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OPTIMAL CONTROL IN THE CLASS OF SMOOTH AND BOUNDED FUNCTIONS

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Abstract: The paper is focused on the optimal control problem with boundary conditions. Unlike the traditional class of piecewise continuous functions, here the admissible controls are defined as continuously differentiable functions with inclusion or amplitude constraints. Admissible variations of control functions are formed using the idea of simultaneous varying. Numerical solution algorithm, obtained as a result, is proved to be convergent to the necessary condition of optimality.

Keywords: Optimal Control, Boundary Value Problem, Successive Approximation Technique.

1. INTRODUCTION

The mathematical theory of optimal processes emerged as an answer to the requirement of solving engineering problems, and the applicability of this theory depends naturally on algorithms for solving the problems of optimal control. In general formulation, a problem of optimal control is aimed at determining the optimal value of an objective functional defined on the profiles of a system of ordinary differential equations subject to given initial conditions. The right-hand endpoint could be either free or could satisfy some constraints. This paper deals with the problem of optimal control where the ODE system is subject to given *boundary* conditions, which is substantially more complex than the problem with initial conditions and includes the latter as a special case. Aside from its importance for pure mathematics, the control problem with boundary conditions has numerous applications, for example, in the problem of choice of optimal compositions for the protections against nuclear radiation (see Fedorenko, 1978, p.268), in the problem of optimization of manufacturing cycles (see Fedorenko, 1978, p.263), in the problem of synthesis of stratified structures under the effect of various waves and temperature factors (Gusev, 1993), etc.

The control problem with boundary conditions has been investigated by the author together with K.Mizukami. At first, there was obtained a necessary condition for optimality of the maximum principle type, which also provided the background for the development of solution techniques (Vasilieva and Mizukami, 1994). Then, this line of research was continued by the authors (2000) when the differential maximum principle has been justified and the idea of combined control variation has been introduced. In parallel, the theory of singular controls has been also proposed (Vasilieva and Mizukami, 1997). Having analyzed various resolved problems of the mentioned type it can be concluded that in many cases the extension of the class of admissible controls from continuous to piecewise continuous is stipulated by the desire to take into account the amplitude constraints for control functions.

The objective of this paper is to develop the optimality condition and optimization technique for a control problem with boundary conditions whose class of admissible controls contains smooth (i.e., continuously differentiable upto any order) functions with inclusion or amplitude constraints. The investigation technique remains the same as before. Namely, the increment of the objective functional together with conjugate BVP is being considered on a certain type of control variation, thus providing the admissibility of varied control under some adjustments of the parameters of variation. In contrast to the classic variation of Lagrange and the needle-shaped variation of McShane it is proposed to use the idea of socalled “inner” or “interior” variation expressed as far back as by M.V.Ostrogradskii and presented in contemporary form, e.g., by Zabello (1990). This idea consists in the simultaneous varying of independent variable and control function. Under such approach, the dominant term of the increment formula determines the necessary condition for optimality, and the formula itself serves as a basis for the development of optimization algorithm which converges to the necessary conditions of optimality.

2. STATEMENT OF THE PROBLEM

Let a controllable process

$$\{\boldsymbol{u}, \boldsymbol{x}\} = \{\boldsymbol{u} = \boldsymbol{u}(t),\ \boldsymbol{u}(t) \in \Re^m; \quad \boldsymbol{x} = \boldsymbol{x}(t),\ \boldsymbol{x}(t) \in \Re^n,\ t \in T = [t_0, t_1]\}$$

be defined by the conditions

$$J(\boldsymbol{u}) = \varphi_0(\boldsymbol{x}(t_0), \boldsymbol{x}(t_1)) + \int_T F(\boldsymbol{x}, \boldsymbol{u}, t)\,\mathrm{d}t \to \min, \quad (1)$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t), \qquad \boldsymbol{\varphi}(\boldsymbol{x}(t_0), \boldsymbol{x}(t_1)) = 0, \quad (2)$$

$$u(\cdot) \in \mathcal{U}. \quad (3)$$

Here the controls $\boldsymbol{u} = \boldsymbol{u}(t),\ t \in T$ are smooth and bounded: $\boldsymbol{u}(\cdot) \in C_q^m(T),\ q = 1, 2, \ldots$. Vector-functions $\boldsymbol{f} = (f_1, \ldots, f_n),\ \boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$ and scalar functions $\varphi_0,\ F$ are continuous with respect to their arguments together with their partial derivatives upto any order q for which all the operations described below are valid. In-addition, it is supposed that for any smooth and bounded admissible control $\boldsymbol{u} = \boldsymbol{u}(t),\ t \in T$, boundary value problem (2) is solvable in the class of smooth and bounded functions $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{u}, t),\ t \in T$.

