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DUALITY THEORY IN FILTERING PROBLEM FOR DISCRETE VOLTERRA EQUATIONS

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Abstract: The filtering problem for discrete Volterra equations is a nontrivial task due to an increasing dimension of the equivalent single-step process model. A difference equation of a moderate dimension is chosen as an approximate model for the original system. Then the reduced Kalman filter can be used as an approximate but efficient estimator. Using the duality theory of convex variational problems, a level of nonoptimality for the chosen filter is obtained. This level can be efficiently computed without exact solving the full filtering problem. *Copyright ©2002 IFAC*

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1. INTRODUCTION

Various processes in motion control, space vehicle dynamics, mechanics, etc. are described by linear Volterra equations (Volterra, 1959; Kolmanovskii and Myshkis, 1999). In the paper, the mean-square filtering problem for discrete Volterra equations is considered. System perturbations and measurement noise are zero-mean white noise processes. Clearly, a discrete Volterra equation can be treated as an extended single-step equation of a corresponding order. So, the classical Kalman filtering technique could be used to solve the problem. However, the exact solving the filtering problem for Volterra equations is a nontrivial task.

In fact, if the amount of measurements is large (that is the case in most applications), then the operation with the matrices of very high dimension is unavoidable that leads to the accumulation of computational errors and to the critical retarding of the calculations. A way to overcome these unfavorable factors is to use simplified filters of moderate dimension. In this case, it is necessary to evaluate the level of nonoptimality for these simplified filters, remaining in the framework of relatively modest computation. In the paper, such approach is developed.

2. PROBLEM STATEMENT

Consider a discrete linear Volterra equation

$$x(j+1) = \sum_{k=0}^{j} A(j,k)x(k) + B(j)u(j), \qquad (1)$$

$$x(0) = x_0, \qquad j = 0, \ldots, N-1,$$

where $x(j) \in \mathbf{R}^n$ is the system state vector; $A(j,k) \in \mathbf{R}^{n\times n}$, $B(j) \in \mathbf{R}^{n\times r}$ and $u(j) \in \mathbf{R}^r$ is a perturbation vector. A prime denotes the transposition sign. The perturbations $u(j)$ are assumed to be a zero-mean white noise process with the covariance matrix $Q(j) \geq 0$:

$$\mathbf{E}u(j) = 0, \qquad \mathbf{E}\left(u(j)u'(k)\right) = Q(j)\delta_{jk}, \qquad (2)$$

$$j,k = 0,\dots,N-1,$$

where δ_{jk} is the Kronecker delta.

The measurements for the state vector have the form

$$z(k) = H'(k)x(k) + \varrho(k), \qquad z(k), \varrho(k) \in \mathbf{R}^m, \qquad (3)$$
$$H(k) \in \mathbf{R}^{n\times m}, \qquad k = 0,\dots,N.$$

The measurement errors $\varrho(k)$ are also assumed to be a zero-mean white noise process with the covariance matrix $R(k) > 0$:

$$\mathbf{E}\varrho(k) = 0, \qquad \mathbf{E}\left(\varrho(k)\varrho'(s)\right) = R(k)\delta_{ks}, \qquad (4)$$

$$k,s = 0,\dots,N.$$

The initial state vector of the system is a zero-mean random variable:

$$\mathbf{E}x(0) = 0, \qquad \mathbf{E}\left(x(0)x'(0)\right) = P_0 > 0 \qquad (5)$$

($x(0)$, u, and ϱ are mutually independent). The problem is to evaluate the scalar quantity $l_* = a'x(N)$, where $a \in \mathbf{R}^n$ is a given vector, with the help of linear functionals

$$\hat{l} = \sum_{i=0}^{N} \Phi'(i)z(i), \qquad \Phi(i) \in \mathbf{R}^m. \qquad (6)$$

