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POLYNOMIAL CONTROLLABILITY IN LINEAR TIME-INVARIANT SYSTEMS: SOME FURTHER RESULTS AND APPLICATIONS TO OPTIMAL CONTROL

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Abstract: Previous studies have shown that a controllable linear system can be driven from a given initial condition to a desired target by means of a polynomial input function whose coefficients are obtained simply by solving a set of linear algebraic equations. This paper shows how the application of the concept of polynomial controllability is useful for solving a suboptimal control problem. In particular we present a simple procedure for searching for a control function that minimizes a quadratic performance measure while the system is transferring between specified end-points. The approach is implemented for both state-space and singular linear time-invariant controllable systems. *Copyright © 2002 IFAC*

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1. INTRODUCTION

In linear system theory it is known that the rich structure of a controllable system allows one not only to transfer the system from its current position to an arbitrarily selected final target, but also to move the system along a *pre-determined path* that connects the initial and the final state vectors and satisfies certain conditions. In this regard a sequence of papers have shown that a controllable linear system is also *polynomial controllable* (Ailon *et al.*, 1986, Ailon and Langholz, 1986, and Aeyels, 1987).

The implication of this result is that a *polynomial input* ensures the transfer of the system along a *polynomial trajectory* and the control function can be computed by solving a set of linear algebraic equations (rather than by the controllability Grammian), the solution of which yields the polynomial coefficients of the desired input. In (Ailon *et al.*, 1986) a bound on the degree of the polynomial input that generates a polynomial trajectory has been proposed. Later (Ailon and Langholz, 1986) improved that result by reducing the bound on the input polynomial degree. Finally, in (Aeyels, 1987) the bound on the degree of the required polynomial input that ensures the transfer of the system along a polynomial trajectory has been lowered to its minimal value.

Those results motivate the present study in which we consider the application of the concept of polynomial controllability to an optimal control problem. We present an approach for steering a linear controllable system from a given initial state to a desired target by means of a polynomial input, while a quadratic performance index is to be minimized. Using the present approach the solution to the optimal control problem is obtained by determining a point in an Euclidean space that minimizes a specific *function*, rather than finding a function that minimizes a predetermined *functional*. Furthermore, the solution is attained simply by solving a set of algebraic equations that determines the required vector of the polynomial coefficients.

A direct extension of the approach for solving the tracking problem while the control objective is to maintain the system state as closely as possible to a desired reference trajectory connecting the initial and final target, is demonstrated. Finally, application of the method to the associated optimal control problems while a linear controllable singular system is under consideration (Lewis, 1986), has been presented as well.

2. PRELIMINARIES

Consider the linear time-invariant state-space system

$$\dot{x}\left(t\right) = Ax\left(t\right) + Bu\left(t\right); t \geq 0, \qquad (1)$$

with $x \in \Re^n$ and $u \in \Re^m$. The initial condition is $x\left(0\right) = x_0$. Define $r \doteq n - m$. We assume that $rankB = n - r$, (i.e., B is a *full column rank matrix,*) and (A, B) is a *controllable pair*. Clearly $r < n$. We assume that $0 < r < n$. (If $r = 0$ then B is square and invertible and the results that will be established for $0 < r < n$ can be extended trivially for this case.) Let the matrix $C \in \Re^{r\times n}$ be a surjective maximum left annihilator of B, i.e.,

$$\begin{aligned} &CB = 0; \\ &\xi^T B = 0 \Leftrightarrow \exists \eta \in \Re^r : \xi^T = \eta^T C, \end{aligned} \qquad (2)$$

where $[\cdot]^T$ is the transpose of $[\cdot]$.

As indicated above, if the problem under consideration is restricted to the derivation of a polynomial control function that drives the system from a given initial state to a desired target along a polynomial trajectory, the integer that determines the polynomial's degree in (Aeyels, 1987) is lower than the one presented in (Ailon and Langholz, 1986). However, here the control objective is more general and we turn to study the application of new tools for minimizing some performance criterions having constraints. In this regard the minimal degree of the polynomial pair $\{x, u\}$ that yields a suboptimal solution is not of prime interest, and it will be more convenience to apply below the various patterns and indices that have been established in the latter reference.