Finally, two types of controls constraints $\mathcal{U}$ will be considered:

$$\mathcal{U} = \mathcal{U}_1 = \{\boldsymbol{u}(\cdot) \in C_q^m(T) : \boldsymbol{u}(t) \in U,\ t \in T, \quad U \subset \Re^m \text{ — compact, int}\, U \neq \emptyset\}; \quad (4)$$

$$\mathcal{U} = \mathcal{U}_2 = \mathcal{U}_1 \cap \{\boldsymbol{u}(\cdot) : g_i(\boldsymbol{u}, \dot{\boldsymbol{u}}) \leq 0, \quad g_i(\boldsymbol{u}, \lambda\dot{\boldsymbol{u}}) = \lambda^p g_i(\boldsymbol{u}, \dot{\boldsymbol{u}}),\ p > 0, \quad \lambda = \lambda(t) > 0,\ t \in T,\ i = 1, 2, \ldots, l\}. \quad (5)$$

3. INCREMENT FORMULA

For two admissible processes, the basic one $\{\boldsymbol{u}, \boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{u})\}$ and the varied one $\{\tilde{\boldsymbol{u}} = \boldsymbol{u} + \Delta\boldsymbol{u}, \tilde{\boldsymbol{x}} = \boldsymbol{x} + \Delta\boldsymbol{x} = \boldsymbol{x}(t, \tilde{\boldsymbol{u}})\}$, the formula for the increment of the functional (1) has been obtained (Vasilieva and Mizukami, 1994). The same work also provides an estimate of the state $\boldsymbol{x}$ caused by an increment of the control $\boldsymbol{u}$. Taking into account the smoothness of $\boldsymbol{u}$, the increment of (1) can be represented in the following form:

$$J(\tilde{\boldsymbol{u}}) - J(\boldsymbol{u}) = -\int_T \left\langle \frac{\partial H(\boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{u}, t)}{\partial \boldsymbol{u}}, \Delta\boldsymbol{u}(t) \right\rangle \mathrm{d}t + \int_T o(\|\Delta\boldsymbol{u}(t)\|)\ \mathrm{d}t, \quad (6)$$

where

$$\frac{o(\alpha)}{\alpha} \to 0, \quad \alpha \to 0.$$

Moreover, it should be noted that there exists some $\mathcal{K} = \text{const} > 0$ such that

$$|\, o(\|\Delta\boldsymbol{u}(t)\|)\,| \leq \mathcal{K}\, \|\Delta\boldsymbol{u}(t)\|^2, \quad t \in T. \quad (7)$$

Here, $H(\boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{u}, t) = \langle \boldsymbol{\psi}(t), \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)\rangle - f(\boldsymbol{x}, \boldsymbol{u}, t)$; $\|\cdot\|$ is the vector's norm and $\langle\cdot\,,\cdot\rangle$ stands for inner product in finite-dimensional Euclidean spaces $\Re^m$ and $\Re^n$. The conjugate vector-function $\boldsymbol{\psi} = \boldsymbol{\psi}(t),\ \boldsymbol{\psi}(t) \in \Re^n$ is a solution profile of the boundary value problem

$$\dot{\boldsymbol{\psi}} = -\frac{\partial H(\boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{u}, t)}{\partial \boldsymbol{x}}, \quad (8)$$

$$-\boldsymbol{B}_0\boldsymbol{\psi}(t_0) + \boldsymbol{B}_1\boldsymbol{\psi}(t_1) + \boldsymbol{B}_0\frac{\partial\varphi_0}{\partial\boldsymbol{x}(t_0)} + \boldsymbol{B}_1\frac{\partial\varphi_0}{\partial\boldsymbol{x}(t_1)} = \boldsymbol{0}, \quad (9)$$

where $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are some numerical $(n \times n)$ matrices which are chosen arbitrarily in order to satisfy the condition

$$\boldsymbol{B}_0\left[\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{x}(t_0)}\right]' + \boldsymbol{B}_1\left[\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{x}(t_1)}\right]' = \boldsymbol{0}. \quad (10)$$

Prime here denotes the transpose of the matrix. It was demonstrated (Vasilieva and Mizukami, 1994) that if the direct BVP (2) is solvable for some admissible process $\{\boldsymbol{u}, \boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{u})\}$ then linear conjugate problem (8)–(10) is also solvable with respect to $\boldsymbol{\psi} = \boldsymbol{\psi}(t, \boldsymbol{u})$.

