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GLOBAL H_∞ CONTROLLER DESIGN FOR DISCRETE-TIME BILINEAR SYSTEMS WITH UNDAMPERED NATURAL RESPONSE [1]

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Abstract: In this paper the problem of H_∞-control design is presented for a class of discrete-time single-input multi-output bilinear systems with undampered natural response. The sufficient conditions for the existence of the global H_∞-controllers are obtained via homogeneous-like state feedback and dynamic output feedback, respectively. The technique used in this paper is based on the concepts of dissipation inequality, LaSalle invariant principle and linear matrix inequality in discrete time.

Keywords: H_∞ control, bilinear system, discrete-time, linear matrix inequality, LaSalle invariant principle.

1. INTRODUCTION

Linear systems and bilinear systems can be considered as the first order and the second order approximation to nonlinear systems, respectively. In this sense, bilinear systems are better mathematical estimators for nonlinear systems than those for linear systems. Bilinear systems can be regarded as a special set of nonlinear systems in control theory, and also as adequate approximation for some real-world systems including engineering applications in nuclear, thermal, and chemical processes, and non-engineering applications in biology, socio-economics, immunology. Detailed reviews of bilinear systems for both theory and applications can be found in (Chabour and J. C. Vivala, 2000; Chen and Tsao, 2000; Chen, 1998; Hanba and Miyasato, 2001; Jerbi, 2001; Khapalov and Mohler, 1998; Lu *et al*, 1998; Mohler, 1991; Rahn, 1996; Stepanenko and Yang, 1996; Tuan and Hosoe, 1997).

It is known that sufficient conditions for the stabilization of bilinear systems can be found in (Jerbi, 2001; Lin and Byrnes, 1994; Lu *et al*, 1998; Stepanenko and Yang, 1996); Also, robust H_∞ control problem for continuous bilinear systems is discussed in (Tuan and Hosoe, 1997), in which the authors assume that unforced systems are robust globally asymptotically stable and a state feedback H_∞ controller is obtained. However H_∞ control problem for discrete-time bilinear systems has not been discussed in the literature so far. Although H_∞ control problems for nonlinear discrete-time systems have been extensively discussed in (Lin and Byrnes, 1996), only local H_∞ performance can be guaranteed by means of local controller in the neighborhood of the origin, and no information is obtained outside this neighborhood.There are certainly some limitations on using the local H_∞ controller design in the applications.

The objective of this paper is to discuss global H_∞ control problem for a class of discrete-time single-input multi-output bilinear systems, where

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the linear state matrix is assumed to be of Lyapunov stability or undampered natural response. Both global state feedback H_∞ controller and global dynamic output feedback H_∞ controller are directly obtained by means of homogeneous-like feedback, which can be easily constructed via optimization method and linear matrix inequalities. In this paper, the dissipative inequality for discrete-time systems is *non-affine* on control input, and a homogeneous-like bounded feedback controller is designed. In addition, LaSalle invariant principle and linear matrix inequality (LMI) in discrete time are employed, and the approach used in this work is quite different from that in (Tuan and Hosoe, 1997), where Lyapunov stability approach and algebraic Riccati equation are applied for a special class of the continuous-time bilinear systems.

Notation: The notation in this paper is quite standard. R^n and $R^{n\times m}$ denote respectively the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. A' denotes the transpose of matrix A. $X \geq Y$ (or $X > Y$ respectively) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (or positive definite respectively). I is the identity matrix with compactible dimension. $l_2[0,+\infty]$ is the space of square summable vector sequence over $[0,+\infty]$. $\|\cdot\|$ will refer to the Euclidean vector norm whereas $\|\cdot\|_{[0,+\infty]}$ denotes the $l_2[0,+\infty]$ norm over $[0,+\infty]$ defined as $\|f\|^2_{[0,+\infty]} = \sum_0^{+\infty} \|f_k\|^2$.