Define an integer $M \geq 2r+1$ and an $(M+1)r \times Mn$ constant matrix W_M as follows

$$W_M = \begin{bmatrix} C & 0 & 0 & \cdots & 0 & 0 \\ -CA & 2C & 0 & \cdots & 0 & 0 \\ 0 & -CA & 3C & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -CA & MC \\ 0 & 0 & 0 & \cdots & 0 & -CA \end{bmatrix} \qquad (3)$$

where I_n is the $n \times n$ identity matrix. Using (3) consider the $[(M+1)r + n] \times Mn$ matrix

$$H_M\left(t\right) = \begin{bmatrix} W_M \\ tI_n \; t^2 I_n \; \cdots \; t^M I_n \end{bmatrix}. \qquad (4)$$

The following results (Ailon and Langholz, 1986) play an essential role in this study.

Lemma 2.1. Controllability of the system (1) implies that for each fixed $t > 0$ the matrix $H_M(t)$ is of full rank for any fixed M satisfying $M \geq 2r+1$.

Theorem 2.1. Consider the controllable system (1) and let $u(t)$ be a control function with

$$u(t) = \left[p_1\left(t\right), \cdots, p_m\left(t\right)\right]^T, t \geq 0 \qquad (5)$$

where $p_i(t)$ are polynomials. Let x_0 and x_f be some arbitrary initial and final states respectively. Then, for any selected $t_f > 0$ a polynomial command $u(t)$ of the form (5) with degree not larger than M where $M \geq 2r+1$, can be computed such that the solution $x(\cdot)$ of the system (1) is a polynomial trajectory of degree M with $x(0) = x_0$ and $x(t_f) = x_f$.

Furthermore, from (Ailon and Langholz, 1986) the procedure to determine a polynomial pair $\{x(\cdot), u(\cdot)\}$ that satisfies the differential equation (1) and the end-point conditions is obtained as follows.

Let

$$x\left(t\right) = x_0 + \sum_{i=1}^{M} d_i t^i, t \geq 0, \qquad (6)$$

where for each i, $d_i = [d_{1i}, d_{2i}, \cdots d_{ni}]^T \in \Re^n$ is a constant vector. The vector $d = [d_1^T, d_2^T, \cdots, d_M^T]^T \in \Re^{Mn}$ of the unknown coefficients in (6) is obtained by solving the algebraic equation

$$H_M\left(t_f\right) d = \begin{bmatrix} CAx_0 \\ 0 \\ \vdots \\ 0 \\ x_f - x_0 \end{bmatrix}, \qquad (7)$$

where the $[(M+1)r+n] \times Mn$ dimensional matrix $H_M\left(t\right)$ is of *full rank* for any fixed $t > 0$.

The polynomial vector-valued function $u(\cdot)$ is obtained from

$$\dot{x}\left(t\right) - Ax\left(t\right) = Bu\left(t\right), \qquad (8)$$

where $x\left(\cdot\right)$ is determined by (6) and (7). Since the coefficients vector d solves (7), $x(t)$ in (6) satisfies (Ailon and Langholz, 1986)

$$C\left[\dot{x}\left(t\right) - Ax\left(t\right)\right] = 0; t \geq 0. \qquad (9)$$

Hence, from (2) for any fixed positive t, $\dot{x}\left(t\right) - Ax\left(t\right)$ belongs to the column space of B and (8) determines uniquely $u(t)$. Furthermore, since the columns of B are linearly independent, (8) yields

$$u(t) = B^{+}\left[\dot{x}(t) - Ax(t)\right], \tag{10}$$

where $B^{+} \doteq (B^T B)^{-1} B^T$ is the *(Moore-Penrose) pseudo-inverse* of B.

Remark 2.1. Let $\alpha \doteq n - r$. Clearly (recall that $0 < r < n$), $\alpha \geq 1$. Then

$$\begin{aligned} Mn - [(M+1)r + n] &= M(r+\alpha) - \\ [(M+1)r + r + \alpha] &= M\alpha - 2r - \alpha. \end{aligned}$$

Hence, if $\alpha = 1$ and $M = 2r+1$ the matrix $H_M(t)$ in (4) is square ($Mn \times Mn$). Since in the general case $\alpha \geq 1$ and the integer M will be selected such that $M > 2r+1$, for each fixed $t > 0$ we have from Lemma 2.1 $rank[H_M(t)] = (M+1)r + n$ (= number of rows< number of columns). For further applications note that the difference between the numbers of columns and rows in H_M increases with M.

