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ALTRUISM AND AGGRESSION IN OPTIMAL CONTROL FOR BEHAVIORAL FIRM MODEL

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Abstract: For a behavioral firm model there suggested closed-loop controls, using the so-called behavior types (normal, altruistic, and aggressive ones), and leading the firm state to a stable Pareto optimal state. The existence of initial states that necessarily require using the abnormal (altruistic and aggressive) behavior types, is proved. The problem of reaching the Pareto optimal set with minimal time of abnormal behaviors is solved.

Keywords: Control system synthesis, behavior firm model, behavior types.

1. INTRODUCTION

In this paper the authors deal with a behavioral firm model described in (de Vries, 1999). This model is based on ideas of behavioral theory of the firm developed in (Cyert and March, 1963). An important feature of this theory is introducing the aspiration levels (Simon, 1987) for criteria of departments. According to (de Vries, 1999), the firm is considered as a system, consisting of three departments: the Production Department (PRD), the Sales Department (SLD), and the Central Management Department (CMD). Each department has its own objective. Control variables are unit price p, chosen by the CMD, and slack of the PRD x. These two variables are considered as parameters characterizing a state of the system.

F. P. de Vries has developed some decision rules for adjusting the values of p and x so that the system reaches a stable state. At the same time, the suggested decision rules sometimes lead the system to stable states, which are not Pareto (see, for example, (Karlin, 1959)) optimal with respect to the mentioned objectives of all departments. Therefore it is essential to consider new possibilities for the system to reach the stable Pareto optimal state from any initial state.

These possibilities are implemented in this paper through the different so-called behavior types for each department, in accordance with (Kleimenov, 1997; Kleimenov and Kryazhimskii, 1998). Four different behavior types for each department can be considered, in general: *normal, altruistic* (with respect to another department), *aggressive* (with respect to another department), and *paradoxical* (with respect to itself). In this paper three behavior types are used: normal, altruistic, and aggressive ones. As a result, the existence of initial states of the system that require using the abnormal (altruistic or aggressive) behavior types, is shown. The problem of reaching the Pareto optimal set with minimal time of abnormal behavior is solved.

2. BEHAVIORAL MODEL OF THE FIRM

In this section elements of the behavioral firm model are presented, according to (de Vries, 1999).

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The state of the firm is defined by a pair of variables (p, x), where $p \geq 0$ is the unit price, and $x \geq 0$ is the level of slack of the PRD (which is 'on-the-job-leisure' by (de Vries, 1999)). The dynamics of the firm state is described by the system of differential equations

$$\begin{cases} \dot{p} = \psi(p)u, \\ \dot{x} = \xi(x)v, \end{cases} \tag{1}$$

where $\psi(p)$ and $\xi(x)$ are differentiable functions, satisfying the conditions: $\psi(p) > 0,\ \xi(x) > 0$ for $p > 0,\ x > 0$ and $\psi(0) = 0,\ \xi(0) = 0$. Controls u and v are restricted: $|u| \leq 1,\ |v| \leq 1$.

Each department of the firm has its own objective, which depends on values of variables p and x, and on positive parameters $\theta, \alpha, \eta, \delta$.

The PRD's interest is that the unit cost, which is calculated as

$$c(p, x) = \left(\frac{\theta}{\alpha} p^{\delta} + \eta\right)(1 + x), \tag{2}$$

is as close as possible to its aspired level, denoted by $\tilde{c}$. Thus, the criterion of the PRD is

$$I_c(p, x) = |c(p, x) - \tilde{c}|. \tag{3}$$

It is assumed that the sales volume always is equal to the volume of demands of consumers and is defined by

$$w(p) = \frac{\alpha}{p^2}. \tag{4}$$

The SLD's interest is that the sales is as close as possible to its aspired level, denoted by $\tilde{w}$. Therefore, the criterion of the SLD is defined by

$$I_w(p) = |w(p) - \tilde{w}|. \tag{5}$$

Finally, the CMD's interest is that the firm's profit, which is calculated by

$$\pi(p, x) = (p - c(p, x))w(p), \tag{6}$$

is as close as possible to its aspired level, denoted by $\tilde{\pi}$. This implies the formula for the criterion of the CMD:

$$I_\pi(p, x) = |\pi(p, x) - \tilde{\pi}|. \tag{7}$$

The purpose of each department is to minimize his own criterion.