The optimal mean-square filtering problem is to find an estimator Φ^0 that minimizes the mean-square estimation error:

$$d(\Phi^0) = \inf_{\Phi\in\mathbf{R}^{m\times(N+1)}} d(\Phi), \qquad (7)$$

$$d(\Phi) = \left\{\mathbf{E}\big(\hat{l} - l_*\big)^2\right\}^{\frac{1}{2}}.$$

For the Volterra equation (1), the resolvent $R(s,t)$ is defined by the equations

$$R(j+1,t) = \sum_{k=t}^{j} A(j,k)R(k,t), \qquad j \geq t,$$
$$(8)$$
$$R(j+1,t) = \sum_{k=t}^{j} R(j+1,k+1)A(k,t), \qquad t \leq j,$$

$$R(t,t) = E_n, \qquad R(t,s) = 0, \quad s > t,$$

where E_n is the identity matrix of order n.

It can be shown from (1), (8) that

$$x(j) = R(j,0)x(0) + \sum_{k=0}^{j-1} R(j,k+1)B(k)u(k). \quad (9)$$

Introduce the quantities:

(1) the function $\xi^{\Phi}(j)$, which is defined by the difference equation for a given $\{\Phi(j)\}_{j=0}^{j=N}$:

$$\xi^{\Phi}(j) = \sum_{k=j}^{N-1} A'(k,j)\xi^{\Phi}(k+1) - H(j)\Phi(j) \quad (10)$$

with the boundary condition

$$\xi^{\Phi}(N) = a - H(N)\Phi(N), \qquad j = 0,\dots,N-1;$$

(2) the functional $J(\Phi_-, \Phi, w)$: $\mathbf{R}^n \times \mathbf{R}^{m\times(N+1)} \times \mathbf{R}^{r\times N} \to \mathbf{R}^1$:

$$J(\Phi_-, \Phi, w) = \Big\{\Phi'_- P_0\, \Phi_- \qquad (11)$$

$$+\sum_{i=0}^{N} \Phi'(i)R(i)\Phi(i) + \sum_{j=0}^{N-1} w'(j+1)Q(j)w(j+1)\Big\}^{\frac{1}{2}}.$$

Using (8) and (10), it can be easily shown that

$$R'(N,k)a - \sum_{i=k}^{N} R'(i,k)H(i)\Phi(i) = \xi^{\Phi}(k), \quad (12)$$

$$k = 0,\dots,N.$$

Direct computation by virtue of (3), (6), (9), (12) result to the formula

$$\hat{l} - l_* = -\xi^{\Phi'}(0)x(0) + \sum_{i=0}^{N} \Phi'(i)\varrho(i) \qquad (13)$$

$$- \sum_{j=0}^{N-1} \xi^{\Phi'}(j+1)B(j)u(j).$$

Then, from (2), (4), (5), (7), (11), and (13),

$$d(\Phi) = J(\Phi_-(\Phi), \Phi, w(\Phi)), \qquad (14)$$

where the quantities $\Phi_-(\Phi)$ and $w(\Phi)$ are defined by the constraints

$$0 = \Phi_- - \xi^{\Phi}(0), \qquad (15)$$

$$0 = w(j+1) - B'(j)\xi^{\Phi}(j+1),$$
$$j = 0,\dots,N-1.$$

Thus the optimal mean-square filtering problem reduces to the following linear-quadratic convex variational problem:

$$J_0 = \inf_{\Phi_-\in\mathbf{R}^n,\,\Phi\in\mathbf{R}^{m\times(N+1)},\,w\in\mathbf{R}^{r\times N}} J(\Phi_-, \Phi, w) \quad (16)$$

under constraints (15).

3. BASIC RELATIONS FOR THE FILTERING PROBLEM SOLUTION

The structure of the filtering problem solution is described by the following basic result that uses the notion of the primal and dual variational problem.