2. PROBLEM FORMULATION

Consider the following discrete-time bilinear systems:

$$\begin{aligned} x_{k+1} &= Ax_k + u_k Bx_k + Dw_k, \\ z_k &= Ex_k + u_k Fx_k, \\ y_k &= Cx_k, \end{aligned} \tag{1}$$

where $x_k \in R^n$, $w_k \in R^q$, $z_k \in R^m$, $y_k \in R^p$ and $u_k \in R$ are the system state, disturbance input, controlled output, measured output and control input, respectively. $A, B \in R^{n\times n}$, $E, F \in R^{m\times n}$, $D \in R^{n\times q}$ and $C \in R^{p\times n}$ are constant matrices with rank $C = p$.

The purpose of this paper is to construct a global state feedback H_∞ controller and a global dynamic output feedback H_∞ controller respectively, such that the resulting closed-loop systems have global H_∞ performance. The resulting closed-loop systems satisfy the following two requirements:

(i) when disturbance input $w_k = 0$ $(\forall k)$, the closed-loop systems are globally asymptotically stable;

(ii) for a given scalar $\gamma > 0$, when the initial condition $x_0 = 0$, then for all $w_k \in l_2[0,+\infty]$ satisfying

$$\|z\|_{[0,+\infty]} \leq \gamma \|w\|_{[0,+\infty]}. \tag{2}$$

Before presenting the main results, we make some basic assumptions and useful preliminary results. The following assumption is similar to those in (Lu *et al*, 1998; Rahn, 1996).

Assumption 2.1. Suppose that the free dynamic system of (1) is Lyapunov stable, *i.e.,* there exists a positive definite constant matrix $P \in R^{n\times n}$ such that $A'PA - P \leq 0$.

The following two lemmas are useful in the sequel, their proofs are trivial and therefore omitted.

Lemma 2.1. Let $T, T_0 \in R^{n\times n}$ be symmetric matrices and c be positive constant, then the following two conditions are equivalent:

(i) $T_0 + cT < 0, \quad T_0 - cT < 0$;

(ii) for any scalar $\rho \in [-c, c], \quad T_0 + \rho T < 0$.

Lemma 2.2. Let $P, S \in R^{n\times n}$ be symmetric matrices, $P \geq 0$ and $\text{rank}(P) = r \geq 1$. If $\{x : x'Px = 0, x \in R^n\} \subset \{x : x'Sx = 0, x \in R^n\}$, then $\sup_{x\in R^n, x'Px\neq 0} \left|\frac{x'Sx}{x'Px}\right| < +\infty$.

3. STATIC STATE FEEDBACK CONTROLLER

The following theorem is the main result of this section, which presents a design for global H_∞ controller by using homogeneous-like state feedback.

Theorem 3.1. Suppose that P satisfies Assumption 2.1 and $\gamma^2 I - D'PD > 0$, then there exists a bounded static state feedback controller $u_k = u(x_k)$ which stabilizes systems (1) with an H_∞-norm γ if there exist $\epsilon > 0$ and $0 < \delta \leq 1$ such that

$$\inf_{\|x\|=1,\ x\in R^n} \Delta(\epsilon, \delta, x) \geq 0, \tag{3}$$

$$\sup_{\|x\|=1,\ x\in \text{Ker}B\cap\text{Ker}F} x'\Delta_\delta x \leq 0, \tag{4}$$

$$\begin{aligned} \Omega &:= \cap_{j=0}^{n-1} [\text{Ker}[(A'PA - P)A^j] \cap \text{Ker}(EA^j)] \\ &= \{0\}, \end{aligned} \tag{5}$$

where

$$\Delta(\epsilon, \delta, x) := (x'\Delta_1 x)^2 - (1+\epsilon)(x'\Delta_\delta x)(x'\Delta_2 x),$$

$$\Delta_\delta := E'E + A'\Delta_P A + (1-\delta)(A'PA - P),$$
$$\Delta_1 := A'PB + E'F + A'\Delta_P B,$$
$$\Delta_2 := B'PB + F'F + B'\Delta_P B,$$
$$\Delta_P := PD(\gamma^2 I - D'PD)^{-1}D'P.$$