The decision rules $u(p, x)$ and $v(p, x)$, considered in (de Vries, 1999), generate certain dynamics of the system (1). But for this dynamics one can find initial states such that trajectories, beginning at them, are attracted to stable states, which are not Pareto optimal with respect to the criteria (3), (5), and (7).

Therefore, the following important problem arises here: given the parameters of the model, describe a set of Pareto optimal states.

3. CONSTRUCTING THE SET OF PARETO OPTIMAL STATES

In this section a description of the set of Pareto optimal states with respect to the criteria $I_c(p, x)$ (3), $I_\pi(p, x)$ (5), and $I_w(p)$ (7), is given.
To this end consider the following three lines on the plane (p, x). At points of these lines the criteria (3), (5), and (7) reach the minimal values, equal to zero. I.e. these lines consist of points, which are the best ones for the PRD, the CMD, and the SLD, respectively.

Solving the equation $I_c(p, x) = 0$, one can obtain the formula of the following curve:

$$x = \tilde{\mu}(p) = \frac{\tilde{c}}{bp^2 + \eta} - 1. \tag{8}$$

Analogously, from the equation $I_\pi(p, x) = 0$, one can found the formula for the following curve:

$$x = \tilde{\nu}(p) = \frac{p + \frac{\eta\tilde{\pi}}{\alpha b}}{bp^2 + \eta} - \frac{\tilde{\pi}}{\alpha b} - 1. \tag{9}$$

Finally, the criterion $I_w(p)$ gets its minimal value at points of the vertical line:

$$p = \tilde{\varphi} = \sqrt{\frac{\alpha}{\tilde{w}}}. \tag{10}$$

These three lines are drawn on Fig. 1.

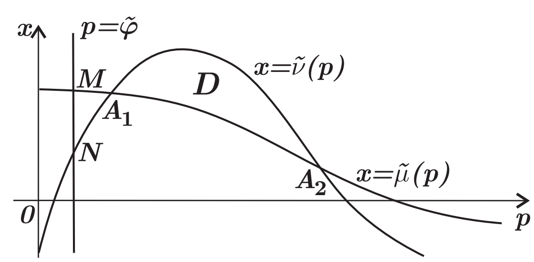

Fig. 1. Lines $x = \tilde{\mu}(p),\ x = \tilde{\nu}(p)$, and $p = \tilde{\varphi}$

In further analysis it will be considered the case when the inequality

$$\tilde{c}\tilde{\pi} < \frac{\alpha}{4} \tag{11}$$

is fulfilled. The complete analysis of all possible cases for parameters $\tilde{c}, \tilde{\pi}, \tilde{w}$ can be found in (Kleimenov and Semenishchev, 2001).

Under the inequality (11) the curves $x = \tilde{\nu}(p)$ and $x = \tilde{\mu}(p)$ have two different common points, denoted by $A_1(p_1, x_1)$ (the left point) and by $A_2(p_2, x_2)$ (the right one). Denote by M the intersection point of the lines $x = \tilde{\mu}(p)$ and $p = \tilde{\varphi}$, and by N the intersection point of the lines $x = \tilde{\nu}(p)$ and $p = \tilde{\varphi}$ (see Fig. 1).

Let $w_1 = \sqrt{\frac{\alpha}{p_1^2}}$, and $w_2 = \sqrt{\frac{\alpha}{p_2^2}}$, where p_1 and p_2 are abscissas of the points A_1 and A_2, respectively. It can be proved that $w_1 > w_2$. The domain situated between lines $x = \tilde{\mu}(p)$ and $x = \tilde{\nu}(p)$, where $p_1 \leq p \leq p_2$, will be denoted by D.