Theorem 1. 1^0. *The solution* $\{\Phi^0_-, \Phi^0, w^0\}$ *of the linear-quadratic problem* (15), (16) *is given by the relations*

$$\Phi^0_- = \xi(0), \tag{17}$$

$$\Phi^0(i) = R^{-1}(i)H'(i)\eta(i), \qquad i = 0, \dots, N,$$

$$w^0(j+1) = B'(j)\xi(j+1), \qquad j = 0, \dots, N-1,$$

where $\xi(j)$, $\eta(j)$ *satisfy the multi-step boundary value problem*

$$\xi(j) = \sum_{k=j}^{N-1} A'(k,j)\xi(k+1) - H(j)R^{-1}(j)H'(j)\eta(j),$$

$$\eta(j+1) = \sum_{k=0}^{j} A(j,k)\eta(k) + B(j)Q(j)B'(j)\xi(j+1), \tag{18}$$

$$j = 0, \dots, N-1,$$

with the boundary conditions

$$\xi(N) = a - H(N)R^{-1}(N)H'(N)\eta(N),$$

$$\eta(0) = P_0\xi(0).$$

2^0. *Boundary value problem* (18) *has a unique solution.*

3^0. *The dual problem to the primal problem* (15), (16) *has the form*

$$J^0 = \sup_{p \in \mathbf{R}^{r\times N},\ \lambda \in \mathbf{R}^n} a'\tilde{x}(N) \tag{19}$$

under the constraint

$$\left\{\tilde{x}'(0)P_0^{-1}\tilde{x}(0) + \sum_{j=0}^{N} \tilde{x}'(j)H(j)R^{-1}(j)H'(j)\tilde{x}(j) + \sum_{j=0}^{N-1} p'(j+1)Q^{-1}(j)p(j+1)\right\}^{\frac{1}{2}} \le 1, \tag{20}$$

where the process $\tilde{x}(j) = \tilde{x}(j; p, \lambda)$ *satisfies the Volterra equation*

$$\tilde{x}(j+1) = \sum_{k=0}^{j} A(j,k)\tilde{x}(k) + B(j)p(j+1),$$

$$p(j+1) \in \mathbf{R}^r, \qquad j = 0, \dots, N-1,$$

with the initial condition $\tilde{x}(0) = \lambda$.

Moreover,

$$J_0 = J^0. \tag{21}$$

4^0. *The solution for the dual problem* (19), (20) *is determined by the formulas*

$$p^0(j+1) = J_0^{-1}Q(j)B'(j)\xi(j+1), \tag{22}$$

$$\lambda^0 = J_0^{-1}P_0\xi(0), \qquad j = 0, \dots, N-1.$$

The proof is based on the general theory of convex variational problems (Ekeland, Temam, 1976).

Note that (17) and (18) imply the equality

$$\xi^{\Phi^0}(j) = \xi(j), \qquad j = 0, \dots, N. \tag{23}$$

It follows from Theorem 1 that the finding of the optimal estimator reduces to the solving of the boundary value problem (18). Unfortunately, the solving of (18) without a substantial extending of dimension remains a serious problem.

In this case, the following approach is proposed. An approximate simplified estimator φ is sought instead of the optimal Φ^0. The approximation quality is defined by the ratio

$$\Delta = d(\varphi)/d(\Phi^0). \tag{24}$$

Obviously, $\Delta \ge 1$. Since Φ^0 is unknown, Δ is unknown. Then the aim is to construct an upper bound Δ^0 for Δ that can be computed without exact solving the optimal problem (15), (16). If Δ^0 is not large, then the use of the estimator φ is justified.

