be written as $\alpha\wedge(\beta\vee\gamma)=(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$.

$$\max\big(\min(\alpha,\beta),\min(\gamma,\delta)\big) = \min\big(\max(\alpha,\gamma),\max(\alpha,\delta), \max(\beta,\gamma),\max(\beta,\delta)\big) \ . \quad (8)$$

- We have

$$\min(\alpha,\beta)+\min(\gamma,\delta) = \min(\alpha+\gamma,\alpha+\delta,\beta+\gamma,\beta+\delta) \quad (9)$$

and

$$\max(\alpha,\beta)+\max(\gamma,\delta) = \max(\alpha+\gamma,\alpha+\delta,\beta+\gamma,\beta+\delta) \ . \quad (10)$$

- The min and max operators are related as follows:

$$\max(\alpha,\beta) = -\min(-\alpha,-\beta) \ . \quad (11)$$

- If $\rho \in \mathbb{R}$ is positive, then

$$\rho\max(\alpha,\beta) = \max(\rho\alpha,\rho\beta) \quad (12)$$
$$\rho\min(\alpha,\beta) = \min(\rho\alpha,\rho\beta) \ . \quad (13)$$

Theorem 4. Any MMPS function $f : \mathbb{R}^n \to \mathbb{R}$ can be rewritten in the min-max canonical form

$$f = \min_{i=1,\dots,K}\max_{j=1,\dots,n_i}(\boldsymbol{\alpha}_{(i,j)}^T\mathbf{x}+\beta_{(i,j)}) \quad (14)$$

or in the max-min canonical form

$$f = \max_{i=1,\dots,L}\min_{j=1,\dots,m_i}(\boldsymbol{\gamma}_{(i,j)}^T\mathbf{x}+\delta_{(i,j)}) \quad (15)$$

for some integers K, L, $n_1,\dots,n_K$, $m_1,\dots,m_L$, vectors $\boldsymbol{\alpha}_{(i,j)}, \boldsymbol{\gamma}_{(i,j)}$, and real numbers $\beta_{(i,j)}, \delta_{(i,j)}$.

PROOF. We will only prove the theorem for the min-max canonical form since the proof for the max-min canonical form is similar.
It is easy to verify that if f_k and f_l are affine functions, then the functions that results from applying the basic constructors of an MMPS function (max, min, +, and scaling — cf. (1)) are in min-max canonical form [3].
Now we use a recursive argument that consists in showing that if we apply the basic constructors of an MMPS function to two (or more) MMPS functions in min-max canonical form, then the result can again be transformed into min-max canonical form. Consider two MMPS functions f and g in min-max canonical form [4]: $f = \min(\max(f_1, f_2),\max(f_3, f_4))$ and $g = \min(\max(g_1,g_2),\max(g_3,g_4))$. In Table 1 it is shown that $\max(f,g)$, $\min(f,g)$, $f+g$ and βf can again be written in min-max canonical form. □

Remark 5. The min-max canonical form (14) is sometimes also called conjunctive normal form, and the max-min canonical form (15) is also called disjunctive normal form.

[3] We allow "void" min or max statements of the form $\min(s)$ or $\max(s)$, which by definition are equal to s for any expression s. Alternatively, we can write $\min(s,s)$ or $\max(s,s)$.

[4] For the sake of simplicity we only consider two min-terms in f and g, each of which consists of the maximum of two affine functions. However, the proof also holds if more terms are considered.

4. MPC FOR MMPS SYSTEMS

In this section we give a short overview of the main results of (De Schutter and van den Boom, 2001*b*) in which we have extended the MPC framework to MMPS systems. Related results can be found in (Bemporad and Morari, 1999). More extensive information on conventional MPC for (linear and nonlinear) discrete-time systems can be found in (Camacho and Bordons, 1995; García and Lee, 2000; Maciejowski, 2002) and the references therein.

We can use the deterministic model (2)–(3) either as a model of an MMPS system, as the equivalent model of a continuous PWA system, or as an approximation of a general smooth nonlinear system. Note that we do not include modeling errors or uncertainty in the model. However, since MPC uses a receding finite horizon approach, we can regularly update the model and the state estimate as new information and measurements become available.