The set of Pareto optimal points with respect to the criteria (3), (5) and (7) depends on the values of aspiration levels $\tilde{c}, \tilde{\pi}, \tilde{w}$; it will be denoted by $P(\tilde{c}, \tilde{\pi}, \tilde{w})$.

Theorem 1. Let the inequality $\tilde{w} \geq w_1$ (or $\tilde{w} \leq w_2$) be fulfilled, and aspiration levels $\tilde{c}$, $\tilde{\pi}$ satisfy the inequality (11). Then the set $P(\tilde{c}, \tilde{\pi}, \tilde{w})$ is the closed curvilinear triangle A_1MN (the triangle A_2MN, respectively).

This theorem was proved in (Kleimenov and Semenishchev, 2001). The curvilinear triangle A_1MN is drawn on Fig. 1.

Now let us analyze the remaining case, when the following inequalities hold:

$$w_2 < \tilde{w} < w_1 \tag{12}$$

Consider an auxiliary curve γ, whose parametric equation is given in (Kleimenov and Semenishchev, 2001). The curve γ consists of all pairs of the points (p^l, x^l) and (p^r, x^r), $p^r > p^l$, such that value of the function $c(p,x)$ (2) (and $\pi(p,x)$ (6), and $I_w(p)$ (5) also) at the point (p^l, x^l) equals to its value at the point (p^r, x^r).

Under the inequalities (12) the vertical line $p = \tilde{\varphi}$ is strictly between points A_1 and A_2. Consider two following sets.

The part of domain D, situated to the left from the line $p = \tilde{\varphi}$ and above the curve γ, will be denoted by D_L. The part of domain D, situated to the right from the vertical line $p = \tilde{\varphi}$ and below the curve γ, will be denoted by D_R. Sets D_L and D_R are depicted on Fig. 2.

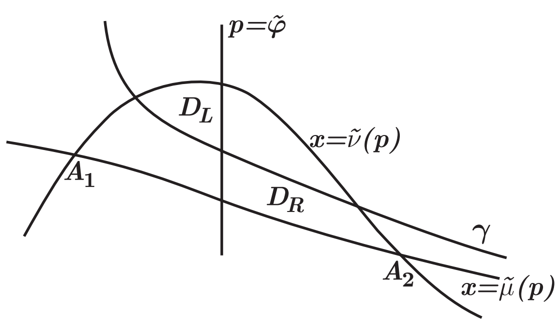

Fig. 2. Sets D_L and D_R

For any $\tilde{w}$, satisfying the inequalities (12), at least one of domains D_L, D_R is non-empty. If both these sets are non-empty, they have a single common point, which is the intersection point of the vertical line $p = \tilde{\varphi}$ and the curve γ.

Theorem 2. Let the inequalities (12) be fulfilled, and aspiration levels $\tilde{c}$, $\tilde{\pi}$ satisfy the inequality (11). Then the set $P(\tilde{c}, \tilde{\pi}, \tilde{w})$ is the union of the sets D_L and D_R.

Remark 3. Under the inequality (11) at least one of the points A_1, A_2 is Pareto optimal one. If, besides that, the equality $\tilde{w} = \frac{w_1 + w_2}{2}$ is fulfilled, then both points A_1 and A_2 are Pareto optimal.

4. CONSTRUCTING THE OPTIMAL CONTROL

Remind that the case, when the inequality (11) holds, is considered only.

Now consider the problem:how to construct decision rules for the PRD and the CMD such that a trajectory of the system, beginning at any initial state, attains the set $P(\tilde{c}, \tilde{\pi}, \tilde{w})$ and is stabilized at some its point.

Firstly, try to solve the problem when each department chooses its control variable in order to minimize its own criterion. Such a behavior of a department will be called a *normal* one.

Now let us define in detail the normal behavior for the PRD and the CMD.