Remark 3.1. Condition (4) implies that systems (1) have H_∞ performance in "uncontrollable mode" or in "input-nulling subset", *i.e.*, $\mathrm{Ker}B \cap \mathrm{Ker}F$. Similar assumption is made in (Gutman, 1981; Hanba and Miyasato, 2001) that the "unstable mode" of the plant is isolated from the region where control does not affect the plant. That is, for stabilization of continuous-time systems: $\dot{x} = Ax + uBx$, (Gutman, 1981; Hanba and Miyasato, 2001) assume that there is a symmetric positive definite matrix P such that for any $x \neq 0$, $x'(PA + A'P)x < 0$, if $x'(PB + B'P)x = 0$.

Proof of Theorem 3.1: Define a Lyapunov function V_k as follows:

$$V_k := x_k' P x_k, \tag{6}$$

and denote $\eta_k := D'P(A + u_k B)x_k - (\gamma^2 I - D'PD)w_k$, after some algebraic manipulations, then

$$\begin{aligned}
&V_{k+1} - V_k + z_k' z_k - \gamma^2 w_k' w_k \\
&= x_k'(A'PA - P)x_k + u_k^2 x_k'(B'PB + F'F)x_k \\
&\quad + 2u_k x_k'(A'PB + E'F)x_k \\
&\quad + x_k' E'E x_k + x_k'(A + u_k B)'\Delta_\delta (A + u_k B)x_k \\
&\quad - \eta_k'(\gamma^2 I - D'PD)^{-1}\eta_k \\
&\leq \delta x_k'(A'PA - P)x_k + u_k^2 x_k' \Delta_2 x_k \\
&\quad + 2u_k x_k' \Delta_1 x_k + x_k' \Delta_\delta x_k.
\end{aligned}$$

Choose the following homogeneous-like state feedback controller

$$u_k = \begin{cases} -\dfrac{x_k'\Delta_1 x_k}{(1+\epsilon)x_k'\Delta_2 x_k}, & x_k \notin \mathrm{Ker}B \cap \mathrm{Ker}F \\ 0, & x_k \in \mathrm{Ker}B \cap \mathrm{Ker}F. \end{cases} \tag{7}$$

Note that Δ_2 is semi-positive definite and $\{x : x'\Delta_2 x = 0, x \in R^n\} = \mathrm{Ker}B \cap \mathrm{Ker}F \subset \{x : x'\Delta_1 x = 0, x \in R^n\}$, when $x_k \notin \mathrm{Ker}B \cap \mathrm{Ker}F$, then from Lemma 2.2 we have $|u_k| \leq \frac{\max|\lambda(\Delta_1 + \Delta_1')|}{2(1+\epsilon)\lambda_{\max}(\Delta_2)} < +\infty$, which implies that control input u_k is bounded. Considering that $\Delta(\epsilon, \delta, x)$ is homogeneous on x, then it follows from (3) that $\Delta(\epsilon, \delta, x_k) = (x_k'\Delta_1 x_k)^2 - (1+\epsilon)(x_k'\Delta_\delta x_k)(x_k'\Delta_2 x_k) \geq 0$, $\forall x_k \in R^n$. Now we have two different cases as follows: a) When $x_k \notin \mathrm{Ker}B \cap \mathrm{Ker}F$,

$$\begin{aligned} &V_{k+1} - V_k + z_k' z_k - \gamma^2 w_k' w_k \\ &\leq \delta x_k'(A'PA - P)x_k - \epsilon u_k^2 x_k' \Delta_2 x_k \leq 0. \end{aligned} \tag{8}$$

b) When $x_k \in \mathrm{Ker}B \cap \mathrm{Ker}F$,

$$\begin{aligned} &V_{k+1} - V_k + z_k' z_k - \gamma^2 w_k' w_k \\ &\leq \delta x_k'(A'PA - P)x_k + x_k'\Delta_\delta x_k \leq 0. \end{aligned} \tag{9}$$

If the initial condition $x_0 = 0$, then it is obvious that, for both cases above, we have

$$\sum_{k=0}^{N} z_k' z_k - \gamma^2 \sum_{k=0}^{N} w_k' w_k \leq V_N \leq 0, \forall N > 0. \tag{10}$$

This implies that the H_∞-norm of the closed-loop systems (1) and (7) is less than or equal to γ.