In MPC we compute at each sample step k an optimal control input that minimizes a cost criterion over the period $[k, k+N_\mathrm{p}-1]$ where N_p is the prediction horizon. Assume that at sample step k the current state can be measured, estimated or predicted using previous measurements. Then we can make an estimate $\hat{\mathbf{y}}(k+j|k)$ of the output of the system (2)–(3) at sample step $k+j$ based on the state $\mathbf{x}(k-1)$ and the future inputs $\mathbf{u}(k+i)$, $i = 0,\dots,j$. Using successive substitution, we obtain an expression of the following form:

$$\hat{\mathbf{y}}(k+j|k) = \mathbf{F}_j(\mathbf{x}(k-1),\mathbf{u}(k),\dots,\mathbf{u}(k+j))$$

for $j = 0,\dots,N_\mathrm{p}-1$. Clearly, $\hat{\mathbf{y}}(k+j|k)$ is an MMPS function of $\mathbf{x}(k-1),\mathbf{u}(k),\dots,\mathbf{u}(k+j)$.

The cost criterion J used in MPC reflects the reference tracking error (J_out) and the control effort (J_in):

$$J(k) = J_\mathrm{out}(k)+\lambda J_\mathrm{in}(k)$$

where λ is a nonnegative real number. Let r contain the reference signal and define the vectors

$$\tilde{\mathbf{u}}(k) = \big[\mathbf{u}^T(k) \ \dots \ \mathbf{u}^T(k+N_\mathrm{p}-1)\big]^T$$
$$\tilde{\mathbf{y}}(k) = \big[\hat{\mathbf{y}}^T(k|k) \ \dots \ \hat{\mathbf{y}}^T(k+N_\mathrm{p}-1|k)\big]^T$$
$$\tilde{\mathbf{r}}(k) = \big[\mathbf{r}^T(k) \ \dots \ \mathbf{r}^T(k+N_\mathrm{p}-1)\big]^T \ .$$

In this paper we consider the following output and input cost functions [5]:

$$J_{\mathrm{out},1}(k) = \|\tilde{\mathbf{y}}(k)-\tilde{\mathbf{r}}(k)\|_1 \quad (16)$$
$$J_{\mathrm{out},\infty}(k) = \|\tilde{\mathbf{y}}(k)-\tilde{\mathbf{r}}(k)\|_\infty \quad (17)$$
$$J_{\mathrm{in},1}(k) = \|\tilde{\mathbf{u}}(k)\|_1 \quad (18)$$
$$J_{\mathrm{in},\infty}(k) = \|\tilde{\mathbf{u}}(k)\|_\infty \ . \quad (19)$$

[5] In conventional MPC usually quadratic cost functions of the form $J_\mathrm{out}(k) = \|\tilde{\mathbf{y}}(k)-\tilde{\mathbf{r}}(k)\|_2^2$ and $J_\mathrm{in}(k) = \|\tilde{\mathbf{u}}(k)\|_2^2$ are used. In a discrete event context, however, other choices are more appropriate (see (De Schutter and van den Boom, 2001*a*; De Schutter and van den Boom, 2001*b*)).

Table 1. The max, min, $+$ and scaling of two MMPS functions in min-max canonical form can again be written in min-max canonical form.

- $\max(f,g) = \max\big[\min\big(\max(f_1,f_2),\max(f_3,f_4)\big),\ \min\big(\max(g_1,g_2),\max(g_3,g_4)\big)\big]$

$$\begin{aligned}
&= \max\big[\max\big(\min(f_1,f_3),\min(f_1,f_4),\min(f_2,f_3),\min(f_2,f_4)\big),\\
&\qquad \max\big(\min(g_1,g_3),\min(g_1,g_4),\min(g_2,g_3),\min(g_2,g_4)\big)\big] \qquad \text{(by (7))}\\
&= \max\big(\min(f_1,f_3),\min(f_1,f_4),\min(f_2,f_3),\min(f_2,f_4),\\
&\qquad \min(g_1,g_3),\min(g_1,g_4),\min(g_2,g_3),\min(g_2,g_4)\big)\\
&= \min\big(\max(f_1,f_1,f_2,f_2,g_1,g_1,g_2,g_2),\max(f_1,f_1,f_2,f_2,g_1,g_1,g_2,g_4),\ldots\\
&\qquad \max(f_3,f_4,f_3,f_4,g_3,g_4,g_3,g_4)\big) \qquad \text{(since max is distributive w.r.t. min)}
\end{aligned}$$