4. CHOICE OF ADMISSIBLE VARIATION

In Section 2 it was proposed to consider two different sets of admissible controls. Now, it is worth to give descriptions of admissible control variations suitable for both (4) and (5).

4.1 *Inclusion Constraints*

Let $\boldsymbol{u} \in \mathcal{U}_1$ according to (4). Then the varied control $\tilde{\boldsymbol{u}} = \boldsymbol{u}_\varepsilon$ can be chosen as

$$\boldsymbol{u}_\varepsilon(t) = \boldsymbol{u}(t + \varepsilon\delta(t)), \qquad \varepsilon \in [0, 1], \quad (11)$$

where $\delta = \delta(t)$ is a smooth real function which satisfies

$$t_0 - t \leq \delta(t) \leq t - t_1, \qquad t \in T. \qquad (12)$$

Proposition 1. If the basic control $\boldsymbol{u} = \boldsymbol{u}(t)$ is admissible in the sense that $\boldsymbol{u} \in \mathcal{U}_1$, i.e. satisfies (4), then the varied control $\tilde{\boldsymbol{u}} = \boldsymbol{u}_\varepsilon$ defined by (11) is also admissible for all $\varepsilon \in [0,1]$ and for any smooth real function $\delta(t)$ satisfying (12).

Apparently, by denoting

$$t_\varepsilon = t + \varepsilon\delta(t) \in T, \qquad \varepsilon \in [0,1]$$

it is concluded that $\boldsymbol{u}_\varepsilon = \boldsymbol{u}(t_\varepsilon) \in U$ due to the fact that $\boldsymbol{u}$ is admissible for all $t \in T$. It is also obvious that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$, $t \in T$ when $\varepsilon \to 0$ since

$$\Delta_\varepsilon \boldsymbol{u}(t) = \boldsymbol{u}_\varepsilon(t) - \boldsymbol{u}(t) = \varepsilon \dot{\boldsymbol{u}}(t)\delta(t) + \hat{\boldsymbol{o}}(\varepsilon), \quad (13)$$

$$\hat{\boldsymbol{o}}(\varepsilon) = (o_1(\varepsilon), \ldots, o_m(\varepsilon)), \ \lim_{\varepsilon \to 0} \frac{o_\iota(\varepsilon)}{\varepsilon} = 0, \ \iota = 1, \ldots, m.$$

4.2 *Homogeneous and Inclusion Constraints*

Let $\boldsymbol{u} \in \mathcal{U}_2$ according to (5). In this case additional homogeneous constraints will complicate the structure of admissible variation.

Proposition 2. If the basic control $\boldsymbol{u} = \boldsymbol{u}(t)$ is admissible in the sense that $\boldsymbol{u} \in \mathcal{U}_2$, i.e. satisfies (5), then the varied control $\tilde{\boldsymbol{u}} = \boldsymbol{u}_\varepsilon$ defined by (11) is also admissible for all $\varepsilon \in [0,1]$ and for any smooth real function $\delta(t)$ which satisfies (12) and

$$\left|\dot{\delta}(t)\right| \leq 1, \qquad t \in T. \qquad (14)$$

In fact, if $\boldsymbol{u}(t) \in U$ and $g_i(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \leq 0$, $i = 1, \ldots, l$, then by virtue of Proposition 1 $\boldsymbol{u}_\varepsilon(t) \in U$, $t \in T$, $\varepsilon \in [0,1]$. Moreover, under homogeneity condition $g_i(\boldsymbol{u}, \lambda\dot{\boldsymbol{u}}) = \lambda^p g_i(\boldsymbol{u}, \dot{\boldsymbol{u}})$, $i = 1, \ldots, l$ with respect to $\dot{\boldsymbol{u}}$, it holds for $t_\varepsilon = t + \varepsilon\delta(t) \in T$ that

$$g_i(\boldsymbol{u}_\varepsilon(t), \dot{\boldsymbol{u}}_\varepsilon(t)) = g_i\left(\boldsymbol{u}(t_\varepsilon), \left(1 + \varepsilon\dot{\delta}(t)\right)\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t_\varepsilon}\right)$$
$$= \left(1 + \varepsilon\dot{\delta}(t)\right)^p g_i\left(\boldsymbol{u}(t_\varepsilon), \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t_\varepsilon}\right) \leq 0,$$
$$i = 1, \ldots, l.$$

since $1 + \varepsilon\dot{\delta}(t) \geq 0$ for all $\varepsilon \in [0,1]$ due to (14).

