15th Triennial World Congress, Barcelona, Spain

REGULATOR PROBLEM FOR LINEAR SYSTEMS WITH CONSTRAINTS ON CONTROL AND ITS INCREMENT

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Abstract: This paper discusses the problem of constraints on both control and its increment for linear systems in state space form, in both the continuous and discrete time domains. For autonomous linear systems with constrained increment, necessary and sufficient conditions are derived, such that the evolution of the system respects the incremental constraints. It is also derived a pole placement technique to solve inverse problem, deriving stabilizing state feedback controllers which respect constraints on both control and its increment. An illustrative example shows the application of the method. *Copyright © IFAC 2002.*

Keywords: Keywords: Linear Systems, Constraints, Control, Increment, Positive Invariance, Pole assignment

1. INTRODUCTION

Usually, real plants or physical plants are subject to constrained variables. The most frequent constraints are of saturation type, that is, limitations on the magnitude of certain variables. Hence, this topic is of continuing interest and one could cite, not exhaustively, (Benzaouia, and Burgat, 1988; Benzaouia, and Hmamed, 1993; Benzaouia, and Mesquine, 1994; Blanchini 1990;1999 and the references Therein). Other type of constraints were introduced while considering predictive control (GPC) and practical applications that is incremental constraints (Dion, et al.,1987; Clarke et al. 1987; Warwick, and Clarke, 1988). In fact, for some processes, the rate of variables change is limited within certain bounds. These limits can arise from physical constraints that, if exceeded, could damage the process. From our knowledge, no work has been published on incremental constraints using state space representations. Henceforth, this paper investigates the problem of stabilizing linear continuous and discrete time systems with constraints on both control and control increment. Necessary and sufficient conditions of positive invariance for incremental domains with respect to autonomous systems are given. Furthermore, a link is done between pole assignment procedure and these conditions to find stabilizing controllers by state feedback.

1.1 *Notations:*

If $x \in \Re^n$ is a vector, $\delta x(.)$ denotes its derivative with respect to time in the continuous time case or $x(t+1)$ in the discrete-time case. Further, for a scalar $a \in \Re$ we define $a^+ = \sup(a, 0)$ and $a^- = \sup(-a, 0)$, and then we note that

$x^+ = (x^+{}_j)$ and $x^- = (x^-{}_j)$ for $j = 1, ..., n$

Furthermore, for a matrix $A = (a)_{ij} and i, j = 1, ..., n$, the Tilde Transforms are defined by

$$\widetilde{A} = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix},$$

where $A^+ = (a^+)_{ij}$ and $A^- = (a^-)_{ij}, i, j = 1, ..., n$ and

$$\widetilde{A_c} = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}$$

where

$$A_1 = \begin{cases} a_{ii} \\ a^+_{ij} \text{ for } i \neq j \end{cases}, \text{ and } A_2 = \begin{cases} 0 \\ a^-_{ij} \text{ for } i \neq j \end{cases}$$

Also, $\sigma(A)$ denotes the spectrum of matrix A; D_s denotes the stability domain for eigenvalues (that is, the left half plane in the continuous-time case or the unit disk in the discrete-time case).

2. PROBLEM STATEMENT

Consider a linear time invariant system represented in the state space by:

$$\delta x(\cdot) = Ax(\cdot) + Bu(\cdot) \tag{1}$$

where $x(\cdot) \in \Re^n$ is the state of the system, $u(\cdot) \in \Re^m$ is the input constrained to evolve in the following domain

$$D_u = \{u(\cdot) \in \Re^m, -u_{\min} \leq u(\cdot) \leq u_{\max} \quad u_{\min}, u_{\max} \in \Re^{m*}_+\}. \tag{2}$$

The control increment is constrained as follows:

i) For discrete-time systems:

$$-\Delta_{\min} \leq u(k+1) - u(k) \leq \Delta_{\max} \tag{3}$$

ii) For continuous-time systems:

$$-\Delta_{\min} \leq \dot{u}(t) \leq \Delta_{\max} \tag{4}$$

We denote

$$U = \begin{bmatrix} u_{\max} \\ u_{\min} \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_{\max} \\ \Delta_{\min} \end{bmatrix}$$

the problem studied in this paper is the following: find a stabilizing state feedback as

$$u(\cdot) = Fx(\cdot), \quad F \in \Re^{mxn} \tag{5}$$

ensuring closed-loop asymptotic stability of the system with non saturating controls that also respects incremental constraints.