4.1 *Normal behavior of the Production Department*

In (de Vries, 1999) there was given the following formula for the control $v = v(p,x)$ in (1):

$$v(p,x) = \begin{cases} 1, & c(p,x) < \tilde{c}, \\ -1, & c(p,x) > \tilde{c}, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

So, the PRD updates the slack x to decrease the value of the criterion $I_c(p,x)$. Note that this decision rule is valid under fixed value of the price p, otherwise the result is unpredictable.
This rule is oriented to the minimization of the PRD's criterion. According to (Kleimenov, 1997; Kleimenov and Kryazhimskii, 1998) we say in this situation about *normal behavior* of the PRD. And the control law, defined by (13), will be called a *normal type control law* of the PRD, and will be denoted by $v^{nr}(p,x)$. By definition of the function $x = \tilde{\mu}(p)$, one can write

$$v^{nr}(p,x) = \text{sgn}(\tilde{\mu}(p) - x). \tag{14}$$

4.2 *Normal behavior of the Central Management Department*

The normal behavior of the CMD is more complicated then the PRD's one. The reason is that the curve $x = \tilde{\nu}(p)$, which is 'ideal' line for the CMD, is not monotonic, as for the PRD (see Fig. 1).

The CMD can change the variable p. Obviously, if the current state (p, x) is to the left from the line $x = \tilde{\nu}(p)$ (for example, at the point A), it is necessary to increase the value of the variable p in order to approach this line (see Fig. 3).

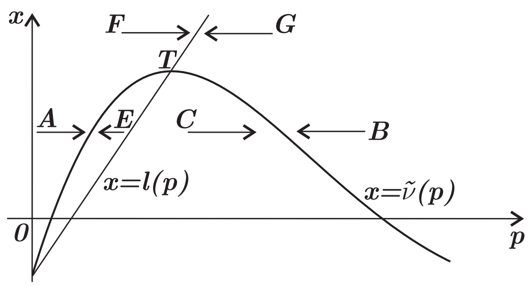

Fig. 3. Normal behavior of the CMD

Analogously, if the current state (p, x) is to the right from the line $x = \tilde{\nu}(p)$ (for example, at the point B), it is necessary to decrease the value of the variable p. The following question arises: what to do when the current state (p, x) is below the curve $x = \tilde{\nu}(p)$ (points C and E) or above it (points F and G, for example). To answer this question, consider the curve, which is a geometric locus of the maximal points of the curves $x = \tilde{\nu}(p)$ for different values of the aspiration level $\tilde{\pi}$. As it was proved in (Kleimenov and Semenishchev, 2001), this curve is a straight line and its equation is $x = l(p) = \frac{1}{2\eta}p - 1$.
The line $x = l(p)$ intersects with the graph of the function $x = \tilde{\nu}(p)$ at two points: at the point $(0, -1)$, and at the point $T(p_T, x_T)$, which is the maximal point of this graph. The abscissa of the point T can be found from equation $\frac{\mathrm{d}\tilde{\nu}(p)}{\mathrm{d}p} = 0$.

Beginning at the points, being below the graph of the function $x = \tilde{\nu}(p)$ (like points C and E), a motion should be directed away from the line $x = l(p)$, then the criterion $I_\pi(p, x)$ will decrease monotonically (provided that x is fixed).

Analogously, beginning at the points F and G, which are above the curve $x = \tilde{\nu}(p)$, a motion should be directed to the line $x = l(p)$.

The normal behavior of the CMD is illustrated on Fig. 3. According to this, the normal type control $u^{nr}(p, x)$ takes the values 1, -1 in the domains just described, and the value 0 at points of the curve $x = \tilde{\nu}(p)$.

4.3 *The normal dynamics of the state of the firm*

Consider the system (1), assuming that both the CMD and the PRD employ the normal behavior, i. e. choose the controls $u = u^{nr}(p, x)$, $v = v^{nr}(p, x)$. Besides that, for the simplicity, assume that the function $\xi(x)$ equals to constant r for $x > 0$, and $\xi(0) = 0$. Analogously, let $\psi(p) = q$ for $p > 0$, and $\psi(0) = 0$.
Then, in the interior of the positive quadrant of the plane (p, x) the system (1) will get the following form:

$$\begin{cases} \dot{p} = qu^{nr}(p, x), \\ \dot{x} = rv^{nr}(p, x). \end{cases} \tag{15}$$

This system has discontinuous right-hand side. Its solutions are understood in sense of (Filippov, 1988). The phase portrait of the system (15) is shown on Fig. 4 (parameters q and r are approximately equal there).