Remark 1. The homogeneity of g_i with respect to $\dot{\boldsymbol{u}}$ is stipulated by the method of formation of admissible variation. Homogeneous constrains are not a rare exception. For instance,

$$\sum_{\iota=1}^{m} \alpha_\iota(\boldsymbol{u}(t))\, \dot{u}_\iota^p(t) \leq 0, \ \ t \in T, \ \ p > 0$$

makes an example of widely used homogeneous constraint.

It is known that in order to use the successive approximation technique based on the maximum principle (Vasilieva and Mizukami, 1994, 2000) the inclusion constraints $\boldsymbol{u}(t) \in U$ should be rather simple because this technique relies on the supposition of solvability of the maximum condition. In other words, the method of successive approximations requires to solve problems of nonlinear programming with respect to U and for all $t \in T$ at every iteration. This part of the technique can be excluded when admissible controls are of the form (4) or (5) and the variation is chosen according to (11), (12) and (14). In fact, it is sufficient to find only one basic admissible control, since the varied control $\boldsymbol{u}_\varepsilon$ will always remain within the class of admissible controls.

5. OPTIMAL CONDITIONS

Increment formula (6) serves as a basis to obtain optimality conditions for the problem (1)–(3) with control constraints (4)–(5). Replacement of $\tilde{\boldsymbol{u}}$ in (6) by admissible variation (11) results in

$$J(\boldsymbol{u}_\varepsilon) - J(\boldsymbol{u}) = -\varepsilon\int_T W(\boldsymbol{u}, t)\delta(t)\,\mathrm{d}t + o(\varepsilon), \quad (15)$$
$$\varepsilon \in [0,1],$$

where for $\boldsymbol{u} \in \mathcal{U}_1$, $\boldsymbol{u} \in \mathcal{U}_2$ and variation (11)

$$W(\boldsymbol{u}, t) = \left\langle \frac{\partial H(\boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{u}, t)}{\partial \boldsymbol{u}}, \dot{\boldsymbol{u}}(t) \right\rangle. \qquad (16)$$

Formula (15) is valid for all $\varepsilon \in [0,1]$ and for all $\delta = \delta(t)$ which satisfy the conditions (12) or (12), (14) for control constraints (4) or (5) respectively. It is also obvious that for the remainder term $\hat{\boldsymbol{o}}(\varepsilon)$ in variation (11), (13) it is fulfilled that

$$\| \hat{\boldsymbol{o}}(\varepsilon) \| \leq \mathcal{K}\varepsilon^2.$$

Theorem 1. Suppose that $\boldsymbol{u}^* = \boldsymbol{u}^*(t)$ is optimal control in problem (1)–(3) and that $\boldsymbol{x}^* = \boldsymbol{x}^*(t)$, $\boldsymbol{\psi}^* = \boldsymbol{\psi}^*(t)$ are the correspondent profiles of direct BVP (2) and conjugate BVP (8)–(10). Then under control constraints (4), (5)

$$W(\boldsymbol{u}^*, t) = 0, \qquad t \in T. \qquad (17)$$

The proof of Theorem 1 is almost immediate and arises out of the increment formula (15) considered for any admissible $\delta(t)$ which has different signs.

Remark 2. Theorem 1 states that the optimality condition in the class of smooth and bounded controls is defined directly by the problem entries and their derivatives (see formulae (16), (17)). When the class of admissible controls is extended from smooth to piecewise continuous functions (so that $\dot{\boldsymbol{u}}(t)$ is not defined for all $t \in T$), the optimality condition for bounded controls obtained in (Vasilieva and Mizukami, 1994) is to be written as

$$\left\langle \frac{\partial H(\boldsymbol{\psi}^*, \boldsymbol{x}^*, \boldsymbol{u}^*, t)}{\partial \boldsymbol{u}}, \boldsymbol{v} - \boldsymbol{u}^* \right\rangle \leq 0 \quad \forall\, \boldsymbol{v} \in U$$

and almost for all $t \in T$. This condition is obviously less useful for numerical calculations than (17) because of the presence of undefined parameter $\boldsymbol{v}$.