4. KALMAN FILTER FOR REDUCED SYSTEM

A natural way to simplify the Volterra equation is to replace the full model by the following reduced model in which the "tails" $y(j-s-1), \dots, y(0)$ are disregarded:

$$y(j+1) = \sum_{k=j-s}^{j} A(j,k)y(k) + B(j)\bar{u}(j),$$

$$j = 0, \dots, N-1, \tag{25}$$

$$y(0) = x_0, \qquad A(j,k) = 0 \ \text{ for } \ k < 0.$$

The noise $\bar{u}(j) \in \mathbf{R}^r$ is a zero-mean white noise process with a covariance matrix $\bar{Q}(j)$:

$$\mathbf{E}\bar{u}(j) = 0, \qquad \mathbf{E}\left(\bar{u}(j)\bar{u}'(k)\right) = \bar{Q}(j)\delta_{jk}, \tag{26}$$

$$j, k = 0, \dots, N-1.$$

The model for the measurements is defined by the equation

$$z(k) = H'(k)y(k) + \bar{\varrho}(k), \tag{27}$$

$$z(k), \bar{\varrho}(k) \in \mathbf{R}^m, \qquad k = 0, \dots, N.$$

The measurement noise $\bar{\varrho}(k)$ is a zero-mean white noise process with a covariance matrix $\bar{R}(k)$:

$$\mathbf{E}\bar{\varrho}(k) = 0, \qquad \mathbf{E}\left(\bar{\varrho}(k)\bar{\varrho}'(s)\right) = \bar{R}(k)\delta_{ks}, \quad (28)$$
$$k, s = 0, \ldots, N.$$

The model equation (25) has the same coefficients as equation (1) but

$$A(j,k) = 0, \qquad j - k > s. \quad (29)$$

The design matrices $\bar{Q}(k)$ and $\bar{R}(k)$ are set by a researcher. In particular, one can put

$$\bar{Q}(j) = \beta_1 Q(j), \qquad \bar{R}(k) = \beta_2 R(k), \quad (30)$$

where β_1, β_2 are positive scalars,

$$j = 0, \ldots, N-1, \qquad k = 0, \ldots, N.$$

Theorem 1 is also valid for the reduced model equation (25) with the additional restriction (29). Thus the related reduced boundary value problem yields an approximate estimator. In contrary to the full problem this reduced boundary value problem for moderate values of s possesses an efficiently computed solution.

In fact, the model system (25)–(30) can be easily represented in a single-step form by introducing the augmented $n(s+1)$-dimensional state vector

$$\mathcal{X}(j) = \left(y'(j), y'(j-1), \ldots, y'(j-s)\right)'.$$

Then the system (25)–(30) turns into the system

$$\mathcal{X}(j+1) = \mathcal{A}(j)\mathcal{X}(j) + \mathcal{B}(j)\bar{u}(j), \quad (31)$$

$$z(k) = \mathcal{H}'(k)\mathcal{X}(k) + \bar{\varrho}(k), \quad (32)$$
$$j = 0, \ldots, N-1, \qquad k = 0, \ldots, N,$$

$$\mathbf{E}\mathcal{X}(0) = 0, \qquad \mathbf{E}\left(\mathcal{X}(0)\mathcal{X}'(0)\right) = \mathcal{P}_0, \quad (33)$$

in which the scalar quantity $l_* = \tilde{a}'\mathcal{X}(N)$ is to be estimated. The augmented matrices (their sizes are $s+1$ times greater) are defined by the following expressions:

$$\mathcal{A}(j) = \begin{pmatrix} A(j,j) & \ldots & A(j,j-s+1) & A(j,j-s) \\ E_n & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & E_n & 0 \end{pmatrix},$$

$$\mathcal{B}(j) = \begin{pmatrix} B(j) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathcal{H}(k) = \begin{pmatrix} H(k) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \tilde{a} = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and

$$\mathcal{P}_0 = \begin{pmatrix} P_0 & 0 & \ldots & 0 \\ 0 & E_n & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & E_n \end{pmatrix}.$$