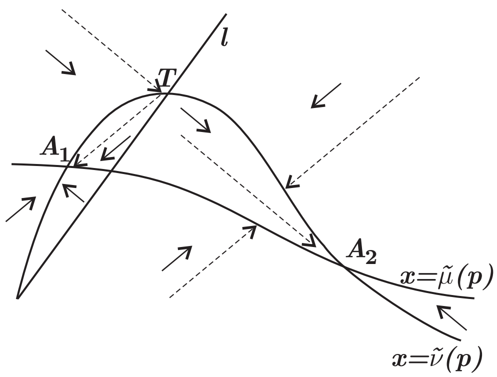

Fig. 4. Normal dynamics of the firm's state

The vector field, defined by the system (15), is piecewise constant. The positive quadrant of the plane (p, x) is divided into several domains, for each of which the right-hand side of the system is a constant vector, as it is shown on Fig. 4. Under some conditions on parameters of the system the trajectories are attracted to one of the points A_1 or A_2. Now the following problem is formulated:

Problem 4. Find conditions on parameters of the system such that all trajectories of the system (15), beginning at any initial state, are attracted to a stable Pareto optimal point.

In (Kleimenov and Semenishchev, 2001) this problem has been considered under the inequality (11) and the assumption that $A_2 \in P(\tilde{c}, \tilde{\pi}, \tilde{w})$. Note that, accordingly to the Remark 3, the point A_1 in this case can either be Pareto optimal one or not. It is proved that there are only two kinds of the conditions, solving Problem 4:

1. (i) The point A_1 is on the curve $x = \tilde{\nu}(p)$ to the left from the point T and

(ii) both the points A_1 and A_2 are Pareto optimal ones.

2. The point A_1 coincides with the point T and $A_2 \in P(\tilde{c}, \tilde{\pi}, \tilde{w})$.

Accordingly to Remark 3, (ii) is fulfilled if and only if $\tilde{w} = \frac{w_1+w_2}{2}$. An analytical form of (i) is quite complicated and will not be given here.

So, for almost all values of the parameters $\tilde{c}$, $\tilde{\pi}$, and $\tilde{w}$ the using of normal behavior only is insufficient to provide the considered system to attain a stable Pareto optimal point from any initial point. The possible way out is to introduce *abnormal* (altruistic and aggressive) behavior.

4.4 *Optimal control through abnormal behavior*

Introduce here abnormal behavior types of the PRD and the CMD.

Besides the normal behavior, the CMD can employ the following four variants of using the altruistic and aggressive behavior types:

u^{al}_{PRD} Altruism with respect to the PRD.
Under this variant the CMD does minimize the criterion $I_c(p,x)$. This implies that the CMD changes the value of p in such a way that the current point approaches the curve $x = \tilde{\mu}(p)$.

u^{ag}_{PRD} Aggression with respect to the PRD.
Under this variant the CMD does maximize the criterion $I_c(p,x)$. This implies that the CMD updates the variable p in order to move away from the curve $x = \tilde{\mu}(p)$.

u^{al}_{SLD} Altruism with respect to the SLD.
Under this variant the CMD does minimize the criterion $I_w(p,x)$. This implies that the CMD changes the value of p in such a way that the current point approaches the line $p = \tilde{\varphi}$.

u^{ag}_{SLD} Aggression with respect to the SLD.
Under this variant the CMD does maximize the criterion $I_w(p,x)$, i. e. updates the variable p in order to move away from the vertical line $p = \tilde{\varphi}$.