3. MAIN RESULTS

Based on the results presented in the previous section we wish to consider further applications of the tools resulting from the concept of the polynomial controllability. Here the control objective is to accomplish the transfer of the system from a given state to a desired target while some selected performance criterions are taken into account.

Thus, define $n \times n$ real constant matrices $Q = Q^T \geq 0$ and $R = R^T > 0$. The design objective is to determine a control function $u(\cdot)$ that transfers the system (1) from $x(0) = x_0$ to $x(t_f) = x_f$ where x_f and t_f are specified final state and time, such that the performance measure

$$J = \frac{1}{2}\int_0^{t_f} \left[x^T(t)\,Qx(t) + u^T(t)\,Ru(t)\right]dt, \tag{11}$$

is minimized. The performance index (11) is standard in the framework of optimal control (Kirk, 1970; Lewis, 1986). In particular if $x_f = 0$, the functional J is associated with the linear regulator control problem. Later on the criterion (11) will be modified to allow greater generality.

Before we present the main results of this study, some considerations are in order concerning the nature of J in (11) while the integrand depends on a pair of polynomial functions x and u.

Since the right-hand side of (11) depends on x_0, d, and t_f, we write $J = J(x_0, d, t_f)$. Assume momentarily that $x_0 = 0$ and consider $J(0, d, t_f)$. Since the integrand in (11) contains quadratic forms, for any fixed $t_f > 0$ the performance index $J(0, t_f, d)$ is a quadratic form of the (coefficients) vector $d = [\delta_i, \cdots, \delta_{Mn}]^T$. That is, we can write $J(0, d, t_f)$ as follows:

$$\begin{aligned} J(0,d,t_f) &= \frac{1}{2}\sum_{i,j} f_{ij}(t_f)\delta_i\delta_j \\ &= \frac{1}{2}(f_{11}(t_f)\delta_1^2 + \cdots + f_{Mn,Mn}(t_f)\delta_{Mn}^2 \\ &\qquad + 2\sum_{i<j} f_{ij}(t_f)\delta_i\delta_j). \end{aligned} \tag{12}$$

The above expression in the variables δ_i is a quadratic polynomial corresponding to the representation (for simplicity we omit the finite time t_f below):

$$\begin{aligned} J(0,d,t_f) &= \frac{1}{2} d^T \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1,Mn} \\ f_{21} & f_{22} & \cdots & f_{2,Mn} \\ \cdots & \cdots & \cdots & \cdots \\ f_{Mn,1} & f_{Mn,2} & \cdots & f_{Mn,Mn} \end{bmatrix} d \\ &\doteq \frac{1}{2} d^T F d. \end{aligned} \tag{13}$$

It is clear that for each $t_f > 0$, $F \geq 0$. We will show that $F > 0$. If $Q > 0$ then any $d \neq 0$ implies in (6) $x(t) \neq 0$, and hence $J(0, d, t_f) \neq 0$. Assume that Q is positive semi-definite and for some $d \neq 0$ the resulting $x(t) \neq 0$ satisfies $x^T(t)Qx(t) = 0$ for all $t \in [0, t_f]$. If $J(0, d, t_f) = \frac{1}{2}d^T F d = 0$ we must have (since by definition $R > 0$) $u(t) = 0$ for all $t \in [0, t_f]$. But observing (10) and (2) this implies that $\dot{x}(t) - Ax(t)$ belongs to the column space of C^T, which contradicts (9).

Using previous observations we conclude that if we remove the restriction $x_0 = 0$, we have

$$\begin{aligned} J(x_0, d, t_f) &= \frac{1}{2}\left(x_0^T Q x_0\right) t_f + \\ &\quad \frac{1}{2} g(x_0, d, t_f) + \frac{1}{2} d^T F d, \end{aligned} \tag{14}$$

where $g(x_0, d, t_f)$ depends linearly on d, and $F = F^T > 0$.

Remark 3.1. The coefficients vector d yields a polynomial vector-valued function $x(\cdot)$ by means of (6), which in its turn determines $u(\cdot)$ in (10). Hence, $d \in \Re^{Mn}$ uniquely defines a polynomial pair $\{x(\cdot), u(\cdot)\}$. Moreover, equation (7) for the unknown vector d is a necessary and sufficient condition that the pair $\{x(\cdot), u(\cdot)\}$ satisfies the state equation (1) together with the end-point conditions $x(0) = x_0$ and $x^*(t_f) = x_f$. Therefore in the framework of this study we shall seek a vector d^* that minimizes $J(x_0, d, t_f)$, subject to the constraint (7).