System (31)-(33) has a standard Kalman form. Thus the optimal linear mean-square estimate for l_* is yielded by the discrete Kalman filter. It follows directly from the Kalman filter theory (Jazwinski, 1970; Matasov, 1999) that the desired simplified estimator φ is given by the formulas

$$\varphi(i) = \mathcal{K}'(i)\Theta'(N,i)\tilde{a}, \qquad i = 0, \ldots, N, \quad (34)$$

$$\Theta(N,i) = \theta(N-1) \cdot \ldots \cdot \theta(i+1) \cdot \theta(i),$$
$$\theta(i) = \left(E_{n(s+1)} - \mathcal{K}(i+1)\mathcal{H}'(i+1)\right)\mathcal{A}(i),$$
$$\Theta(N,N) = E_{n(s+1)}, \qquad i = 0, \ldots, N-1,$$

where the Kalman gain $\mathcal{K}(i)$ is determined by the classical equations

$$\mathcal{K}(j) = \mathcal{P}(j|j-1)\mathcal{H}(j) \times \left[\mathcal{H}'(j)\mathcal{P}(j|j-1)\mathcal{H}(j) + \bar{R}^{-1}(j)\right]^{-1} \quad (35)$$

and

$$\mathcal{P}(j|j) = \left(E - \mathcal{K}(j)\mathcal{H}'(j)\right)\mathcal{P}(j|j-1),$$

$$\mathcal{P}(0|-1) = \mathcal{P}_0, \qquad j = 0, \ldots, N,$$

$$\mathcal{P}(j|j-1) = \mathcal{A}(j-1)\mathcal{P}(j-1|j-1)\mathcal{A}'(j-1) + \mathcal{B}(j-1)\bar{Q}(j-1)\mathcal{B}'(j-1), \qquad j = 1, \ldots, N.$$

By virtue of (23) the estimator φ generates an approximation to the function $\xi(j)$ by the formula $\hat{\xi}(j) = \xi^\varphi(j)$. Similarly, the approximations to the dual elements $(p^0(j+1), \lambda^0)$ from (22) can be constructed by the equalities

$$\hat{p}^0(j+1) = \nu\, Q(j)B'(j)\xi^\varphi(j+1), \quad (36)$$

$$\hat{\lambda}^0 = \nu\, P_0\xi^\varphi(0), \qquad j = 0, \ldots, N-1,$$

where ξ^φ is given by (10) and ν is an appropriate positive scalar.

So, the estimator φ, the function ξ^φ, and the dual elements $(\hat{p}^0, \hat{\lambda}^0)$ can be computed by a simple and reliable algorithm.

5. LEVEL OF NONOPTIMALITY FOR SIMPLIFIED FILTER

In accordance with the proposed approach the estimator φ obtained for the system (25)–(30) is used instead of the optimal one.

Theorem 2. *Let φ be the filter for the reduced model. Then the following inequalities hold:*

$$1 \leq \Delta \leq \Delta^0, \qquad \Delta^0 = \frac{J^\varphi \cdot k^\varphi}{|a'\tilde{x}^\varphi(N)|}, \quad (37)$$

where

$$J^\varphi = \left\{\xi^{\varphi\prime}(0)P_0\xi^\varphi(0) + \sum_{i=0}^{N}\varphi'(i)R(i)\varphi(i) \quad (38)\right.$$

$$+\sum_{j=0}^{N-1} \xi^{\varphi\prime}(j+1)B(j)Q(j)B'(j)\xi^{\varphi}(j+1)\Big\}^{\frac{1}{2}},$$

$$k^{\varphi} = \Big\{\xi^{\varphi\prime}(0)P_0\xi^{\varphi}(0) \tag{39}$$

$$+\sum_{i=0}^{N} \tilde{x}^{\varphi\prime}(i)H(i)R^{-1}(i)H'(i)\tilde{x}^{\varphi}(i)$$

$$+\sum_{j=0}^{N-1} \xi^{\varphi\prime}(j+1)B(j)Q(j)B'(j)\xi^{\varphi}(j+1)\Big\}^{\frac{1}{2}},$$

and the auxiliary process $\tilde{x}^{\varphi}(j)$, $j = 0,\ldots,N$, is defined by the Volterra equation

$$\tilde{x}^{\varphi}(j+1) = \sum_{k=0}^{j} A(j,k)\tilde{x}^{\varphi}(k) \tag{40}$$

$$+ B(j)Q(j)B'(j)\xi^{\varphi}(j+1)$$

with the initial value $\tilde{x}^{\varphi}(0) = P_0\xi^{\varphi}(0)$.