Next, we show the internal stability of the closed-loop systems (1) and (7). Let $w_k = 0, \forall k$. If $x_k \notin \mathrm{Ker}B \cap \mathrm{Ker}F$, then it follows from (8) that $V_{k+1} - V_k \leq \delta x_k'(A'PA - P)x_k - \epsilon u_k^2 x_k'\Delta_2 x_k - z_k' z_k \leq 0$. If $x_k \in \mathrm{Ker}B \cap \mathrm{Ker}F$, then $u_k = 0$ and from (9), we have that $V_{k+1} - V_k \leq \delta x_k'(A'PA - P)x_k - x_k'E'Ex_k + x_k'\Delta_\delta x_k \leq 0$. Then the resulting closed-loop systems are Lyapunov stable. Furthermore, $V_{k+1} - V_k = 0$ implies that $x_k'(A'PA - P)x_k = 0$, $u_k = 0$ and $z_k = 0$. Then $(A'PA - P)x_k = 0$, $Ex_k = 0$ and $x_k = A^k x_0$, $k = 0, 1, 2, \cdots$, which implies $x_0 \in \Omega = \{0\}$. It follows from LaSalle invariant principle that the resulting closed-loop systems are globally asymptotically stable when $w_k = 0$, $\forall k$.

Remark 3.2. From the above proof, we can see that Ω contains the maximal invariant subset of $V_{k+1} - V_k = 0$ $(k = 1, 2, \cdots)$, we introduce condition (5) to guarantee the global asymptotic stability of the resulting closed-loop systems by means of LaSalle invariant principle.

Remark 3.3. It is easy to see that, for any $\epsilon > 0$ and $0 < \delta \leq 1$, we have $\inf_{\|x\|=1,\ x\in R^n} \Delta(\epsilon, 0, x) \geq \inf_{\|x\|=1,\ x\in R^n} \Delta(\epsilon, \delta, x)$ and $\sup_{\|x\|=1, x\in \mathrm{Ker}B\cap\mathrm{Ker}F} x'\Delta_0(0)x \leq \sup_{\|x\|=1,\ x\in\mathrm{Ker}B\cap\mathrm{Ker}F} x'\Delta_\delta x$, which means that the following condition (11) is less conservative than (3) and (4).

$$\begin{aligned} \inf_{\|x\|=1,\ x\in R^n} \Delta(\epsilon, 0, x) &\geq 0, \\ \sup_{\|x\|=1,\ x\in\mathrm{Ker}B\cap\mathrm{Ker}F} x'\Delta_0(0)x &\leq 0. \end{aligned} \tag{11}$$

If (3) and (4) are replaced by (11), from the proof of Theorem 3.1, we can obtain the global asymptotic stability of the resulting closed-loop systems by LaSalle invariant principle if

$$\Omega_0 := \cap_{j=0}^{n-1} \mathrm{Ker}(EA^j) = \{0\}. \tag{12}$$

However, (5) is less conservative than (12). It can be seen from the numerical example (26) in this paper that condition (12) is not satisfied, while condition (5) is satisfied (see also Remark 5.1). In addition, if $\delta > 0$, the invariant subset of $V_{k+1} - V_k = 0$ is smaller than that in the case when

$\delta = 0$, which can also be seen from (5) and (12). Usually, both $\delta > 0$ and $\epsilon > 0$ are chosen to be as small as possible to make that (3) and (4) easy to satisfy. Therefore the purpose of introducing ϵ and δ in Theorem 3.1 is to guarantee the maximal invariant subset as small as possible.