- $\min(f,g) = \min\big[\min\big(\max(f_1,f_2),\max(f_3,f_4)\big),\ \min\big(\max(g_1,g_2),\max(g_3,g_4)\big)\big]$

$$= \min\big(\max(f_1,f_2),\max(f_3,f_4),\max(g_1,g_2),\max(g_3,g_4)\big)$$

- $f+g = \min\big(\max(f_1,f_2),\max(f_3,f_4)\big) + \min\big(\max(g_1,g_2),\max(g_3,g_4)\big)$

$$\begin{aligned}
&= \min\big(\max(f_1,f_2)+\max(g_1,g_2),\max(f_1,f_2)+\max(g_3,g_4),\\
&\qquad \max(f_3,f_4)+\max(g_1,g_2),\max(f_3,f_4)+\max(g_3,g_4)\big) \qquad \text{(by (9))}\\
&= \min\big(\max(f_1+g_1,f_1+g_2,f_2+g_1,f_2+g_2),\\
&\qquad \max(f_1+g_3,f_1+g_4,f_2+g_3,f_2+g_4),\\
&\qquad \max(f_3+g_1,f_3+g_2,f_4+g_1,f_4+g_2),\\
&\qquad \max(f_3+g_3,f_3+g_4,f_4+g_3,f_4+g_4)\big) \qquad \text{(by (10))}
\end{aligned}$$

- $\beta f = \beta \min\big(\max(f_1,f_2),\max(f_3,f_4)\big)$

$$= \begin{cases}
\min\big(\max(\beta f_1,\beta f_2),\max(\beta f_3,\beta f_4)\big) \quad \text{(by (12) and (13))} & \text{if } \beta \geqslant 0\\
-|\beta|\min\big(\max(f_1,f_2),\max(f_3,f_4)\big) & \text{if } \beta < 0\\
\quad = -\min\big(\max(|\beta|f_1,|\beta|f_2),\max(|\beta|f_3,|\beta|f_4)\big) & \text{(by (12) and (13))}\\
\quad = \max\big(-\max(|\beta|f_1,|\beta|f_2),-\max(|\beta|f_3,|\beta|f_4)\big) & \text{(by (11))}\\
\quad = \max\big(\min(-|\beta|f_1,-|\beta|f_2),\min(-|\beta|f_3,-|\beta|f_4)\big) & \text{(by (11))}\\
\quad = \max\big(\min(\beta f_1,\beta f_2),\min(\beta f_3,\beta f_4)\big) & \\
\quad = \min\big(\max(\beta f_1,\beta f_3),\max(\beta f_1,\beta f_4),\max(\beta f_2,\beta f_3),\max(\beta f_2,\beta f_4)\big) & \text{(by (8))}
\end{cases}$$

Since we have $|x| = \max(x,-x)$ for all $x \in \mathbb{R}$, it is easy to verify that these cost functions are also MMPS functions.

In practical situations, there will be constraints on the input and output signals (caused by limited capacity of buffers, limited transportation rates, saturation, etc.) In general this is reflected in a nonlinear constraint of the form

$$\mathbf{C}_\mathrm{c}(k,\mathbf{x}(k-1),\tilde{\mathbf{u}}(k),\tilde{\mathbf{y}}(k)) \geqslant 0 \ . \tag{20}$$

The MPC problem at sample step k consists in minimizing $J(k)$ over all possible future input sequences subject to the constraints. To reduce the complexity of the optimization problem a control horizon N_c is introduced in MPC, which means that the input is taken to be constant beyond sample step $k+N_\mathrm{c}$:

$$\mathbf{u}(k+j) = \mathbf{u}(k+N_\mathrm{c}-1) \ \text{ for } j = N_\mathrm{c},\ldots,N_\mathrm{p}-1. \tag{21}$$

Alternatively, we can set the input rate constant as was done in (De Schutter and van den Boom, 2001*b*):

$$\Delta\mathbf{u}(k+j) = \Delta\mathbf{u}(k+N_\mathrm{c}-1) \ \text{ for } j = N_\mathrm{c},\ldots,N_\mathrm{p}-1\,, \tag{22}$$

where $\Delta\mathbf{u}(k) = \mathbf{u}(k) - \mathbf{u}(k-1)$. In addition to a decrease in the number of optimization parameters and thus also the computational burden, a smaller control horizon N_c also gives a smoother control signal, which is often desired in practical situations.

MPC uses a receding horizon principle. This means that after computation of the optimal control sequence $\mathbf{u}(k),\mathbf{u}(k+1),\ldots,\mathbf{u}(k+N_\mathrm{c}-1)$, only the first control sample $\mathbf{u}(k)$ will be implemented, subsequently the horizon is shifted one sample, next the model and the state are updated using new information from the measurements, and a new MPC optimization is performed for sample step $k+1$.