The idea of the proof. The proof is essentially based on the duality theory (Ekeland and Temam, 1976). It follows from (14), (15), and (24) that the level of nonoptimality Δ can be rewritten in the form

$$\Delta = J(\Phi_-(\varphi), \varphi, w(\varphi))/J_0 = J^{\varphi}/J_0. \tag{41}$$

In order to get the upper bound for the level of nonoptimality (41) a lower bound should be constructed for the optimal estimation error J_0. In accordance with the duality relation (21), any admissible pair (p,λ) (that is a pair satisfying (20)) generates such a lower bound. The one-dimensional manifold (36) is taken to obtain a "good" lower bound. In other words, the dual problem (19), (20) is replaced by an appropriate one-dimensional problem. Choosing the best value for the scalar parameter ν, the required bound is constructed.

The algorithm for calculating Δ^0 consists of the following steps:

Step 1. The calculation of the Kalman gains $\{\mathcal{K}(j)\}_{j=0}^{N-1}$ defined by (35);

Step 2. The calculation of the model filter $\{\varphi(i)\}_{i=0}^{N}$ by formulas (34);

Step 3. The calculation of the function $\{\xi^{\varphi}(j)\}_{j=0}^{N}$ by difference equation (10);

Step 4. The calculation of the auxiliary process $\{\tilde{x}^{\varphi}(j)\}_{j=0}^{N}$, in accordance with Volterra equation (40);

Step 5. The calculation of the quadratures J^{φ} and k^{φ} by (38) and (39);

Step 6. The calculation of Δ^0 by formula (37).

Table 1. Example 1: values of Δ^0; filter order $s = 8$

N	120	160	200	240	280	320	360	400
Δ^0	1.22	1.34	1.48	1.64	1.82	2.06	2.24	2.41
Δ^0_{opt}	1.21	1.33	1.47	1.62	1.79	1.97	2.16	2.36
β_1	0.8	0 .7	0.7	0.7	0.7	0.7	0.7	0.7
β_2	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

It should be emphasized that all these operations can be easily implemented.

6. NUMERICAL EXAMPLES

In this section, two simple examples of linear discrete Volterra equations are investigated. (The computation was performed by A.B. Bashkov.)

Example 1. Consider a stationary scalar equation of the form

$$x(j+1) = \sum_{k=0}^{j} \alpha^{j-k+1}x(k) + u(j), \tag{42}$$

$$x(0) = x_0, \qquad j = 0,\ldots,N-1,$$

and the measurements

$$z(k) = x(k) + \varrho(k), \qquad k = 0,\ldots,N,$$

where α is a specified scalar, x_0 is a zero-mean random variable with a given variance σ^2. The sequences $u(j)$ and $\varrho(k)$ are scalar zero-mean white noise processes with constant variances Q and R, respectively. The quantity $x(N)$ is to be evaluated. Obviously, this is a special case of system (1), (3) with $n = r = m = 1$,

$$A(j,k) = \alpha^{j-k+1}, \quad B(j) = 1, \quad H(k) = 1, \quad a = 1.$$

The following parameters are taken for simulation:

$$\alpha = 0.5, \quad \sigma^2 = 100, \quad Q = 1, \quad R = 1.$$

The values of Δ^0 for various N with two values of the filter order s are shown in Tables 1 and 2. The quantities Δ^0 for $\beta_1 = \beta_2 = 1.0$ are presented in the first row. The quantities Δ^0 that were optimized in scales β_1, β_2 are indicated in the second row. The corresponding optimal values of β_1, β_2 are cited in the third and fourth row. Note that though $\alpha = 0.5$, the estimated process (42) does not decay: $\mathbf{E}x(N) = 0$, $\mathbf{E}x^2(N) \sim N$ for large N.

