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15th Triennial World Congress, Barcelona, Spain

MIXED H_2/H_∞ CONTROL FOR LINEAR SINGULAR SYSTEMS

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Abstract: This paper considers the mixed H_2/H_∞ con trol problem for linear singular systems, the sufficient condition to the existence of a mixed H_2/H_∞ output feedback controller is given in terms of LMIs, the controller guarantees that the closed-loop system is impulse-free and stable, satisfies H_∞ performance, and minimizes the H_2 performance subject to these three constraints. *Copyright ©2002 IFA C*

Keywords: Linear system, singular system, mixed H_2/H_∞ control, output feedbac k, linear matrix inequalities.

1. INTRODUCTION

The singular system is a natural representation of linear dynamical systems, it is a more general system than the standard state-space system, so it is v ery important to study an y property of singular systems (Dai,1989). H_2, H_∞ control problems for singular systems hav e been studied based on model-matching (Tak aba and Katay ama, 1998) and the J-spectral factorization (Morihira et al., 1993; Takaba et al., 1994). In recent years, LMIs ha v e widely been applied to solve the H_∞ con trol problem for standard state-space systems (Iwasaki and Skelton, 1994), and the mixed H_2/ H_∞ etc. multiobjective control design problems (Scherer et al.,1997; Chilali and Gahinet, 1996). Since the calculability of LMIs, such LMIs hav e been extended to H_∞ control (Masubuchi et al., 1997; Rehm and Augöwer, 1999; Ma et al., 2000) and robust H_2 state feedback control (Takaba, 1998) for singular systems, but for the mixed H_2/H_∞ output feedback control of singular systems, there is few research.

In this paper, the mixed H_2/H_∞ output feedback control for linear singular systems is considered in terms of LMIs, the sufficient condition that the problem is feasible is giv en. The paper is structured as follows: The necessary preliminaries and the description of problem is given in Section 2. In Section 3, the output feedback con trollers design and the sufficient condition based on LMIs is giv en. Section 4 and Section 5 are example and conclusions, respectively.

2. PRELIMINARIES

This paper considers a linear singular system

$$\left\{\begin{array}{ll} E\dot{x} & = Ax + B_1 w + B_2 u \\ z_0 & = C_0 x + D_{02} u \\ z_1 & = C_1 x + D_{12} u \\ y & = C_2 x + D_{21} w \end{array}\right. \tag{1}$$

where $x \in R^n$ is the state variable, $u \in R^p$ is the control input, $w \in R^q$ is the exogenous input, $z_0 \in R^s$, $z_1 \in R^m$ is the con trolledoutput, $y \in R^l$ is the measured output. $E \in R^{n\times n}$ is a consant singular matrix, that is rank$E = r < n$. Other coefficient matrices are appropriate dimension consant.

First, consider the following singular system

$$E\dot{x} = Ax + Bw, \quad z = Cx, \tag{2}$$

where $x \in R^n$ is the state variable, $w \in R^q$ is the exogenous input, $z \in R^m$ is the controlled output, $E \in R^{n\times n}$ is consant, and rank$E = r < n$.

Definition 1 (Masubuchi et al., 1997). A pair (E, A) is said to be admissible, if it is regular, impulse-free and stable.

Let $G(s) = C(sE - A)^{-1}B$ be the transfer function of the system (2) from w to z, then the H_2 norm is defined by (Takaba and Katayama, 1998)

$$\|G(s)\|_2 = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\{G^*(j\omega)G(j\omega)\}dw\right]^{1/2}.$$

Lemma 1 (Takaba, 1998). The singular system (2) is admissible, and for given $\gamma > 0$, the transfer function $G(s)$ satisfies $\|G(s)\|_\infty < \gamma$, if and only if there exists a matrix $X \in R^{n\times n}$ such that

$$XE^T = EX^T \geq 0, \tag{3a}$$

$$\mathcal{B}_1(\gamma, X, A, B, C) = \begin{bmatrix} XA^T + AX^T & XC^T & B \\ CX^T & -\gamma I & 0 \\ B^T & 0 & -\gamma I \end{bmatrix} < 0. \tag{3b}$$