Theorem 3.1. Fix an integer $M > 2r + 1$ and a constant $t_f > 0$, and select arbitrarily initial *and final states x_0 and x_f respectively. Then there exists a unique d^* such*

$$J(x_0, d^*, t_f) < J(x_0, d, t_f) \qquad (15)$$

for any $d \neq d^*$, subject to the constraint (7).

Proof. We show first that there exists a point d^* which is a *strict local minimum* of $J(x_0, d, t_f)$ subject to the constraint (7). Observing the constraint equation it appears that any d that satisfies (7) is a *regular point*. This conclusion follows from the fact that $H_M(t_f)$ is surjective and hence, if $h_i(d)$ is the $i-th$ row of $H_M(t_f)$ multiplied by d the gradient vectors $\nabla h_1(d)$, $\nabla h_2(d)$, ..., $\nabla h_{(M+1)r+n}(d)$, are linearly independent.

Observing (7) and (14), the *first-order necessary condition* for the constrained optimal problem, together with the constraint equation, are found to be

$$\begin{aligned} \nabla J(x_0, d, t_f) + \lambda^T \nabla H_M(d) = 0 \\ H_M(t_f)\, d - c = 0, \end{aligned} \qquad (16)$$

where $\lambda \in \Re^{(M+1)r+n}$ is some constant vector, $\lambda^T \nabla H_M$ is the gradient of $\lambda^T H_M$, and $c \in \Re^{(M+1)r+n}$ is a constant vector representing the right hand-side of (7). Note that (we use the notations of (Luenberger, 1989)) λ is the *Lagrangian* associated with constrained problem, and

$$\nabla W(z) = \left[\frac{\partial w_i(z)}{\partial z_j} \right],$$

where w_i is the $i-th$ row of W.

Next we claim that there exists a unique vector d that satisfies (16). Recalling (7) and (14) we can re-write (16) as follows

$$\begin{bmatrix} F & H_M^T \\ H_M & 0 \end{bmatrix} \begin{bmatrix} d \\ \lambda \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}, \qquad (17)$$

where (recall that $g(\cdot,\cdot,\cdot)$ depends linearly on d) $2b = -\partial g(x_0, d, t_f)/\partial d$ is independent of d.

To assert the claim we must show that the matrix on the left-hand side of (17) is nonsingular. To this end we recall that $H_M(t_f)$ is surjective and that there exists a square nonsingular matrix $L \in \Re^{Mn \times Mn}$ such that

$$H_M L = \begin{bmatrix} H_\# & 0 \end{bmatrix} \qquad (18)$$

where $H_\#(t_f) \in \Re^{((M+1)r+n)\times((M+1)r+n)}$ is nonsingular for any fixed $t_f > 0$. Define

$$Q \doteq \begin{bmatrix} L & 0 \\ 0 & I_{(M+1)r+n} \end{bmatrix}. \qquad (19)$$

We have

$$Q^T \begin{bmatrix} F & H_M^T \\ H_M & 0 \end{bmatrix} Q = \begin{bmatrix} L^T F L & L^T H_M^T \\ H_M L & 0 \end{bmatrix}. \qquad (20)$$

Partitioning the symmetric matrix $L^T F L \doteq \Phi$ into four blocks and applying (18) we have from (20)

$$\begin{bmatrix} L^T F L & L^T H_M^T \\ H_M L & 0 \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12}^T & H_\#^T \\ \Phi_{12} & \Phi_{22} & 0 \\ H_\# & 0 & 0 \end{bmatrix}. \qquad (21)$$

Since the matrices Φ and F are congruent and $F = F^T$ is positive definite, so is Φ. Therefore the symmetric matrix Φ_{22} is positive definite.

Recalling that $H_\#$ is a square nonsingular matrix and applying a sequence of elementary operations it appears that the matrix in (21) is singular if and only if Φ_{22} is singular, which contradicts the positive definiteness of Φ, and hence of F. This asserts our claim and we assign the unique vector d that solves (16) by d^*. Therefore the *first-order necessary condition* for d^* to be a local minimum point subject to the equality constraints has been established.

Recalling that the *Hessian* of $J(x_0, d, t_f)$ is $F = F^T > 0$ and the Hessian of the function $\lambda^T(H_M(t_f)d - c)$ (associated with the constraint) is zero, the *second-order sufficient condition* for the constrained problem is satisfied. Since the necessary and the sufficient conditions are fulfilled, d^* is a *strict local minimum* of J subject to the constraint $H_M(t_f)d - c = 0$ (Luenberger, 1989, Chap. 10).