Example 2. Now consider a two-dimensional Volterra equation

$$x(j+1) = \sum_{k=0}^{j} \alpha^{j-k+1} \begin{pmatrix} w & 1 \\ 0 & 1 \end{pmatrix} x(k) + u(j), \tag{43}$$

Table 2. Example 1: values of Δ^0; filter order $s = 9$

N	120	160	200	240	280	320	360	400
Δ^0	1.07	1.11	1.16	1.23	1.30	1.39	1.48	1.58
Δ^0_{opt}	1.06	1.11	1.16	1.22	1.29	1.38	1.47	1.56
β_1	0.9	0 .9	0.8	0.8	0.8	0.7	0.7	0.7
β_2	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Table 3. Example 2: values of Δ^0 for $N = 100$ and $w = 0.8$

s	8	9	10	11	12	13
Δ^0	9.74	4.50	2.33	1.43	1.12	1.03
Δ^0_{opt}	3.03	2.41	1.77	1.33	1.11	1.03
β_1	42.0	22.0	9.0	4.0	2.0	1.2
β_2	1.0	1.0	1.0	1.0	1.0	1.0

$$x(0) = x_0, \qquad x = (x_1, x_2)', \qquad j = 0, \ldots, N-1,$$

and the scalar measurements

$$z(k) = x_1(k) + \varrho(k), \qquad k = 0, \ldots, N.$$

The aim is to estimate the component $x_2(N)$.

Here α is a specified scalar, x_0 is a zero-mean random vector with a given covariance matrix P_0, the sequences $u(j)$ and $\varrho(k)$ are zero-mean white noise processes with a covariance matrix Q and a variance R. This is a special case of system (1), (3) with $n = r = 2$, $m = 1$,

$$A(j,k) = \alpha^{j-k+1} \begin{pmatrix} w & 1 \\ 0 & 1 \end{pmatrix}, \qquad B(j) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$H(k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The following values for the parameters are set:

$$\alpha = 0.5, \qquad P_0 = \begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R = 1.$$

Tables 3 and 4 represent the quantities of Δ^0 for $N = 100$ and various values of filter order s with $w = 0.8$ and $w = 1.0$, respectively. The zero-mean solutions of (43) are also unbounded: $\mathbf{E}x_1^2(N) \sim N$ for $w = 0.8$; $\mathbf{E}x_1^2(N) \sim N^3$ for $w = 1.0$.

It follows from Tables 1-4 that the reduced filters of moderate orders can be successfully used for the filtering in Volterra equations. Moreover, in some cases, the scales β_1, β_2 can be adjusted to improve the nonoptimality level. Note that the developed approach reduces hundreds of times the implementation time relative to the direct optimal Kalman filter.

Table 4. Example 2: values of Δ^0 for $N = 100$ and $w = 1.0$

s	14	15	16	17	18	19
Δ^0	9.43	4.66	2.44	1.48	1.14	1.03
Δ^0_{opt}	6.38	3.73	2.22	1.45	1.13	1.03
β_1	70.0	21.0	7.0	2.5	1.4	1.1
β_2	1.0	1.0	1.0	1.0	1.0	1.0

The filters presented in Section 4 seem to be the most natural ones to start the design. They were chosen to emphasize the main idea. One can propose more sophisticated schemes to simplify the filters that yield the same nonoptimality levels but operate with systems of smaller orders.

7. CONCLUSION

Efficient filtering algorithms for linear discrete Volterra equations are proposed. On the one hand, the filtering algorithms are quite easy for implementation. On the other hand, the levels of nonoptimality for these algorithms are constructed. It is important that the levels of nonoptimality can be easily computed without solving the original full-dimensional filtering problem. Thus a useful tool for analyzing the filtering problem in Volterra equations is developed.

Similar approach was earlier used for various estimation and control problems, in particular, for dynamic systems with delay (Matasov, 1999).

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