So, the CMD has five different behavior variants, including the normal behavior one. For each of these variants one can find the corresponding control function of the variables p, x. Consider the set of these functions:

$$U = \{u^{nr}(p,x), u^{al}_{PRD}(p,x), u^{ag}_{PRD}(p,x), u^{al}_{SLD}(p,x), u^{ag}_{SLD}(p,x)\} \quad (16)$$

Now describe abnormal behavior types for the PRD. Since the PRD controls the variable x, it has no influence on the SLD, therefore it has only two abnormal behavior variants.

v^{al}_{CMD} Altruism with respect to the CMD.
It means that the PRD does minimize the criterion $I_\pi(p,x)$. This implies that the PRD tries to move the current state (p,x) to the curve $x = \tilde{\nu}(p)$.

v^{ag}_{CMD} It means that the PRD does maximize the criterion $I_\pi(p,x)$. This implies that the PRD tries to move the current state (p,x) away from the curve $x = \tilde{\nu}(p)$.

Different behavior variants of the PRD are realized by control functions of the variable p, x. The set of these functions is

$$V = \{v^{nr}(p,x), v^{al}_{CMD}(p,x), v^{ag}_{CMD}(p,x).\} \quad (17)$$

Consider a problem of optimal control for the system (1) under the restrictions $u \in U,\ v \in V$.

Definition 5. A pair $(u(p,x), v(p,x))$ will be called an admissible control for the system (1), if

1. $u(p,x) \in U,\ v(p,x) \in V$ for all $p \geq 0,\ x \geq 0$;
2. a trajectory of the system (1), generated by controls $u = u(p,x)$, $v = v(p,x)$ and beginning at any initial state (p_0, x_0), attains a stable Pareto optimal point.

Trajectories of the system (1), generated by admissible control, will be called *admissible* ones.
If for an initial state an admissible trajectory includes a part that is generated by abnormal control of at least one department, then such part of the trajectory will be called *abnormal* one.
Denote by $T^{abnr}(p_0, x_0; u(\cdot,\cdot), v(\cdot,\cdot))$ the time of using abnormal behavior by at least one department along the trajectory, beginning at the initial state (p_0, x_0), and generated by admissible control $(u(p,x), v(p,x))$. Due to quite simple structure of the trajectories, the abnormal time can be calculated by geometrical and analytical means.

The following problem of minimizing the time of using the abnormal behavior is formulated.

Problem 6. Find an admissible control $(u^*(\cdot,\cdot), v^*(\cdot,\cdot))$ minimizing the functional $T^{abnr}(p_0, x_0; u(\cdot,\cdot), v(\cdot,\cdot))$ for all initial points (p_0, x_0) of the positive quadrant of the plane (p,x):

$$T^{abnr}(p_0, x_0; u^*(\cdot,\cdot), v^*(\cdot,\cdot)) = \min_{u(\cdot,\cdot)\in U,\ v(\cdot,\cdot)\in V} T^{abnr}(p_0, x_0; u(\cdot,\cdot), v(\cdot,\cdot))$$

The solution of Problem 6 is called an optimal control.

An optimal control has been constructed in form of synthesis, and is shown on Fig. 5. Corresponding trajectories are drawn with dash lines. Note