Lemma 2 (Takaba, 1998). If the singular system (2) is admissible, and $C = \hat{C}E$, then the H_2 norm of the system (2) is given by

$$\|G(s)\|_2 = \{\mathrm{Tr}(\hat{C}EX^T\hat{C}^T)\}^{1/2},$$

where $X \in R^{n\times n}$ is a constant matrix satisfying

$$EX^T = XE^T,\ AX^T + XA^T + BB^T = 0. \tag{4}$$

Lemma 3. If there exists a matrix X_* satisfying

$$EX_*^T = X_*E^T \geq 0,\ AX_*^T + X_*A^T + BB^T < 0, \tag{5}$$

then, (1) (E, A) is admissible; (2) The inequality $EX^T \leq EX_*^T$ holds for any solution X of (4).

Lemma 4. If there exist matrices X, Z satisfying

$$EX^T = XE^T \geq 0, \tag{6a}$$

$$\mathcal{B}_2(X, A, B) = \begin{bmatrix} AX^T + XA^T & B \\ B^T & -I \end{bmatrix} < 0, \tag{6b}$$

$$\mathcal{B}_3(X, Z, E, \hat{C}, M) = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}^T \begin{bmatrix} Z & \hat{C}EX^T \\ XE^T\hat{C}^T & EX^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} > 0, \tag{6c}$$

then the system (2) is admissible, and the H_2 norm satisfies $\|G(s)\|_2^2 < \mathrm{Tr}(Z)$, where $\hat{C}$ and M satisfy that $C = \hat{C}E$, $\mathrm{Im}M = \mathrm{Im}E$.

Proof. First, the admissibility of the system (2) is obvious from (6a), (6b) and Lemma 3. Next prove that $\|G(s)\|_2^2 < \mathrm{Tr}(Z)$. The singular-values decomposition of E

$$UEV = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix},$$

where U, V are orthogonal matrices, $\Sigma_r = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, σ_i, $i = 1, 2, \cdots, r$ are singular-values of E. Accordingly, write

$$UXV = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix},\quad \hat{C}U^T = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix},$$

from (6a) and (6b), it yields

$$\begin{bmatrix} \Sigma_r X_1^T & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1\Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$
$$UXV = \begin{bmatrix} X_1 & X_2 \\ 0 & X_4 \end{bmatrix},\ \det X \neq 0,\ \Sigma_r X_1^T > 0,$$

and by $\mathrm{Im}M = \mathrm{Im}E$, it is obtained

$$UM = \begin{bmatrix} M_1 \\ 0 \end{bmatrix},\quad M_1 \in R^{r\times r},\quad \det M_1 \neq 0.$$

Consider (6c), we get

$$\begin{bmatrix} I & 0 \\ 0 & UM \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} Z & \hat{C}EX^T \\ XE^T\hat{C}^T & EX^T \end{bmatrix}$$
$$\cdot \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & UM \end{bmatrix} > 0,$$

this is

$$\begin{bmatrix} I & 0 \\ 0 & [M_1^T \ \ 0] \end{bmatrix} R_Z \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} M_1 \\ 0 \end{bmatrix} \end{bmatrix} > 0,$$

where

$$R_Z = \begin{bmatrix} Z & \hat{C}_1\Sigma_r X_1^T & 0 \\ X_1\Sigma_r\hat{C}^T & \Sigma_r X_1^T & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

since M_1 is nonsingular shows that

$$\begin{bmatrix} Z & \hat{C}_1\Sigma_r X_1^T \\ X_1\Sigma_r\hat{C}^T & \Sigma_r X_1^T \end{bmatrix} > 0,$$

then $Z - \hat{C}_1\Sigma_r X_1^T\hat{C}^T > 0$, i.e. $\hat{C}EX^T\hat{C}^T < Z$, together with Lemma 2, Lemma 3, the conclusion is obtained. □

In this paper, the mixed H_2/H_∞ control problem for the linear singular system (1) is finding a controller $u = k(s)y$ such that

(1) the closed-loop system is impulse-free,
(2) the closed-loop system is stable,
(3) $\|T_{z_1w}(s)\|_\infty < \gamma$, (7)
(4) $\|T_{z_0w}(s)\|_2$ is minimized
subject to these (1) -(3) constraints,

where $T_{z_1w}(s)$, $T_{z_0w}(s)$ are the transfer functions from w to z_1, and w to z_0, respectively, $\gamma > 0$ is given.