Next, we shall show that d^* is a strict global minimum point of J. Clearly (16), or equivalently (17), gives a total of $Mn + [(M+1)r+n]$ equations in the $Mn+[(M+1)r+n]$ variables comprising d and λ. Since $H_M(t_f)d - c = 0$ and H_M is surjective, $(M+1)r+n$ components of d depend linearly on the other $Mn-[(M+1)r+n]$ components. (As illustrated in Remark 2.1 for $M > 2r+1$ the integer $Mn - [(M+1)r+n]$ is strictly positive.) In this way we may represent the performance index J by $\bar{J}(x_0, \bar{d}, t_f)$ where $\bar{d} \in \Re^{Mn-[(M+1)r+n]}$. Since $\bar{d}^*$ minimizes $\bar{J}$ where d^* minimizes J subject to the constraint $H_M(t_f)d - c = 0$ and vice versa, if we show that the Hessian of $\bar{J}$, denoted by $\bar{F}$ is positive definite, we complete the proof. But this is obvious because by evaluating the functional $\bar{J}$ we obtain a function of $\bar{d}$ which is similar to the right-hand side of (14); that is, as far as the vector $\bar{d}$ is concerned it contains terms of three types: those which are independent of $\bar{d}$, terms which depend linearly on $\bar{d}$, and a quadratic polynomial corresponding to the representation $\frac{1}{2}\bar{d}^T \bar{F} \bar{d}$. ◇◇

To further generalize the results obtained thus far we consider a tracking control problem as follows. Suppose that the system motion between a given initial position x_0 and a desired final target $x(t_f) = x_f$ should follow as closely as possible a predetermined trajectory $\sigma(\cdot)$ connecting the two points. In the present case the performance measure to be minimized is

$$J = \frac{1}{2}\int_0^{t_f} ([x(t)-\sigma(t)]^T Q[x(t)-\sigma(t)] + u^T(t)Ru(t))dt, \tag{22}$$

where $\sigma : [0,t_f] \to \Re^n$ is uniformly continuous. In view of the approach of this study we are looking for a polynomial input that allows us to present a suboptimal solution to the present control problem.

Let $||\sigma||_{L_\infty} = \sup_{t\in[0,t_f]} ||\sigma(t)||$, where $||\cdot||$ is the Euclidean norm. In view of Weierstrass's Theorem (Lang, 1968, Chap. XI) σ can be uniformly approximated by polynomials on $[0,t_f]$. Hence, fix an $\epsilon > 0$ sufficiently small and take an integer $M > 2r+1$ such that the polynomial

$$s(t) = s_0 + \sum_{i=1}^{M} a_i t^i \tag{23}$$

with $a_i \in \Re^n$ satisfies

$$\|s(t) - \sigma(t)\|_{L_\infty} \le \epsilon; \forall t \in [0,t_f]. \tag{24}$$

Using (23)-(24) we modify the performance index J in (22) as follows

$$\Im = \frac{1}{2}\int_0^{t_f} ([x(t)-s(t)]^T Q[x(t)-s(t)] + u^T(t)Ru(t))dt. \tag{25}$$

In the framework of the paper approach it is required now to find a polynomial pair $\{x,u\}$ (of degree M) that minimizes $\Im$ subject to the constraint (7).

Evaluating the functional (25) we have (similarly to (14)) that $\Im$ contains three types of term: those which depend exclusively on x_0 and a_i (see (23)), and are independent of d; terms which depend linearly on d, and the quadratic term $\frac{1}{2}d^T Fd$ which is independent of x_0 and a_i. From here the suboptimal solution follows from Theorem 3.1 and its constructive proof.

4. APPLICATIONS TO SINGULAR SYSTEMS

In this section we apply the tools developed in the previous section to a linear singular system model. Some preliminary observations concerning the model of singular systems, are to be represented first.

We consider the model

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{26}$$

where $E \in \Re^{n\times n}$ is a singular matrix. It is assumed that (26) is solvable, i.e., $\det(sE-A) \neq 0$ for almost all s. The singular system is *c-controllable* if any state is reachable from any initial state. (We consider only the concept of c-controllable singular systems. The type of *r-controllable* singular systems, will not be treated here.) The system (26) is c-controllable if and only if (Yip and Sincovec, 1981) for any finite s the augmented matrices

$$[sE-A]\,;[E \quad B] \tag{27}$$

are of full rank.