For the case when $\epsilon = \delta = 0$, the following theorem presents different sufficient conditions to guarantee the existence of H_∞ controller, the proof is similar to Theorem 3.1, where Lyapunov stability theorem is used instead of LaSalle invariant principle.

Theorem 3.2. Suppose that P satisfies Assumption 2.1 and $\gamma^2 I - D'PD > 0$, then there exists a bounded static state feedback controller $u_k = u(x_k)$ which stabilizes systems (1) with an H_∞-norm γ if the following conditions are satisfied.

$$\inf_{\|x\|=1,\ x\in R^n} \Delta(0,0,x) > 0, \tag{13}$$

$$\sup_{\|x\|=1,\ x\in \mathrm{Ker}B\cap\mathrm{Ker}F} x'\Delta_0(0)x < 0. \tag{14}$$

Remark 3.4. Compared with inequalities (3) and (4) in Theorem 3.1, (13) and (14) in Theorem 3.2 are strict inequalities, while an additional sufficient condition (5) is needed in Theorem 3.1.

4. DYNAMIC OUTPUT FEEDBACK CONTROLLER

Before presenting the main result, we give some assumptions and preliminary results.

Assumption 4.1. There exist $0 < \mu_1 \leq 1$ and $\mu_2 > 0$ such that

$$\inf_{\|x\|=1,\ x\in R^n} \Gamma(\mu_1,\mu_2,x) \geq 0, \tag{15}$$

$$\sup_{\|x\|=1, x\in \mathrm{Ker}B\cap\mathrm{Ker}F} x'\Gamma_0 x \leq 0, \tag{16}$$

where

$$\begin{aligned}
&\Gamma(\mu_1,\mu_2,x) := (x'\Gamma_1 x)^2 - (x'\Gamma_0 x)(x'\Gamma_2 x),\\
&\Gamma_0 := 2E'E + A'PA + (1-\mu_1)(A'PA - P),\\
&\Gamma_1 := 2A'PB + 2E'F,\\
&\Gamma_2 := 2(\mu_2+1)(B'PB + F'F).
\end{aligned}$$

Noticing that $\{x : x'\Gamma_2 x = 0, x \in R^n\} = \mathrm{Ker}B \cap \mathrm{Ker}F \subset \{x : x'\Gamma_1 x = 0, x \in R^n\}$, then from Assumption 4.1 and Lemma 2.2, we have

Lemma 4.1. Under Assumption 4.1, then there exist $0 < \mu_1 \leq 1$ and $\mu_2 > 0$ such that

$$\sup_{x\notin \mathrm{Ker}B\cap\mathrm{Ker}F} \left| \frac{x'\Gamma_1 x}{x'\Gamma_2 x} \right| := c < +\infty. \tag{17}$$

The following theorem is the main result of this section, which presents a design for global H_∞ controller by using homogeneous-like dynamic output feedback.

Theorem 4.1. Under Assumptions 2.1, 4.1, then there exists a bounded dynamic output feedback controller which stabilizes systems (1) with an H_∞-norm γ if the following conditions are satisfied:

(i) $\Omega = \{0\}$;

(ii) the following LMIs on $X \in R^{n\times n}$, $Y \in R^{n\times p}$ and $M \in R^{p\times p}$ are solvable:

$$\begin{aligned}
&\Phi(-c,X,Y) < 0, \quad \Phi(c,X,Y) < 0,\\
&CX - MC = 0,
\end{aligned} \tag{18}$$

where $\Psi_1 = (A+\rho B)X - YC$, $\Psi_2 := (E+\rho F)X$, $\Psi_3 := -X + \gamma^{-2}DD'$ and

$$\Phi(\rho,X,Y) := \begin{pmatrix} -X & \Psi_1' & \Psi_2' & C'Y' \\ \Psi_1 & \Psi_3 & 0 & 0 \\ \Psi_2 & 0 & -\frac{1}{2}I & 0 \\ YC & 0 & 0 & -\frac{1}{2}P^{-1} \end{pmatrix}.$$