3. CONTROLLERS DESIGN

The realization form of the controller $u = k(s)y$ is assumed as follows:

$$E_k \dot{x}_k = A_k x_k + B_k y, \quad u = C_k x_k, \tag{8}$$

here, we design a rth-order controller for the system (1), without loss of generality, let

$$E_k = E, \tag{9}$$

then closed-loop system formed by the controller (8), (9) and the system (1) is

$$E_c \dot{x}_c = A_c x_c + B_c w, \ z_0 = C_{c0} x_c, \ z_1 = C_{c1} x_c, \tag{10}$$

where $x_c = [x^T, \ x_k^T]^T$, and

$$\begin{cases} E_c = \begin{bmatrix} E & \\ & E \end{bmatrix}, \ A_c = \begin{bmatrix} A & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix}, \\ B_c = \begin{bmatrix} B_1 \\ B_k D_{21} \end{bmatrix}, \ C_{c0} = \begin{bmatrix} C_0 & D_{02} C_k \end{bmatrix}, \\ C_{c1} = \begin{bmatrix} C_1 & D_{12} C_k \end{bmatrix}. \end{cases} \tag{11}$$

In order to consider the closed-loop system (10), make the following assumption for the system (1):

Assumption 1. In the system (1), $C_0 = \hat{C}_0 E$.

From Lemma 1, Lemma 4, if

$$C_{c0} = \hat{C}_c E_c, \tag{12}$$

and there exist $X_c \in R^{2n\times 2n}$, $Z \in R^{s\times s}$ satisfying

$$X_c E_c^T = E_c X_c^T \geq 0, \tag{13a}$$

$$\mathcal{B}_1(\gamma, X_c, A_c, B_c, C_{c1}) < 0, \tag{13b}$$

$$\mathcal{B}_2(X_c, A_c, B_c) < 0, \tag{13c}$$

$$\mathcal{B}_3(X_c, Z, E_c, \hat{C}_c, M_c) > 0, \tag{13d}$$

where

$$\mathrm{Im} M_c = \mathrm{Im} E_c, \tag{13e}$$

then the closed-loop system (10) satisfies (1)-(3) in (7), and $\|T_{z_0 w}\|_2^2 < \mathrm{Tr}(Z)$, i.e. the mixed H_2/H_∞ control problem for the system (1) is feasible.

From (13b), X_c is a nonsingular matrix, when X_c is partitioned as

$$\begin{cases} X_c = \begin{bmatrix} X & X_2 \\ X_3 & X_4 \end{bmatrix}, \quad X \in R^{n\times n}, \\ X_i \in R^{n\times n}, \quad i = 2,3,4, \end{cases} \tag{14}$$

without loss of generality, we can assume that X, X_2, X_3 in (14) are nonsingular. If not, then make some modification (Ma et al., 2000; Masubuchi et al., 1997). Define matrices

$$\begin{cases} T_1 = \begin{bmatrix} I & 0 \\ 0 & XX_3^{-1} \end{bmatrix}, T_2 = \begin{bmatrix} I & 0 \\ 0 & X_2^{-1}X \end{bmatrix}, \\ T_3 = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_q \end{bmatrix}, T_4 = \begin{bmatrix} T_1 & 0 \\ 0 & I_q \end{bmatrix}, \end{cases} \tag{15}$$

$$\bar{X}_c = T_1 X_c T_2 = \begin{bmatrix} X & X \\ X & XX_3^{-1}X_4X_2^{-1}X \end{bmatrix}, \tag{16}$$

$$\bar{E}_c = T_1 E_c T_2^{-T} = \begin{bmatrix} E & \\ & XX_3^{-1}EX_2^T X^{-T} \end{bmatrix}$$

$$\overset{(13a)}{=} \begin{bmatrix} E & \\ & E \end{bmatrix}, \tag{17a}$$

$$\bar{A}_c = T_1 A_c T_2^{-T} = \begin{bmatrix} A & B_2 \bar{C}_k \\ \bar{B}_k C_2 & \bar{A}_k \end{bmatrix}, \tag{17b}$$