Note that if the system is c-controllable and C satisfies (2), then the matrix CE is surjective. In fact if this is not the case, there is a nonzero vector $\eta \in \Re^r$ such that $\eta^T CE = 0$ and $\eta^T C \neq 0$ which means that $\eta^T C\,[E \quad B] = 0$, i.e., the system is not c-controllable.

Take $M \ge 2r+1$ and define an $(M+1)r \times Mn$ constant matrix $\tilde{W}_M$ as follows

$$\tilde{W}_M = \begin{bmatrix} \tilde{C} & 0 & 0 & \cdots 0 & 0 \\ -CA & 2\tilde{C} & 0 & \cdots 0 & 0 \\ 0 & -CA & 3\tilde{C} & \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \; \vdots & \vdots \\ 0 & 0 & 0 & \cdots -CA & M\tilde{C} \\ 0 & 0 & 0 & \cdots 0 & -CA \end{bmatrix} \tag{28}$$

where $\tilde{C} \doteq CE$, and consider the matrix

$$\tilde{H}_M(t) = \begin{bmatrix} \tilde{W}_M \\ tI_n \; t^2 I_n \; \cdots \; t^M I_n \end{bmatrix}. \tag{29}$$

We have the following results (Ailon and Berman, 1989).

Lemma 4.1. C-controllability of the system (26) implies that for each fixed $t > 0$ the matrix $\tilde{H}_M(t)$ is of full rank for any fixed M satisfying $M \ge 2r+1$.

Theorem 4.1. Consider the c-controllable system (26) and let $u(t)$ be a control function with

$$u(t) = [p_1(t), \cdots, p_m(t)]^T, t \ge 0 \tag{30}$$

where $p_i(t)$ are polynomials. Let x_0 and x_f be some arbitrary initial and final states respectively. Then, for any selected $t_f > 0$ a polynomial command $u(t)$ of the form (30) with degree not larger than M where $M \ge 2r+1$, can be computed such that the solution $x(\cdot)$ of the system (26) is a polynomial trajectory of degree M with $x(0) = x_0$ and $x(t_f) = x_f$.

Using Lemma 4.1 and Theorem 4.1 we proceed as follows. Let $x(\cdot)$ be given by (6). Replace H_M in (7) with $\tilde{H}_M$ in (29), and let equation (10) be replaced by

$$u(t) = B^{+}\left[E\dot{x}(t) - Ax(t)\right], \qquad (31)$$

where $B^{+} \doteq (B^TB)^{-1}B^T$ is the pseudo-inverse of B. Then, whether the performance measure (11) or (22) is under consideration, the procedures established in Section 3 for obtaining suboptimal solutions to the constrained minimization problem for controllable regular state-space systems, can be applied straightforwardly to c-controllable singular systems.

5. CONCLUDING REMARKS

The paper presents a simple procedure for obtaining a suboptimal solution to a constrained minimization of a quadratic performance index for time-invariant linear controllable systems. While the standard solution to the problem requires the application of calculus of variations associated with minimizing a quadratic functional subject to boundary conditions and differential equations as constraints, the proposed approach exhibits a suboptimal solution which is achieved by minimizing a function of several unknowns, subject to a set of linearly independent equations. The approach is based on the concept of polynomial controllability. By increasing the polynomial's degree, the resulting suboptimal solution becomes more accurate with respect to the optimal solution. Applications of the approach to singular systems have been considered.

As in many other solution methods established for the constrained optimal control problem (Kirk, 1970, Chap.6; Lewis, 1986, Chap. 3), the proposed technique determines an open-loop suboptimal control, that is, a specification of the optimal control as a function of time and the boundary conditions, not of the current state.

In this paper no constraints have been imposed on the system input. However in real situations there are constraints on the control signals and the state variables. Under this condition the sub-optimal control problem under consideration is associated with the evaluation of an *admissible input function* which satisfies the control constraints and generates an *admissible state trajectory* (that satisfies equation (7)) and mimimizes a performance index J. Further developments and research in this direction is currently conduced. In this regard we indicate here the paper of Sussmann, 1987, where the concept of a polynomial map has been studied in connection with the problem of small-time local controllability of the optimal time function.

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