Remark 3 Necessary condition for optimality (17) holds trivially within subsegments $T_* \subset T$, $\mathrm{mes}\, T_* > 0$, where $\dot{\boldsymbol{u}}^*(t) = \boldsymbol{0}$, i.e., $\boldsymbol{u}^*(t) = \mathrm{const}$.

6. OPTIMIZATION ALGORITHMS

Variation of the type (11) is often referred to as "inner" or "interior" variation due to the following feature. If a given control function is adjusted by means of such variation, the resulting function remains within the limits of control domain. In order to use interior variations in practice, it is of great utility to point out an appropriate way to choose the function $\delta(t)$.

Lemma 1 Conditions (12) are fulfilled for

$$\left.\begin{aligned} \delta(t) = \delta_1(t) &= \frac{(t - t_0)(t_1 - t)}{\mathcal{M}(t_1 - t_0)}\, a(t), \\ \mathcal{M} &\geq \max_{t \in T} |a(t)|, \end{aligned}\right\} \quad (18)$$

where $a = a(t)$, $t \in T$ is some arbitrary smooth real function. Moreover,

$$\delta(t_0) = \delta(t_1) = 0. \quad (19)$$

Proof. A mere glance reveals that condition (19) holds. Therefore, it is sufficient to note that $\delta_1(t)$ has the same sign as $a(t)$. Therefore, if $a(t) \geq 0$ then $t_0 - t \leq \delta_1(t)$, and if $a(t) \leq 0$ then $\delta_1(t) \leq t_1 - t$. It remains to show that
(a) if $a(t) \geq 0$ then $\delta_1(t) \leq t_1 - t$;
(b) if $a(t) \leq 0$ then $t_0 - t \leq \delta_1(t)$.
Item (a) holds due to

$$\frac{(t - t_0)(t_1 - t)}{\mathcal{M}(t_1 - t_0)} a(t) \leq \frac{(t_1 - t_0)(t_1 - t)}{\mathcal{M}(t_1 - t_0)} a(t) \leq t_1 - t$$

since $\frac{a(t)}{\mathcal{M}} \leq 1$ and $t - t_0 \leq t_1 - t_0$. On the other hand, item (b) holds due to

$$\frac{(t - t_0)(t_1 - t)}{\mathcal{M}(t_1 - t_0)} a(t) = \frac{(t_0 - t)(t_1 - t)}{\mathcal{M}(t_1 - t_0)} |a(t)| \geq t_0 - t$$

since $\frac{|a(t)|}{\mathcal{M}} \leq 1$ and $t_1 - t \leq t_1 - t_0$, $t_0 - t \leq 0$. □

Lemma 2 Conditions (12), (14), (19) are fulfilled for

$$\delta(t) = \delta_2(t) = \frac{(t - t_0)(t_1 - t)}{\mathcal{M} \cdot \mathcal{L}(t_1 - t_0)}\, a(t), \quad (20)$$

$$\left.\begin{aligned} \mathcal{M} &\geq \max_{t \in T} |a(t)|, \\ \mathcal{L} &= \max_{t \in T} \{(t_1 + t_0 - 2t)\, a(t) \\ &\quad + (t - t_0)(t_1 - t)\dot{a}(t)\} \frac{1}{\mathcal{M}(t_1 - t_0)}, \end{aligned}\right\} \quad (21)$$

where $a = a(t)$, $t \in T$ is some arbitrary smooth real function.

Proof. First, it is easy to check that (20) satisfies the conditions of Lemma 1. Moreover, it can be noted that

$$\mathcal{L} = \max_{t \in T} \left|\dot{\delta}_1(t)\right|,$$

where $\delta_1(t)$ is defined by (18). Then, having calculated the derivative of $\delta_2(t)$, it is obtained that

$$\left|\dot{\delta}_2(t)\right| = \left| \frac{\dot{\delta}_1(t)}{\max_{t \in T} \left|\dot{\delta}_1(t)\right|} \right| \leq 1.$$