$$\bar{B}_c = T_1 B_c == \begin{bmatrix} B_1 \\ \bar{B}_k D_{21} \end{bmatrix}, \tag{17c}$$

$$\bar{C}_{ci} = C_{ci} T_2^{-T} = \begin{bmatrix} C_i & D_{i2}\bar{C}_k \end{bmatrix}, \ i = 0, 1, \tag{17d}$$

where

$$\begin{cases} \bar{A}_k = XX_3^{-1} A_k X_2^T X^{-T}, \\ \bar{B}_k = XX_3^{-1} B_k, \ \bar{C}_k = C_k X_2^T X^{-T}. \end{cases} \tag{17e}$$

Since the closed-loop system $(\bar{E}_c, \bar{A}_c, \bar{B}_c, \bar{C}_{ci})$ and (E_c, A_c, B_c, C_{ci}), $i = 0, 1$ have the same transfer functions, and by (12), (15), (17), (13), it is known that

$$\begin{aligned} &\bar{C}_{c0} = C_{c0} T_2^{-T} = \hat{C}_c E_c T_2^{-T} \\ = \ &\hat{C}_c T_1^{-1} \bar{E}_c = \bar{C}_c \bar{E}_c, \quad \bar{C}_c = \hat{C}_c T_1^{-1}, \end{aligned} \tag{18a}$$

$$\begin{aligned} &\bar{X}_c \bar{E}_c^T = T_1 (X_c E_c^T) T_1^T \\ = \ &T_1 (E_c X_c^T) T_1^T = \bar{E}_c \bar{X}_c^T \geq 0, \end{aligned} \tag{18b}$$

$$\begin{aligned} &\mathcal{B}_1(\gamma, \bar{X}_c, \bar{A}_c, \bar{B}_c, \bar{C}_{c1}) \\ = \ &T_3 \mathcal{B}_1(\gamma, X_c, A_c, B_c, C_{c1}) T_3^T < 0, \end{aligned} \tag{18c}$$

$$\mathcal{B}_2(\bar{X}_c, \bar{A}_c, \bar{B}_c) = T_4 \mathcal{B}_2(X_c, A_c, B_c) T_4^T < 0, \tag{18d}$$

$$\mathcal{B}_3(\bar{X}_c, Z, \bar{E}_c, \bar{C}_c, M_c) > 0, \tag{18e}$$

((18e) is derived by manipulations for (13d) similar to (28c) in the following, omitted), so the closed-loop system $(\bar{E}_c, \bar{A}_c, \bar{B}_c, \bar{C}_{ci})$ and (E_c, A_c, B_c, C_{ci}), $i = 0, 1$ satisfy (7) simultaneity. Note that

$$\begin{bmatrix} I & 0 \\ -X_3X^{-1} & I \end{bmatrix} \begin{bmatrix} X & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} I & -X^{-1}X_2 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} X & 0 \\ 0 & X_4 - X_3X^{-1}X_2 \end{bmatrix}, \tag{19}$$

so $X_4 - X_3X^{-1}X_2$ is nonsingular. Let

$$\begin{aligned} &XX_3^{-1}X_4X_2^{-1}X - X = \\ &XX_3^{-1}(X_4 - X_3X^{-1}X_2)X_2^{-1}X = S^{-1}, \end{aligned} \tag{20}$$

then $\bar{X}_c$ can be written as

$$\bar{X}_c = \begin{bmatrix} X & X \\ X & S^{-1} + X \end{bmatrix}. \tag{21}$$

Set

$$Y^T = S + X^{-1}, \tag{22}$$

by (21), (22) yields

$$\bar{X}_c^{-1} = \begin{bmatrix} Y^T & -S \\ -S & S \end{bmatrix}. \quad (23)$$

Compared the parameter matrices of the closed-loop system $(\bar{E}_c, \bar{A}_c, \bar{B}_c, \bar{C}_{ci})$ with that of (E_c, A_c, B_c, C_{ci}), $i = 0, 1$, the difference is only between the controller parameters $\bar{A}_k$, $\bar{B}_k$, $\bar{C}_k$ and A_k, B_k, C_k. So, without loss of generality, E_c, A_c, B_c, C_{c0}, C_{c1}, A_k, B_k, C_k and X_c in (10), (8) and (14) can be look as $\bar{E}_c$, $\bar{A}_c$, $\bar{B}_c$, $\bar{C}_{c0}$, $\bar{C}_{c1}$, $\bar{A}_k$, $\bar{B}_k$, $\bar{C}_k$ and $\bar{X}_c$ in (17) and (21). In the following discussion is as so.