3. PRELIMINARY RESULTS

Consider the linear time invariant autonomous system

$$\delta z(\cdot) = Hz(\cdot), \quad z(t_o) = z_o \tag{6}$$

where $z \in \Re^m$ is the state constrained to evolve in domain

$$D_z = \{z \in \Re^m, -z_{\min} \leq z(\cdot) \leq z_{\max} \quad z_{\min}, z_{\max} \in \Re^{m*}_+\} \tag{7}$$

Consider also that the state increment is constrained as follows:

For discrete time systems:

$$-\Delta_{\min} \leq z(k+1) - z(k) \leq \Delta_{\max} \tag{8}$$

For continuous time systems:

$$-\Delta_{\min} \leq \dot{z}(t) \leq \Delta_{\max} \tag{9}$$

First, recall the definition of positive invariance of domain D_z which is very useful for the sequel.

Definition 1. Domain D_z given by (7) is positively invariant with respect to motion of system (6) if for all initial condition $z_o \in D_z$, the trajectory of the system $z(t, t_o, z_o) \in D_z$ for all $t > t_o$

We give also a technical lemma that will be related to a pole placement procedure to find stabilizing controllers for systems with constrained control and increment.

Lemma 2. The evolution of the autonomous system (6) respects incremental constraints if and only if matrix H satisfies:

$$\widetilde{(H - I)}Z \leq \Delta \quad \text{for the discrete-time case} \tag{10}$$

$$\widetilde{H} \; Z \leq \Delta \quad \text{for the continuous-time case} \tag{11}$$

where

$$Z = \begin{bmatrix} z_{\max} \\ z_{\min} \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_{\max} \\ \Delta_{\min} \end{bmatrix}$$

Proof. *If part* Consider the discrete time case and assume that condition (10) is satisfied. Hence, it is possible to write:

$$\begin{aligned} z(k+1) - z(k) &= Hz(k) - z(k) \\ &= (H - I)z(k) \\ &= G \; z(k), \end{aligned}$$

but it is known that

$$-z_{\min} \leq z(k) \leq z_{\max} \tag{12}$$

Next, decompose matrix $G = G^+ - G^-$, premultiplying (12) by G^+ and $-G^-$, respectively, gives

$$-G^+ z_{\min} \leq G^+ z(k) \leq G^+ z_{\max} \tag{13}$$

$$-G^- z_{\max} \leq -G^- z(k) \leq G^- z_{\min} \qquad (14)$$

addition of inequalities (13) and (14) enables us to write:

$$-G^+ z_{\min} - G^- z_{\max} \leq Gz(k) \leq G^+ z_{\max} + G^- z_{\min}$$

according to condition (10)

$$-\Delta_{\min} \leq -G^+ z_{\min} - G^- z_{\max} \leq Gz(k) \leq G^+ z_{\max} + G^- z_{\min} \leq \Delta_{\max}$$

which is equivalent to

$$-\Delta_{\min} \leq z(k+1) - z(k) \leq \Delta_{\max}.$$

In the continuous time case,

$$\dot{z}(t) = H\, z(t)$$

following the same reasoning, replacing matrix G by matrix H, and condition (10) by condition (11), it is easy to obtain

$$-\Delta_{\min} \leq \dot{z}(t) \leq \Delta_{\max}$$

Only if part, Consider the continuous time case. Assume that the derivative of $z(t)$ respects the constraints and that condition (11) is not satisfied for an index $1 \leq i \leq n$ such that

$$[\widetilde{H}\ Z]_i > \Delta_i \qquad (15)$$
$$[H^+ z_{\max} + H^- z_{\min}]_i > \Delta^i_{\max}$$

The following state vector for the system can be selected

$$\phi(t) = \begin{cases} z^j_{\max} \text{ if } h_{ij} > 0 \\ 0 \text{ if } h_{ij} = 0 \\ -z^j_{\min} \text{ if } h_{ij} < 0 \end{cases}, j = 1, ..., n$$

It is easy to check that $\phi(t)$ is an admissible state for the system. Calculation of the i^{th} component of the derivative of this state gives

$$\begin{aligned} [\frac{d}{dt}\phi(t)]_i &= [H\phi(t)]_i \\ &= \sum_{j=1}^{n} h_{ij}\phi_j(t) \\ &= [H^+ z_{\max} + H^- z_{\min}]_i \end{aligned}$$

taking into account inequality (15), it is possible to write

$$[\frac{d}{dt}\phi(t)]_i > \Delta^i_{\max}$$

which contradicts the assumption. The discrete time part could be easily deduced replacing matrix H by matrix G in the necessary part. ■

Evolution of the autonomous system (6) will respect both constraints on the state $z(t)$ and constraints on its increment if domain D_z given by (7) is positively invariant and conditions given in the previous lemma are satisfied.