This entirely proves all the statements of Lemma 2. □

It should be noted that the choice of $\delta_1(t)$, $\delta_2(t)$ according to (18), (20)–(21) will establish non-negativity of the dominant term of the increment in (15) for $a(t) = W(\boldsymbol{u}, t)$. This allows to introduce two nonnegative functionals $\mu_j(\boldsymbol{u})$, $j = 1, 2$. Each functional is determined by the corresponding control constraints (4) or (5) respectively:

$$\mu_1(\boldsymbol{u}) = \int_T W(\boldsymbol{u}, t)\, \delta_1(t)\, \mathrm{d}t \quad (22)$$

where $\delta_1(t)$ is defined by (18) for $a(t) = W(\boldsymbol{u}, t)$;

$$\mu_2(\boldsymbol{u}) = \int_T W(\boldsymbol{u}, t)\, \delta_2(t)\, \mathrm{d}t \quad (23)$$

where $\delta_2(t)$ is defined by (20) for $a(t) = W(\boldsymbol{u}, t)$. Then, for admissible variation (13), the following formula is correct for both types of control constraints (4) and (5):

$$J(\boldsymbol{u}_\varepsilon) - J(\boldsymbol{u}) = -\varepsilon\, \mu_j(\boldsymbol{u}) + o(\varepsilon), \quad (24)$$
$$\mu_j(\boldsymbol{u}) \geq 0, \qquad j = 1, 2,$$

where by virtue of estimate (7)

$$|\, o(\varepsilon)\, | \leq \mathcal{K}\, \varepsilon^2, \quad \mathcal{K} = \mathrm{const} > 0. \quad (25)$$

Theorem 2. Let $\boldsymbol{u}^* = \boldsymbol{u}^*(t)$ be optimal control in problem (1)–(3) with one type of control constraints (4) or (5), i.e., $\boldsymbol{u} \in \mathcal{U}_j,\ j = 1,2$. Then, for corresponding j, it holds that

$$\mu_j(\boldsymbol{u}^*) = 0, \quad j = 1,2. \tag{26}$$

The result of Theorem 2 is proved by the increment formula (24).

Numerical algorithms for problem (1)–(3) have the same structure for both types of control constraints (4) and (5). Both algorithms are designed to be convergent to the necessary conditions for optimality (26). This implies that a numerical solution $\boldsymbol{u}^* = \boldsymbol{u}^*(t)$ will not be necessarily optimal but always extremal control.

Let an admissible control $\boldsymbol{u}^k \in \mathcal{U}_j,\ j \in \{1,2\},\ k = 0$ be given. It should be emphasized that $\boldsymbol{u}^k = \boldsymbol{u}^k(t)$ must not contain constant sections, i.e., $\boldsymbol{u}^k(t) \neq \text{const},\ t \in T_k,\ \text{mes}\, T_k > 0$ (see Remark 3). Then one should integrate numerically the direct BVP (2) and conjugate BVP (8)–(10) and store their profiles $\boldsymbol{x}^k = \boldsymbol{x}(t, \boldsymbol{u}^k),\ \boldsymbol{\psi}^k = \boldsymbol{\psi}(t, \boldsymbol{u}^k)$. After that, for $j \in \{1,2\}$ subject to (4) or (5), the corresponding $\mu_j(\boldsymbol{u}^k) \geq 0$ must be calculated using formulae (22) or (23) respectively. If $\mu_j(\boldsymbol{u}^k) = 0$, then by virtue of Theorem 1, the control function $\boldsymbol{u}^k = \boldsymbol{u}^k(t)$ is a possible solution of the problem (1)–(3) and thus the algorithm is depleted. Otherwise, it is supposed that

$$\mu_j(\boldsymbol{u}^k) > 0, \quad j \in \{1,2\}. \tag{27}$$

Next stage of the solution process is dedicated to construction of the control variation. First step in this direction is to define smooth real function $\delta_1(t)$ or $\delta_2(t)$ according to (18) or (20)–(21) for $a(t) = W(\boldsymbol{u}^k, t)$ and then to construct one-parameter family of admissible controls $\boldsymbol{u}^k_\varepsilon = \boldsymbol{u}^k_\varepsilon(t),\ t \in T,\ \varepsilon \in [0,1]$ using formula (11): $\boldsymbol{u}^k_\varepsilon \in \mathcal{U}_j,\ j \in \{1,2\}$. Second step is to solve the problem of one-parameter minimization

$$\varepsilon_k = \arg \min_{\varepsilon \in [0,1]} J\left(\boldsymbol{u}^k_\varepsilon\right) \tag{28}$$

and then to determine next approximation as

$$\boldsymbol{u}^{k+1}(t) = \boldsymbol{u}^k_{\varepsilon_k}(t), \quad k = 0,1,2,\ldots. \tag{29}$$