Set

$$\begin{cases} T_5 = \begin{bmatrix} Y^T & -S \\ I & 0 \end{bmatrix}, T_7 = \begin{bmatrix} T_4 & 0 \\ 0 & I_q \end{bmatrix}, \\ T_6 = \begin{bmatrix} T_4 & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_q \end{bmatrix}, \end{cases} \quad (24)$$

then

$$T_5 X_c E_c^T T_5^T = T_5 E_c X_c^T T_5^T \geq 0, \quad (25a)$$

$$T_6 \mathcal{B}_1(\gamma, X_c, A_c, B_c, C_{c1}) T_6^T < 0, \quad (25b)$$

$$T_7 \mathcal{B}_2(X_c, A_c, B_c,) T_7^T < 0, \quad (25c)$$

$$\begin{bmatrix} I & 0 \\ 0 & M_c \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & T_5^{-1} \end{bmatrix} Q_Z \cdot \begin{bmatrix} I & 0 \\ 0 & T_5^{-T} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & M_c \end{bmatrix} > 0, \quad (25d)$$

$$Q_Z = \begin{bmatrix} Z & \hat{C}_c E_c X_c^T T_5^T \\ T_5 X_c E_c^T \hat{C}_c^T & T_5 E_c X_c^T T_5^T \end{bmatrix}. \quad (25e)$$

Make the singular-values decomposition of E

$$UEV = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}, \quad (26)$$

accordingly, write

$$\begin{cases} \bar{A} = UAV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\ \bar{B}_1 = UB_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \\ \bar{B}_2 = UB_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \\ \bar{C}_0 = \hat{C}_0 U^T = \begin{bmatrix} \hat{C}_{01} & \hat{C}_{02} \end{bmatrix}, \\ \bar{C}_1 = C_1 V = [C_{11} \ \ C_{12}], \\ \bar{C}_2 = C_2 V = [C_{21} \ \ C_{22}]. \end{cases} \quad (27)$$

For the mixed H_2/H_∞ control design of the system (1), we have

Theorem 1. Assume that Assumption 1 holds, then the mixed H_2/H_∞ control problem for the system (1) is feasible, if there exist matrices X_0, Y_0, X_2, X_4, Y_3, Y_4, Z, $\bar{L}$, $\bar{W}_B$, K_1, W_{C21}, W_{C22}, and X_4, $\begin{bmatrix} Y_4^T & I \\ I & X_4 \end{bmatrix}$ are nonsingular, such that the following LMIs

$$\begin{bmatrix} \bar{R}_1 & \bar{L} & \bar{C}_1^T & \bar{R}_3 \\ \bar{L}^T & \bar{R}_2 & \bar{R}_4 & \bar{B}_1 \\ \bar{C}_1 & \bar{R}_4^T & -\gamma I & 0 \\ \bar{R}_3^T & \bar{B}_1^T & 0 & -\gamma I \end{bmatrix} < 0, \quad (28a)$$

$$\begin{bmatrix} \bar{R}_1 & \bar{L} & \bar{R}_3 \\ \bar{L}^T & \bar{R}_2 & \bar{B}_1 \\ \bar{R}_3^T & \bar{B}_1^T & -I \end{bmatrix} < 0, \quad (28b)$$

$$\begin{bmatrix} Z & \hat{C}_{01}\Sigma_r & \bar{R}_5 \\ \Sigma_r \hat{C}_{01}^T & Y_0 & \Sigma_r \\ \bar{R}_5^T & \Sigma_r & X_0 \end{bmatrix} > 0, \quad (28c)$$