Positive invariance conditions have already been proposed in (Benzaouia, and Burgat, 1988; Benzaouia and Hmamed, 1993). Using these conditions it is possible to derive the following result:

Lemma 3. Domain (7) is positively invariant with respect to motion of system (6) and increment constraints (3) are respected if and only if

$$\begin{cases} (\widetilde{H - I})Z \leq \Delta \\ \widetilde{H}\, Z \leq Z \end{cases}, \text{ for the discrete-time case}$$

$$\begin{cases} \widetilde{H} Z \leq \Delta \\ \widetilde{H}_c\, Z \leq 0 \end{cases}, \text{ for the continuous-time case}$$

Proof. For the increment constraints, conditions can be derived from the previous lemma, and the positive invariance conditions are given in (Benzaouia, and Burgat, 1988; Benzaouia, and Hmamed, 1993). ■

Relating the previous lemma to a pole placement procedure makes possible to solve the problem stated above. Recall the pole assignment procedure used in the so-called Inverse Procedure for constrained control (Benzaouia, 1994). Consider the time invariant system given by (1). Without loss of generality (see Remark below), assume that matrix A possesses $(n-m)$ stable eigenvalues. Resolution of equation

$$XA + XBX = HX \qquad (16)$$

gives us a state feedback assigning spectrum of matrix H ($\sigma(H) \subset D_s$) together with the stable part of spectrum of matrix A in closed loop. For this equation to have a valid solution, matrix H must satisfy the following conditions:

$$\begin{cases} \sigma(H) \cap \sigma(A) = \emptyset \\ B\zeta_i \neq 0,\ i = 1, ..., m \\ \zeta_i, i = 1, ..., m \text{ are linearly independent} \end{cases} \qquad (17)$$

for ζ_i such that $H\zeta_i = \lambda_i\zeta_i$, that is ζ_i eigenvectors of matrix H. There exits a unique solution to equation (16) if and only if

$$\{\, \chi_1 \cdots \chi_{n-m}\ \chi_{n-m+1} \cdots \chi_n \,\}$$

are linearly independent, where $\chi_i, i = n - m + 1, ..., n$ are eigenvectors associated to stable eigenvalues of matrix A, and $\chi_i, i = 1, ..., m$ are computed by

$$\chi_i = (\lambda_i I_n - A)^{-1} B\zeta_i,\ i = 1, ..., m$$

Hence, the solution is given by:

$$F = \begin{bmatrix} \zeta_1 \cdots \zeta_m\ 0 \ldots 0 \end{bmatrix} \begin{bmatrix} \chi_1 \cdots \chi_{n-m}\ \chi_{n-m+1} \cdots \chi_n \end{bmatrix}^{-1} \qquad (18)$$

Remark 4. Without loss of generality, it was assumed that the system presents $(n-m)$ stable eigenvalues. If the system matrix does not satisfy such requirement, it is always possible to augment the representation as follows: let v be a vector of fictitious inputs such that

$$v \in \Re, -v_{\min} \leq v \leq v_{\max}$$
$$-\Delta^v_{\min} \leq \delta v \leq \Delta^v_{\max}$$

where $v_{\min}$ and $v_{\max}$ are freely chosen constraints on the fictitious inputs. In this case, vectors U and Δ become:

$$U = \begin{bmatrix} u_{max} \\ v_{max} \\ u_{min} \\ v_{min} \end{bmatrix}, \Delta = \begin{bmatrix} \Delta_{max} \\ \Delta^v_{max} \\ \Delta_{min} \\ \Delta^v_{min} \end{bmatrix}$$

The augmented system is then given by

$$\delta x(\cdot) = Ax(\cdot) + \begin{bmatrix} B & 0 \end{bmatrix} \begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix} \qquad (19)$$

It is true that this augmentation limits the domain of linear behavior of the closed-loop system, but it is always possible to soften the fictitious limitations to enlarge the domain. It is easy to see that for the square system obtained the problem of (n_m) stable eigenvalues is eliminated and controllability is not changed.