Theorem 3 Suppose that $J(\boldsymbol{u})$ in the problem (1)–(3) is bounded from below for both types of control constraints (4) and (5). Then the sequence of admissible controls $\{\boldsymbol{u}^k\}$ generated by the algorithm (27)–(29) is a strictly relaxational one, i.e., $J\left(\boldsymbol{u}^{k+1}\right) < J\left(\boldsymbol{u}^k\right),\ k = 0,1,2,\ldots$ and convergent to the necessary condition of optimality (26) in the sense that

$$\lim_{k\to\infty} \mu_j\left(\boldsymbol{u}^k\right) = 0, \quad j \in \{1,2\}. \tag{30}$$

Proof. To begin with, the increment formula (24) should be examined for $\boldsymbol{u} = \boldsymbol{u}^k,\ \boldsymbol{u} = \boldsymbol{u}^k_\varepsilon$ taking into account the estimate (25):

$$J\left(\boldsymbol{u}^k_\varepsilon\right) - J\left(\boldsymbol{u}^k\right) \leq -\varepsilon\, \mu_j\left(\boldsymbol{u}^k\right) + \mathcal{K}\varepsilon^2.$$

By virtue of inequality (27), strict relaxation for small $\varepsilon > 0$ becomes obvious. Hence, taking into consideration the minimization problem (28)

$$J(\boldsymbol{u}^{k+1}) - J(\boldsymbol{u}^k) \leq -\varepsilon\, \mu_j(\boldsymbol{u}^k) + \mathcal{K}\varepsilon^2, \tag{31}$$
$$\varepsilon \in [0,1], \quad j \in \{1,2\}.$$

Inequality (31) can be transformed into

$$0 \leq \varepsilon\, \mu_j(\boldsymbol{u}^k) \leq J(\boldsymbol{u}^k) - J(\boldsymbol{u}^{k+1}) + \mathcal{K}\varepsilon^2. \tag{32}$$

Due to the relaxation and boundedness of $J(\boldsymbol{u})$ from below

$$0 \leq J(\boldsymbol{u}^k) - J(\boldsymbol{u}^{k+1}) \to 0, \quad k \to \infty.$$

Then passing to the limit in (32)

$$0 \leq \varepsilon \left[\lim_{k\to\infty} \mu_j(\boldsymbol{u}^k)\right] \leq \mathcal{K}\varepsilon^2,$$
$$\varepsilon \in [0,1], \quad j \in \{1,2\}.$$

Last inequality is valid only if (30) holds. □

Remark 4. The choice of $\delta(t)$ according to formulae (18), (20)–(21) will obviously narrow down the possibilities of variation since in that case $\delta_1(t_0) = \delta_1(t_1) = 0$ even though it is not required by the general form of $\delta = \delta(t)$ provided by (12) where $\delta(t_0) \geq 0$ and $\delta(t_1) \leq 0$. In order to extend the possibilities of variation, function $\delta(t)$ satisfying (12) can be chosen as some arbitrary real function which carry the same sign as $W(\boldsymbol{u}, t)$. Such choice will also guarantee non-negativity of the dominant term of the increment (15). On the other hand, the choice of $\delta(t)$ by formulae (18), (20)–(21) is fully justified if $\mathcal{U}_1$ includes additional conditions, ad modum $\boldsymbol{u}(t_0) = \boldsymbol{u}^0,\ \boldsymbol{u}(t_1) = \boldsymbol{u}^1$, which have frequent occurrence in many practical problems simulating various dynamic processes (see Gusev, 1993).

The following example illustrates the application of the solution procedure described above.

Example. Consider a simplified version of the problem (1)–(3):

$$\begin{cases} \dot{x}_1 = x_2, & x_1(0) = 1, \\ \dot{x}_2 = x_1 + u(t), & x_2(1) = 0, \end{cases} \quad t \in T = [0,1],$$

$$J(u) = [3x_2(0) + 2.16]^2 + [10x_1(1) - 5.8]^2 \to \min,$$

$$u(t) \in \mathcal{U} = \{u(\cdot) \in C_q(T) : |u| \leq 1,\ u(0) = -1,\ u(1) = 1\},$$

where $\boldsymbol{x} = (x_1, x_2) \in \Re^2$, $t \in T$ and the control function is scalar $\boldsymbol{u}(t) = u(t)$. In this particular example, optimal control is $u^*(t) = (2t - 1)^3$, with the correspondent states $x_1(1, u^*) = 0.58$, $x_2(0, u^*) = -0.72$ providing that $J(u^*) = 0$, $W(u^*, t) = 0$. It should be noted that optimal control $u^*(t) = (2t-1)^3$ is also singular in the sense of the maximum principle if the end-points of admissible controls are left free (Vasilieva and Mizukami, 1997).