hold. Where

$$\begin{cases} \bar{R}_1 = \bar{A}^T \bar{Y} + \bar{Y}^T \bar{A} - \bar{C}_2^T \bar{W}_B^T - \bar{W}_B \bar{C}_2, \\ \bar{R}_2 = \bar{X}\bar{A}^T + \bar{A}\bar{X}^T + \bar{W}_C^T \bar{B}_2^T + \bar{B}_2 \bar{W}_C, \\ \bar{R}_3 = \bar{Y}^T \bar{B}_1 - \bar{W}_B D_{21}, \bar{R}_4 = \bar{X}\bar{C}_1^T + \bar{W}_C^T D_{12}^T, \\ \bar{R}_5 = \hat{C}_{01} X_0 + \hat{D}_{02} K_1, \end{cases} \quad (29a)$$

$$\begin{cases} \bar{X} = UXV = \begin{bmatrix} X_0 \Sigma_r^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}, \\ \bar{Y} = UYV = \begin{bmatrix} \Sigma_r^{-1} Y_0 & 0 \\ Y_3 & Y_4 \end{bmatrix}, \\ \bar{W}_C = P \begin{bmatrix} K_1 & 0 \\ W_{C21} & W_{C22} \end{bmatrix}, \\ D_{02} P = \begin{bmatrix} \hat{D}_{02} & 0 \end{bmatrix}, \end{cases} \quad (29b)$$

where P is a nonsingular matrix, $\hat{D}_{02}$ has full column rank,

$$\begin{cases} W_B = SB_k, \quad W_C = C_k X^T, \\ L = Y^T A X^T + Y^T B_2 W_C \\ \quad -W_B C_2 X^T - S A_k X^T + A^T, \end{cases} \quad (29c)$$

$$\bar{W}_B = V^T W_B, \ \bar{W}_C = W_C U^T, \ \bar{L} = V^T L U^T. \quad (29d)$$

Proof. By Lemma 1, Lemma 4 it is known that if (12), (13) hold, then the mixed H_2/H_∞ control problem for the system (1) is feasible. By the former discussion, (13) is equivalent to (25), so it only need to prove that (12), (25) are equivalent to (28). First, by (25a), (25b) yields

$$= \begin{bmatrix} E^T & 0 \\ 0 & E \\ Y^T & I \\ I & X \end{bmatrix} \begin{bmatrix} Y & I \\ I & X^T \\ E & 0 \\ 0 & E^T \end{bmatrix} \geq 0, \quad (30)$$

and X, $\begin{bmatrix} Y^T & I \\ I & X \end{bmatrix}$ are nonsingular, let

$$\begin{cases} T_8 = \begin{bmatrix} V^T & 0 \\ 0 & U \end{bmatrix}, T_9 = \begin{bmatrix} T_8 & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_q \end{bmatrix}, \\ T_{10} = \begin{bmatrix} T_8 & 0 \\ 0 & I_q \end{bmatrix}, \end{cases} \quad (31a)$$

$$\begin{cases} \bar{X} = UXV = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \\ \bar{Y} = UYV = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}, \end{cases} \tag{31b}$$

pre- and postmultiply the inequality (30) by T_8 and T_8^T, respectively, using X, $\begin{bmatrix} Y^T & I \\ I & X \end{bmatrix}$ are nonsingular, this yields

$$\begin{bmatrix} Y_0 & \Sigma_r \\ \Sigma_r & X_0 \end{bmatrix} > 0, \quad X_3 = 0, \quad Y_2 = 0, \tag{32a}$$

$$\det X_4 \neq 0, \quad \det \begin{bmatrix} Y_4^T & I \\ I & X_4 \end{bmatrix} \neq 0, \tag{32b}$$

where

$$X_0 = X_1\Sigma_r, \quad Y_0 = \Sigma_r Y_1. \tag{32c}$$

Pre- and postmultiply the inequality (25b) by T_9 and T_9^T, respectively, carrying out (29c), (29d), and (32a), (32c), this yields the LMI (28a); Pre- and postmultiply the inequality (25c) by T_{10} and T_{10}^T, respectively, the LMI (28b) is obtained similarly.