4. MAIN RESULTS

With this background, we are now able to solve the problem stated in section 2. Consider a stabilizable linear time invariant system with constraints on both control and increments of the control, that is

$$\delta x(\cdot) = Ax(\cdot) + Bu(\cdot) \qquad (20)$$

where $x(\cdot) \in \Re^n$ is the state of the system, $u(\cdot) \in \Re^m$ the input constrained to evolve in the following domain

$$D_u = \{u(\cdot) \in \Re^m, -u_{\min} \leq u(\cdot) \leq u_{\max}$$
$$u_{\min}, u_{\max} \in \Re^{m*}_+\}$$

and the increment of the control is constrained as follows:

For discrete-time systems

$$-\Delta_{\min} \leq u(t+1) - u(t) \leq \Delta_{\max}$$

For continuous-time systems

$$-\Delta_{\min} \leq \dot{u}(t) \leq \Delta_{\max}$$

Using the state feedback

$$u(\cdot) = Fx(\cdot),\ F \in \Re^{mxn} \qquad (21)$$

such that

$$\sigma(A + BF) \in D_s \qquad (22)$$

the following domain of linear behaviour is induced in the state space

$$D_F = \{x \in \Re^n, -u_{\min} \leq Fx(\cdot) \leq u_{\max}$$
$$u_{\min}, u_{\max} \in \Re^{m*}_+\} \qquad (23)$$

If the state does not leave the domain (23), the control signal does not violate the constraints. That is, the set D_F is positively invariant with respect to motion of system (20). This gives the following result:

Proposition 5. System (20) with state feedback (21)-(22) is asymptotically stable at the origin with constraints on both the control and its increment if there exists a matrix $H \in \Re^{mxm}$ satisfying conditions (17) such that:

i)

$$FA + FBF = HF$$

iia)

$$\begin{cases} (\widetilde{H - I})U \leq \Delta \\ \widetilde{H}\,U \leq U \end{cases}, \text{ for the discrete-time case} \qquad (24)$$

iib)

$$\begin{cases} \widetilde{H}U \leq \Delta \\ \widetilde{H}_c\,U \leq 0 \end{cases}, \text{ for the continuous-time case} \qquad (25)$$

where $U = [u^t_{\max} \quad u^t_{\min}]^t$ for all initial state $x_o \in D_F$.

Proof. Introduce the following change of coordinates:

$$z = F\,x$$

it is possible to write

$$\begin{aligned} \delta z(\cdot) &= F\delta x(\cdot) \\ &= F(A + BF)x(\cdot) \\ &= HFx(\cdot) \\ &= Hz(\cdot) \end{aligned} \qquad (26)$$

With this transformation, domain D_F is transformed to domain D_z given by (7). Further, with conditions (24) and (25), it is easy to note that domain D_z is positively invariant with respect to the system (26) while the constraints on the increment of the control are respected. Bearing in mind that $\sigma(A + BF) \in D_s$ and that the linear behaviour is guaranteed, one can conclude to the asymptotic stability of the closed-loop system. ■

Remark 6. It is worth noting that conditions (24) and (25) do not affect the set of positive invariance D_F. However, they present an additional constraint on the pole assignment problem.

Algorithm:

Step 1. Check if matrix A has $(n - m)$ stable eigenvalues, if not augment the matrix B with $n - m$ null column.

Step 2. Choose matrix $H \in \Re^{mxm}$ or $H \in \Re^{nxn}$, if the system is augmented, according to (17)-(24) or (25).
Step 3. Compute the gain matrix F or F_a by using (18)
Step 4. Use F or extracted F from F_a for the control.

Example 7. Consider the double integrator in the continuous-time state space representation by (Gutman, and Hagander, 1985):

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

Control constraints are given by;

$$u_{min} = 1; u_{max} = 10$$

Assume that control increments are constrained as follows:

$$\Delta_{min} = 25; \quad \Delta_{max} = 20$$

As discussed in the Remark above, the system can be augmented with fictitious constrained inputs v in domain $[-v_{min} \;\; v_{max}]$. Select the following matrix H:

$$H = \begin{bmatrix} -2 & 0.5 \\ 0 & -0.5 \end{bmatrix}$$

which satisfies all the required conditions (17). Further, the fictitious constraints are selected such that conditions (25) are satisfied.