Proof: It follows from (18) that X is nonsingular, thus the fact that C is full rank and $CX - MC = 0$ in (18) implies that M is nonsingular, see (Crusius and Trofino, 1999). Let $L = YM^{-1}$. Consider the full-order dynamic output feedback controller of the form

$$\begin{aligned}
&\xi_{k+1} = A\xi_k + u_k B\xi_k + L(y_k - C\xi_k), \\
&u_k = \begin{cases} -\dfrac{\xi_k'\Gamma_1\xi_k}{\xi_k'\Gamma_2\xi_k}, & \text{for } \xi_k \notin \mathrm{Ker}B\cap\mathrm{Ker}F \\ 0, & \text{for } \xi_k \in \mathrm{Ker}B\cap\mathrm{Ker}F. \end{cases}
\end{aligned} \tag{19}$$

It follows from Lemma 4.1 that we have $|u_k| \leq c$, that is, the control input u_k in (19) is bounded. Denote the error state $e_k = x_k - \xi_k$, then it follows from (1) and (19) that

$$e_{k+1} = (A - LC + u_k B)e_k + Dw_k. \tag{20}$$

Let $Q = X^{-1}$, then choose the following Lyapunov function $V_k = \xi_k' P \xi_k + e_k' Q e_k$. Then

$$\begin{aligned}
\Pi := &V_{k+1} - V_k + z_k' z_k - \gamma^2 w_k' w_k \\
= &[(A+u_kB)\xi_k + LCe_k]'P[(A+u_kB)\xi_k \\
&+LCe_k] - \xi_k'P\xi_k + [(A-LC+u_kB)e_k \\
&+Dw_k]'Q[(A-LC+u_kB)e_k + Dw_k] \\
&-e_k'Qe_k + [Ee_k + E\xi_k + u_k(Fe_k + F\xi_k)]' \\
&\cdot[Ee_k + E\xi_k + u_k(Fe_k + F\xi_k)] - \gamma^2 I
\end{aligned}$$

Noticing that

$$2e_k'(A - LC + u_kB)'QDw_k$$

$$\le e_k'(A - LC + u_k B)QD(\gamma^2 I - D'QD)^{-1}D'Q$$
$$\cdot(A - LC + u_k B)e_k + w_k'(\gamma^2 I - D'QD)w_k,$$
$$2\xi_k'(A + u_k B)'PLCe_k$$
$$\le \xi_k'(A + u_k B)'P(A + u_k B)\xi_k + e_k'C'L'PLCe_k,$$
$$2\xi_k'(E + u_k F)'(E + u_k F)e_k$$
$$\le \xi_k'(E + u_k F)'(E + u_k F)\xi_k$$
$$+e_k'(E + u_k F)'(E + u_k F)e_k.$$

Then

$$\Pi \le \xi_k'\Phi_1\xi_k + e_k'\Phi_2 e_k, \tag{21}$$

where $\Phi_1(u_k) := \mu_1(A'PA - P) - 2\mu_2 u_k^2(B'PB + F'F) + u_k^2\Gamma_2 + 2u_k\Gamma_1 + \Gamma_0$, $\Phi_2(u_k) := 2C'L'PLC + 2(E + u_k F)'(E + u_k F) - Q + (A - LC + u_k B)'[Q + QD(\gamma^2 I - D'QD)^{-1}D'Q](A - LC + u_k B)$.