Next, consider the form of the matrix $\bar{W}_C$. Since (12) and Assumption 1 hold, $\hat{C}_c$ can be written as $\hat{C}_c = \begin{bmatrix} \hat{C}_0 & \hat{C}_k \end{bmatrix}$, then by

$$\begin{bmatrix} C_0 & D_{02}C_k \end{bmatrix} = \begin{bmatrix} \hat{C}_0 & \hat{C}_k \end{bmatrix} \begin{bmatrix} E & \\ & E \end{bmatrix},$$

it yields

$$D_{02}C_k = \hat{C}_k E. \tag{33}$$

Postmultiply (33) by X^T, then

$$\begin{aligned} D_{02}W_C &= D_{02}PP^{-1}W_C \overset{(29b)}{=} [\hat{D}_{02} \;\; 0]P^{-1}W_C \\ &= \hat{C}_k E X^T \overset{(30)}{=} \hat{C}_k X E^T, \end{aligned} \tag{34}$$

where $\hat{D}_{02}$, P are shown as (29b). Set

$$W_C = P \begin{bmatrix} W_{C1} \\ W_{C2} \end{bmatrix}, \tag{35}$$

from (35) yields

$$\hat{D}_{02}W_{C1} = \hat{C}_k X E^T. \tag{36}$$

Write $N_0 = \mathrm{Ker}E^T$, then $\hat{D}_{02}W_{C1}N_0 = 0$, since $\hat{D}_{02}$ has full column rank, $W_{C1}N_0 = 0$, namely $\mathrm{Ker}E^T \subseteq \mathrm{Ker}W_{C1}$, this is

$$W_{C1} = \bar{W}_{C1}E^T. \tag{37}$$

Thus,

$$\begin{aligned} W_{C1} &= \bar{W}_{C1}VV^TE^TU^TU \\ &= \begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} U \\ &= \begin{bmatrix} \bar{K}_1\Sigma_r & 0 \end{bmatrix} U, \end{aligned} \tag{38}$$

where $\bar{W}_{C1}V = \begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix}$. Synthesize (35), (38), and let $W_{C2}U^T = [W_{C21} \;\; W_{C22}]$, $K_1 = \bar{K}_1\Sigma_r$, then

$$W_C = P \begin{bmatrix} K_1 & 0 \\ W_{C21} & W_{C22} \end{bmatrix} U, \tag{39}$$

i.e. $\bar{W}_C$ has the form (29b).

Last, consider the equivalence of the inequality (25d) and (28c). Since Assumption 1 and (12) hold, it follows that

$$Q_Z = \begin{bmatrix} Z & \begin{bmatrix} \hat{C}_0E & R_6 \end{bmatrix} \\ \begin{bmatrix} E^T\hat{C}_0^T \\ R_6^T \end{bmatrix} & \begin{bmatrix} E^TY & E^T \\ E & EX^T \end{bmatrix} \end{bmatrix}, \tag{40}$$

where

$$R_6 = \hat{C}_0EX^T + D_{02}W_C, \tag{41}$$

by (13e) yields

$$\begin{cases} \bar{M}_c = \begin{bmatrix} U & \\ & U \end{bmatrix} M_c = [M_{c1}^T \;\; 0 \;\; M_{c2}^T \;\; 0]^T \\ M_{c1} \in R^{r\times 2r}, M_{c2} \in R^{r\times 2r}, \det \begin{bmatrix} M_{c1} \\ M_{c2} \end{bmatrix} \neq 0. \end{cases} \tag{42}$$

Set

$$T_{11} = \begin{bmatrix} I & \\ & T_8 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T_5^{-T} \end{bmatrix} \begin{bmatrix} I & \\ & M_c \end{bmatrix}, \tag{43}$$

by (22), (30), it yields $ES^{-T} = S^{-1}E^T$, then

$$V^TS^{-T}U^T = \begin{bmatrix} S_1 & 0 \\ S_3 & S_4 \end{bmatrix}, \tag{44}$$

together with (31b), (32a), (42), it follows that

$$T_{11} = \begin{bmatrix} I & 0 \\ 0 & -S_1M_{c2} \\ 0 & * \\ 0 & M_{c1} + Y_1S_1M_{c2} \\ 0 & * \end{bmatrix}, \tag{45}$$