In order to perform an iteration of the optimization algorithm it is convenient to define analytically several useful quantities. First, in this example

$$H(\boldsymbol{\psi}, \boldsymbol{x}, u, t) = \psi_1 x_2 + \psi_2 x_1 + \psi_2 u(t).$$

There is no difficulty to determine the conjugate BVP according to (8)–(10)

$$\begin{cases} \dot{\psi}_1 = -\psi_2, & \psi_1(1) = -20\,[10x_1(1) - 5.8]\,, \\ \dot{\psi}_2 = -\psi_1, & \psi_2(0) = 6\,[3x_2(0) + 2.16]\,. \end{cases}$$

whose boundary conditions depend on the missing end-points of the state system. These end-points can be obtained using the matrix representation of the solution of linear BVP (Vasilieva and Mizukami, 1994):

$$x_1(1,u) = 0.65 - 0.32\left[\int_0^1 \mathrm{e}^t\, u(t)\,\mathrm{d}t - \int_0^1 \mathrm{e}^{-t}\, u(t)\,\mathrm{d}t\right],$$

$$x_2(0,u) = -0.76 - 0.12\int_0^1 \mathrm{e}^t u(t)\,\mathrm{d}t - 0.88\int_0^1 \mathrm{e}^{-t} u(t)\,\mathrm{d}t.$$

Thus, function (16) turns into

$$W(u,t) = \psi_2(t)\dot{u}(t),$$

where

$$\psi_2(t,u) = 0.32\,\psi_2(0)\,\left[0.37\,\mathrm{e}^t + 2.72\,\mathrm{e}^{-t}\right] - 0.32\,\psi_1(1)\,\left[\mathrm{e}^t - \mathrm{e}^{-t}\right].$$

Let the initial approximation be given by $u^0(t) = -2t^2 + 4t - 1$. Under this control $x_1(1, u^0) = 0.42$, $x_2(0, u^0) = -0.96$ and $J(u^0) = 3.08$. Function $\delta = \delta^0(t)$ is defined by (18) for $a(t) = W(u^0, t) \le 0$. For manual illustrative computations the function δ^0 can be simplified as follows:

$$\delta^0(t) = t^2 - t \le 0, \qquad t \in T = [0, 1].$$

Then by virtue of (11)

$$u^0_\varepsilon(t) = -2\left[t + \varepsilon\delta^0(t)\right]^2 + 4\left[t + \varepsilon\delta^0(t)\right] - 1,\ \varepsilon \in [0,1]$$

and for $\varepsilon = 1$ it is obtained that $u^1(t) = -2t^4 + 4t^2 - 1$, $x_1(1, u^1) = 0.61$, $x_2(0, u^1) = -0.84$, and $J(u^1) = 0.22 < J(u^0) = 3.08$.

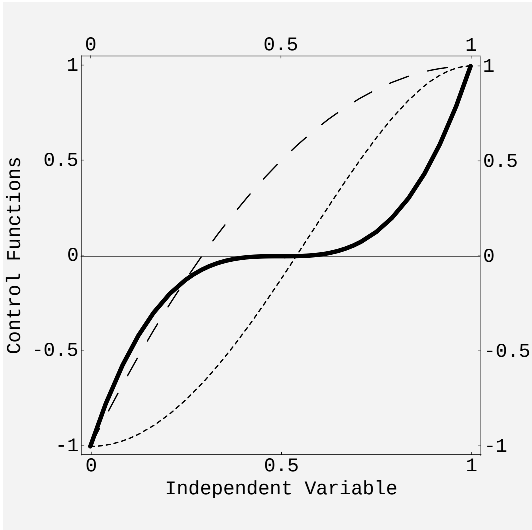

Fig. 1. Smooth control functions, Example 1.

Figure 1 shows the control functions. Here $u^0(t)$ is given by dashed line, dotted line stands for $u^1(t)$, and $u^*(t)$ is drawn using thick solid line. On the sixth iteration of the computer implementation, $u^6(t)$ coincides with $u^*(t)$ within the limits of given precision of computations.

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