$$v_{min} = 4; v_{max} = 25; \;\; \Delta^v_{min} = 15; \Delta^v_{max} = 5.$$

that is,

$$\widetilde{H}_c U = [-7.5 \;\; -12.5 \;\; 0 \;\; -2]^t \leq 0$$
$$\widetilde{H} U = [14.5 \;\; 2 \;\; 22 \;\; 12.5]^t$$
$$\leq [20 \;\; 5 \;\; 25 \;\; 15]^t$$

Solution of equation (16) leads to the following augmented gain matrix F_a:

$$F_a = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}$$

Note that the effective gain matrix F can be extracted from the previous matrix.

$$F = \begin{bmatrix} -1 & -2 \end{bmatrix}$$

The closed-loop dynamics are given by:

$$A + BF = \begin{bmatrix} -0.5 & 0 \\ -1 & -2 \end{bmatrix}$$

It is easy to note here that $\sigma(A + BF) = \sigma(H)$.
The obtained set of positive invariance, as shown in Figure 1, with the augmented matrix F_a is given by:

$$D_{F_a} = \{x \in \Re^n \mid -g_{min} \leq F_a x \leq g_{max}\}$$

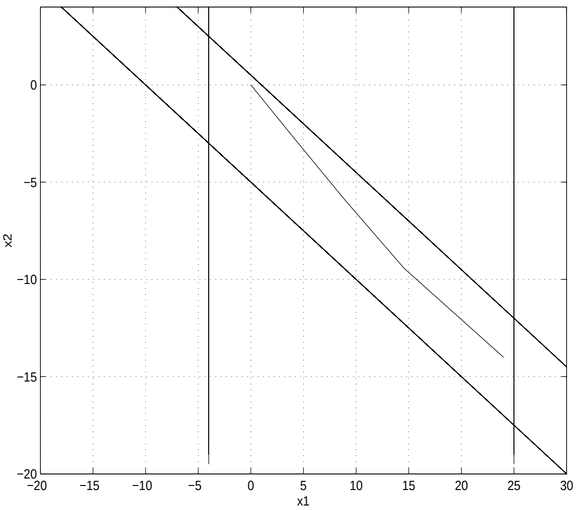

Fig. 1. Set of positive invariance D_{F_a} with a trajectory emanating from $x_o = [24 \;\; -14]^t$

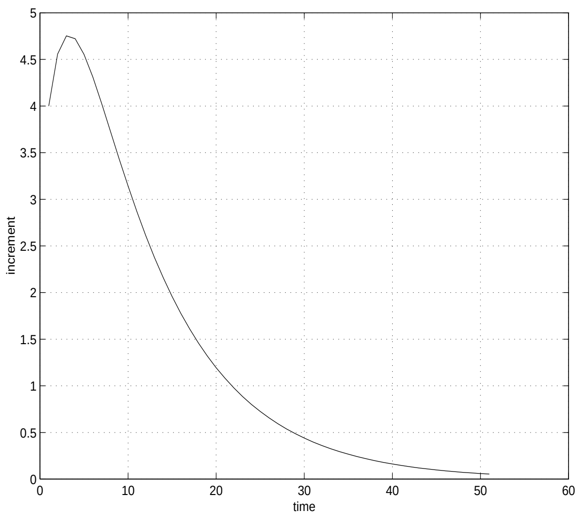

Fig. 2. Evolution of the control increment

where $g^t_{min} = [u^t_{min} \;\; v^t_{\text{min}}]$,
and $g^t_{max} = [u^t_{max} \;\; v^t_{\text{max}}]$.

Figure 1 represents the set of positive invariance and a trajectory emanating from the initial state $x_o = [24 - 14]^t$, while Figure 2 shows the evolution of the control increment, which, it is possible to see, respects the constraints $\Delta_{min} = -25$ and $\Delta_{max} = 20$.

5. CONCLUSION

In this paper, the regulator problem for linear systems with constraints on both control and its increment in the state space representation has been studied. Application of necessary and sufficient conditions, established for linear autonomous systems such that their motion respects incremental constraints, is the key to solve this problem. The link of the so called inverse procedure, the pole assignment method for constrained control, to the previous conditions is the cornerstone of this work. In fact, this link enables to give a simple algorithm to compute a regulator respecting constraints on both control and its increment.

ACKNOWLEDGEMENTS

This work is the product of collaboration between Unversity Cadi Ayyad Morocco and University of Valladoli, Spain under grant N 23P/2000

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