5. ALGORITHMS FOR THE MMPS-MPC OPTIMIZATION PROBLEM

5.1 *Nonlinear optimization*

In general the MMPS-MPC optimization problem is a nonlinear, non-convex optimization problem. In (De Schutter and van den Boom, 2001*b*) we have discussed some algorithms to solve the MMPS-MPC optimization problem: we can use multi-start nonlinear optimization based on sequential quadratic programming (SQP), or we can use a method based on the extended linear complementarity problem (ELCP). However, both methods have their disadvantages. If we use the SQP approach, then we usually have to consider a large number of initial starting points and perform several optimization runs to obtain (a good approximation of) the global minimum. In addition, the objective functions that appear in the MMPS-MPC optimization problem are non-differentiable and PWA (if we use the cost criteria given in (16)–(19) or in (De Schutter and van den Boom, 2001*a*)), which makes the SQP algorithm less suitable for them. The main disadvantage of the ELCP approach is that the execution time of this algorithm increases exponentially as the size of the problem increases. This implies that this approach is not feasible if N_c or the number of inputs and outputs of the system are large.

An alternative option consists in transforming the MMPS system into a mixed-logic (MLD) system (Bemporad and Morari, 1999) since MMPS systems are equivalent to MLD systems (Heemels *et al.*, 2001*a*). The main difference between MLD-MPC and MMPS-MPC is that MLD-MPC requires the solution of *mixed integer-real* optimization problems. In general, these are also computationally hard optimization problems.

In the next section we will present another method to solve the MMPS-MPC optimization problem that is similar to the cutting-plane method used in convex optimization.

5.2 *A new algorithm*

We assume that the cost criteria given in (16)–(19) are used [6]. Recall that these objective functions (and any linear combination of them) are MMPS functions. The same holds for the estimate of future output $\tilde{\mathbf{y}}(k)$. So if we substitute $\tilde{\mathbf{y}}(k)$ in the expression for $J(k)$, we finally obtain an MMPS function of $\tilde{\mathbf{u}}(k)$ as objective function. From Theorem 4 it follows that this objective function can be written in min-max canonical form as follows (where — for the sake of simplicity of notation — we drop the index k):

$$J = \min_{i=1,\ldots,\ell} \max_{j=1,\ldots,n_i} (\boldsymbol{\alpha}_{(i,j)}^T \tilde{\mathbf{u}} + \beta_{(i,j)})$$

for appropriately defined integers ℓ, $n_1, \ldots, n_\ell$, vectors $\boldsymbol{\alpha}_{(i,j)}$ and integers $\beta_{(i)}$. Note that in general the expression obtained by straightforwardly applying the manipulations of the proof of Theorem 4 will contain a large number of affine arguments $\boldsymbol{\alpha}_{(i,j)}^T \tilde{\mathbf{u}} + \beta_{(i,j)}$. However, many of these terms are redundant [7] and can thus be removed. This reduces the number of affine arguments. Also note that the transformation into canonical form only has to be performed once — provided that we explicitly consider all arguments that depend on k as additional variables when performing the transformation, — and that it can be done off-line.

The derivation below is similar to the cutting-plane algorithm for convex optimization (see, e.g., (Boyd and Barratt, 1991)). Hence, it requires constraints that are linear (or convex) in $\tilde{\mathbf{u}}$. Note that the control horizon constraints (21) and (22) satisfy this condition. However, even if the original MPC constraint (20) is linear in $\tilde{\mathbf{u}}(k)$ and $\tilde{\mathbf{y}}(k)$, then in general this constraint is not linear any more after substitution of $\tilde{\mathbf{y}}(k)$. Therefore, from now on we assume that (after substitution of $\tilde{\mathbf{y}}(k)$) there are only linear [8] constraints on the input $\tilde{\mathbf{u}}(k)$:

$$\mathbf{P}\tilde{\mathbf{u}} + \mathbf{q} \geqslant \mathbf{0} \ . \tag{23}$$

Note that in general $\mathbf{P}$ and $\mathbf{q}$ may depend on $\mathbf{x}(k-1)$ and k, but for the sake of simplicity of notation we do not explicitly indicate this dependence. In practice constraints of the form (23) occur if we have to guarantee that the control signal $\tilde{\mathbf{u}}(k)$ or the control signal rate $\Delta\tilde{\mathbf{u}}(k)$ stay within certain bounds.