Similar to the proof of Theorem 3.1, Assumption 4.1 also implies that $\Gamma(\mu_1, \mu_2, \xi_k) = (\xi_k'\Gamma_1\xi_k)^2 - (\xi_k'\Gamma_0\xi_k)(\xi_k'\Gamma_2\xi_k) \ge 0$, $\forall \xi_k \in R^n$. In addition, when $\xi_k \notin \mathrm{Ker}B \cap \mathrm{Ker}F$, $\xi_k'\Phi_1\xi_k = \mu_1\xi_k'(A'PA - P)\xi_k - 2\mu_2 u_k^2\xi_k'(B'PB + F'F)\xi_k \le 0$; when $\xi_k \in \mathrm{Ker}B \cap \mathrm{Ker}F$, $\xi_k'\Phi_1\xi_k = \mu_1\xi_k'(A'PA - P)\xi_k + \xi_k'\Gamma_0\xi_k \le 0$, which implies that for all $\xi_k \in R^n$, $\xi_k'\Phi_1\xi_k \le 0$.

Next, we show that $\Phi_2 < 0$. It follows from Lemma 2.1 that $\Phi(u_k, X, Y) < 0$. Noticing that $X = Q^{-1}$ and the following matrix equality: $(Q^{-1} - \gamma^{-2}DD')^{-1} = Q + QD(\gamma^2 I - D'QD)^{-1}D'Q$, then it follows from the Schur complement and some algebraic manipulations that that $\Phi_2 < 0$. Therefore

$$\Pi \le \xi_k'\Phi_1\xi_k + e_k'\Phi_2 e_k \le 0. \tag{22}$$

Following the same proof for Theorem 3.1, we can obtain the results, therefore it is omitted here.

Similar to Theorem 3.2 and the proof of Theorem 4.1, we have the following result.

Theorem 4.2. Under Assumptions 2.1, then there exists a bounded dynamic output feedback controller which stabilizes systems (1) with an H_∞-norm γ if LMIs (18) are solvable and

$$\inf_{\|x\|=1,\ x\in R^n} \Gamma(0, 0, x) > 0, \tag{23}$$

$$\sup_{\|x\|=1, x\in \mathrm{Ker}B\cap\mathrm{Ker}F} x'\Gamma_0 x\,|_{\mu_1=0} < 0, \tag{24}$$

where

$$c = \sup_{x\notin \mathrm{Ker}B\cap\mathrm{Ker}F} \left|\frac{x'\Gamma_1 x}{x'\Gamma_2 x}\right|_{\mu_1=\mu_2=0}. \tag{25}$$

Remark 4.1. Compared with inequalities (15) and (16) in Theorem 4.1, (23) and (24) are strict inequalities, while an additional sufficient condition $\Omega = \{0\}$ is needed in Theorem 4.1.

Remark 4.2. The objective of introducing two parameters μ_1 and μ_2 in Theorem 4.1 is similar to that in Remark 3.3. In the proof of Theorem 4.1, the separation principle is used to design control input gain and observer respectively. The LMI sufficient conditions for output feedback design in the Theorem 4.1 is motivated by (Crusius and Trofino, 1999).

5. NUMERICAL EXAMPLE

Consider the following example:

$$\begin{aligned} x_{1k+1} &= x_{1k} + x_{1k}u_k + 0.1w_k, \\ x_{2k+1} &= 0.1x_{2k} + x_{2k}u_k + 0.2w_k, \\ z_k &= 0.1x_{1k} + 0.1x_{2k} + x_{1k}u_k, \\ y_k &= x_{1k}. \end{aligned} \tag{26}$$

That is,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0.1 \end{pmatrix},\quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad C = (\,1 \quad 0\,),$$
$$D = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix},\quad E = (\,0.1 \quad 0.1\,),\quad F = (\,1 \quad 0\,).$$