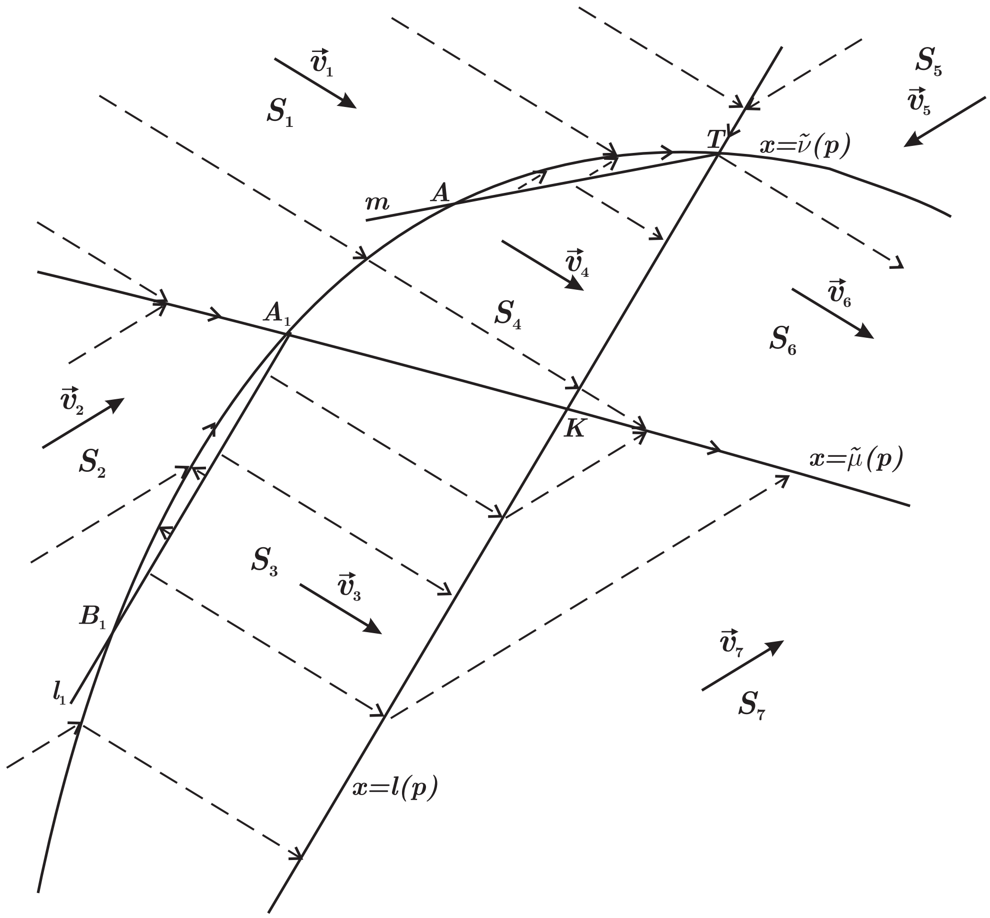

Fig. 5. Structure of the optimal control

that, besides the rectilinear parts of trajectories, there exist the parts of sliding mode on curves $x = \tilde{\mu}(p), x = \tilde{\nu}(p)$.

In several domains ($S_1,\ S_2,\ S_5 - S_7$) the optimal control of each department is the normal type control. In some domains the departments should employ altruism (in S_3 the optimal control is $v_3 = (u^{al}_{SLD}, v^{nr})$), and aggression (in S_4 the optimal control is $v_4 = (u^{al}_{SLD}, v^{ag}_{CMD})$).

5. REFERENCES

Cyert, R. and J. March (1963). *A Behavioral Theory of the Firm.* Prentice-Hall. Englewood Cliffs.

de Vries, F. P. (1999). The behavioral firm and its internal game: Evolutionary dynamics of decision making. *IIASA Interim Report* **IR-99-036**, 25.

Filippov, A. F. (1988). *Differential Equations with Discontinuous Right-Hand side.* Kluwer Academic Publishers.

Karlin, S. (1959). *Mathematical Methods and Theory in Games, Programming, and Economics.* Addison-Wesley Publishing Company. Reading, Massachusetts.

Kleimenov, A. F. (1997). Solutions in a nonantagonistic positional differential game. *Prikl. Mat. Mekh.* **61, no. 5**, 739–746.

Kleimenov, A. F. and A. A. Semenishchev (2001). Constructing the set of pareto optimal points in a multicriterial behavioral firm model (in russian). *Dep. VINITI* **752-B2001**, 87.

Kleimenov, A. F. and A. V. Kryazhimskii (1998). Normal behavior, altruism and aggression in cooperative game dynamics. *IIASA Interim Report* **IR-98-076**, 48.

Simon, H. A. (1987). Behavioral economics. In: *The New Palgrave: A Dictionary of Economics* (J. Eatwell, M. Milgate and P. Newman, Eds.). Vol. 1. MacMillan. London.