where $*$ expresses the needless matrix blocks. Write

$$\begin{aligned} T_{12} &= \begin{bmatrix} -S_1M_{c2} \\ M_{c1} + Y_1S_1M_{c2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -S_1 \\ I & Y_1S_1 \end{bmatrix} \begin{bmatrix} M_{c1} \\ M_{c2} \end{bmatrix}, \end{aligned} \tag{46}$$

obviously, T_{12} is nonsingular. Thus, by (40), (43),(45), (46), it follows that inequality (25d) is equivalent to

$$\begin{aligned} &T_{11}^T \begin{bmatrix} I & \\ & T_8 \end{bmatrix} Q_Z \begin{bmatrix} I & \\ & T_8^T \end{bmatrix} T_{11} = \begin{bmatrix} I & \\ & T_{12}^T \end{bmatrix} \\ &\cdot \begin{bmatrix} Z & \hat{C}_{01}\Sigma_r & \bar{R}_5 \\ \Sigma_r\hat{C}_{01}^T & Y_0 & \Sigma_r \\ \bar{R}_5^T & \Sigma_r & X_0 \end{bmatrix} \begin{bmatrix} I & \\ & T_{12} \end{bmatrix} > 0, \end{aligned} \tag{47}$$

where $\bar{R}_5$ is shown as $(29a)$, since T_{12} is nonsingular shows that $(28c)$ holds. It is evident that the inequality $(28c)$ contains the inequality $(32a)$, synthesize the above discussion, the conclusion of Theorem is obtained.

Theorem 1 is LMIs with constraints, the mixed H_2/H_∞ controller for the system (1) can be obtained by the following

(1) Solve LMIs (28) to obtain X_0, Y_0, X_2, X_4, Y_3, Y_4, Z, $\bar{L}$, $\bar{W}_B$, K_1, W_{C21}, W_{C22}, such that $\mathrm{Tr}(Z)$ is minimum, for a given $\gamma > 0$.

(2) If X_4, $\begin{bmatrix} Y_4^T & I \\ I & X_4 \end{bmatrix}$ are nonsingular, find S by (22), and find A_k, B_k, C_k by (29).

(3) If X_4, $\begin{bmatrix} Y_4^T & I \\ I & X_4 \end{bmatrix}$ are singular, take a scalar α small enough such that $\hat{X}_4 = X_4 + \alpha I$, $I - \hat{X}_4 Y_4^T$ are nonsingular, and also satisfies (28).

(4) If the controller formed by the obove (E, A_k, B_k, C_k) is impulse-free, then the controller guarantees that the closed-loop system (10) satisfies (7). If it is nonimpulse-free, then based on the singular-values decomposition of E

$$\begin{cases} E &= U^T \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^T, \\ A_k &= U^T \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} V^T, \end{cases} \tag{48}$$

where $\bar{A}_{22}$ is singular, similar to Masubuchi et al. (1997), take μ is a scalar small enough such that $\bar{A}_{22} + \mu I$ is nonsingular and not to invalidate $(13b)$, $(13c)$.

4. EXAMPLE

Consider the linear singular system

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 3 & 3 & 0 \\ -1.5 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ D_{02} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.2 & 0.2 & 0 \end{bmatrix}, \ D_{12} = \begin{bmatrix} -1 & 1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix}, \ D_{21} = 5.$$

Let $\gamma = 1$, and solve the LMIs (28), then the controller matrices

$$A_k = \begin{bmatrix} -7.2299 & 9.2077 & 0.5887 \\ -2.0594 & -1.0126 & 1.0000 \\ 2.7821 & -0.3397 & -0.1551 \end{bmatrix},$$

$$B_k = \begin{bmatrix} -1.4952 & 0.0024 & 0.2008 \end{bmatrix}^T,$$

$$C_k = \begin{bmatrix} -0.5625 & 0 & 0 \\ -2.1718 & -0.3323 & 0.0775 \end{bmatrix}$$

are obtained by $(29c)$, the controller guarantees that the closed-loop system is impulse-free, and stable, $\|T_{z_1 w}\|_\infty < 1$, $\|T_{z_0 w}\|_2 < 1.415$.

5. CONCLUSIONS

In this paper, the mixed H_2/H_∞ output feedback control problem for linear singular systems is discussed, the sufficient condition that the problem is feasible is given in terms of LMIs.

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