Let the H_∞ norm constraint $\gamma = 1$. At first, we consider state feedback H_∞ controller design. Choose $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then it is easy to have that $\Omega \subset \mathrm{Ker}(A'PA - P) \cap \mathrm{Ker}E = \{0\}$, which implies that (5) holds. Choose small parameters ϵ and δ as $\epsilon = 0.1$ and $\delta = 0.01$, it is easy to check that condition (3) holds by means of Optimization toolbox in Matlab. Then the state feedback controller (7) can be given as follows:

$$u_k = \begin{cases} \alpha_k, & \text{for } x_{1k}^2 + x_{2k}^2 \ne 0, \\ 0, & \text{for } x_{1k}^2 + x_{2k}^2 = 0, \end{cases} \tag{27}$$

where

$$\alpha_k := -\frac{1.1105x_{1k}^2 + 0.1232x_{1k}x_{2k} + 0.1042x_{2k}^2}{2.116x_{1k}^2 + 0.0464x_{1k}x_{2k} + 1.1463x_{2k}^2}.$$

In this case, we have $|u| \le 0.5045$. Let $w_k = \frac{1}{k+1}$, $(k = 0, 1, 2\cdots)$, $x_{10} = x_{20} = 0$, then $\frac{\|z\|_{[0,+\infty]}}{\|w\|_{[0,+\infty]}} = 0.0209 \le \gamma = 1$.

Remark 5.1. As a contrast, choose $\delta = 0$ we have that

$$\Omega_0 := \cap_{j=0}^{n-1}\mathrm{Ker}(EA^j) = \mathrm{span}\ \left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \ne \{0\},$$

which implies that we cannot apply the sufficient conditions (11) and (12) in Remark 3.3 to obtain the global asymptotic stability of the resulting closed-loop systems.

Next we consider the dynamic output feedback H_∞ controller design. Choose $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and small parameters as $\mu_1 = 0.01$ and $\mu_2 = 0.1$, similarly we can check that (15) and (16) hold. In this case, we have $c = 0.5025$. A triple of solutions for LMIs (18) can be given as follows:

$$X = \begin{pmatrix} 0.2077 & 0 \\ 0 & 0.3154 \end{pmatrix}, \quad Y = \begin{pmatrix} 0.1879 \\ 0.0394 \end{pmatrix},$$

$$M = 0.2077.$$

Then $L = (\,0.9048 \quad 0.1896\,)'$, and dynamic output feedback controller can be given as follows:

$$\begin{aligned} \xi_{1k+1} &= \xi_{1k} + \xi_{1k}u_k + 0.9048(y_k - \xi_{1k}), \\ \xi_{2k+1} &= 0.1x_{2k} + x_{2k}u_k + 0.1896(y_k - \xi_{1k}), \\ u_k &= \begin{cases} \beta_k, & \text{for } \xi_{1k}^2 + \xi_{2k}^2 \neq 0, \\ 0, & \text{for } \xi_{1k}^2 + \xi_{2k}^2 = 0, \end{cases} \end{aligned}$$

where

$$\beta_k := -\frac{2.2\xi_{1k}^2 + 0.2\xi_{1k}\xi_{2k} + 0.2\xi_{2k}^2}{4.4\xi_{1k}^2 + 2.2\xi_{2k}^2}.$$

Let $w_k = \frac{1}{k+1}$, $(k = 0, 1, 2\cdots)$, $x_{10} = x_{20} = 0$, then $\frac{\|z\|_{[0,+\infty]}}{\|w\|_{[0,+\infty]}} = 0.0204 \leq \gamma = 1$.

6. CONCLUSIONS

Global H_∞-control problem for discrete-time bilinear systems is first discussed in this paper. Bounded state feedback controller and bounded dynamic output feedback controller are designed to guarantee the global H_∞ performance for the resulting closed-loop systems, respectively. The techniques used are dissipation inequality, differential games, LaSalle invariant principle and linear matrix inequality in discrete time. In this paper, two different types (*(i)* $\epsilon > 0$ *and* $\delta > 0$, *(ii)* $\epsilon = \delta = 0$) of sufficient conditions are presented for discrete-time bilinear systems by state feedback controllers, similar discussion is also presented for output feedback case. Extension of the present results to MIMO discrete-time bilinear systems is an interesting topic for further study.

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