To obtain the optimal MPC input signal at sample step k, we have to solve an optimization problem of the following form:

$$\min_{\tilde{\mathbf{u}}} \min_{i=1,\ldots,\ell} \max_{j=1,\ldots,n_i} (\boldsymbol{\alpha}_{(i,j)}^T \tilde{\mathbf{u}} + \beta_{(i,j)})$$
$$\text{subject to } \mathbf{P}\tilde{\mathbf{u}} + \mathbf{q} \geqslant \mathbf{0} .$$

or equivalently

$$\min_{i=1,\ldots,\ell} \min_{\tilde{\mathbf{u}}} \max_{j=1,\ldots,n_i} (\boldsymbol{\alpha}_{(i,j)}^T \tilde{\mathbf{u}} + \beta_{(i,j)})$$
$$\text{subject to } \mathbf{P}\tilde{\mathbf{u}} + \mathbf{q} \geqslant \mathbf{0} \ . \tag{24}$$

Now let $i \in \{1, \ldots, \ell\}$ and consider

$$\min_{\tilde{\mathbf{u}}} \max_{j=1,\ldots,n_i} (\boldsymbol{\alpha}_{(i,j)}^T \tilde{\mathbf{u}} + \beta_{(i,j)})$$
$$\text{subject to } \mathbf{P}\tilde{\mathbf{u}} + \mathbf{q} \geqslant \mathbf{0} \ .$$

It is easy to verify that this problem is equivalent to the following linear programming (LP) problem:

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[6] The result below also holds for any other cost criterion that is an MMPS function of $\tilde{\mathbf{y}}(k)$ and $\tilde{\mathbf{u}}(k)$. So it follows from Theorem 1 that any continuous PWA norm function can also be used.

[7] E.g., since they appear twice, or since there are other arguments in the max (min) expression that are always larger (smaller) than the given argument.

[8] The optimization algorithm used below, which is based on the cutting plane algorithm for convex optimization, can also deal with convex constraints. So we can also allow convex constraints instead of (23).

$$\begin{aligned} &\min\ t \\ &\text{subject to}\ \ t \geqslant \boldsymbol{\alpha}_{(i,j)}^T \tilde{\mathbf{u}} + \beta_{(i)} \quad \text{for } j = 1, \ldots, n_i \\ &\qquad\qquad \mathbf{P}\tilde{\mathbf{u}} + \mathbf{q} \geqslant \mathbf{0}\ . \end{aligned} \tag{25}$$

This LP problem can be solved efficiently using (variants of) the simplex method or an interior-point algorithm (see, e.g., (Nesterov and Nemirovskii, 1994; Wright, 1997)).
To obtain the solution of (24), we solve (25) for $i = 1, \ldots, \ell$ and afterward we select the solution $\tilde{\mathbf{u}}_{(i)}^{\text{opt}}$ for which $\max_{j=1,\ldots,n_i} (\boldsymbol{\alpha}_{(i,j)}^T \tilde{\mathbf{u}}_{(i)}^{\text{opt}} + \beta_{(i,j)})$ is the smallest [9]. This results in an algorithm to solve the MMPS-MPC problem that is more efficient than the SQP or the ELCP approach.

6. CONCLUSIONS

We have shown that continuous piecewise-affine (PWA) systems are equivalent to max-min-plus-scaling (MMPS) systems. This result is a refinement of previous results since it does not require the introduction of auxiliary variables or additional MMPS constraints. Next, we have considered model predictive control for PWA and MMPS systems. In general, this leads to nonlinear non-convex optimization problems. We have presented a method based on canonical forms for MMPS functions and similar to the cutting-plane convex optimization algorithm to solve these optimization problems. More specifically, the approach consists in solving several linear programming problems and afterward selecting the solution that yields the smallest objective function. This results in a method that is more efficient than just applying nonlinear optimization as was done in previous research.

Topics for future research include: a thorough investigation and comparison of the performance and the efficiency of the different optimization algorithms that have been considered above, investigation and characterization of the computational complexity of the transformation into the canonical form, investigation and characterization of the (average) number of linear programming problems and the number of inequalities they contain, and extension of our results to include modeling errors and noise in a stochastic or an ℓ_∞ framework.

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[9] If we use a primal-dual simplex method or an interior-point method to solve the LP problems, we can improve the efficiency of the approach even further by stopping the optimization if we obtain a lower bound for the objective function of the current LP problem that is larger than the smallest final objective function of the LP problems that have already been solved.

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