15th Triennial World Congress, Barcelona, Spain

STABILITY OF CONTINUOUS-TIME LINEAR SYSTEMS WITH MARKOVIAN JUMPING PARAMETERS AND CONSTRAINED CONTROL

A.BENZAOUIA*, E. K. BOUKAS and N. DARAOUI***

* *Research Unit: Constrained and Robust Regulation, Department of Physics, Faculty of Science Semlalia, P.B 2390, Marrakech,Morocco.*
E-mail.: benzaouia@ucam.ac.ma
** *Mechanical Engineering Department, l'Ecole Polytechnique de Montréal, P.O. Box 6079, Montréal, Québec, Canada H3C 3A7.*
Email.: boukas@meca.polymtl.ca

Abstract: This work is devoted to the study of linear continuous-time systems with Markovian jumping parameters and constrained control. The constraints used in this paper are of symmetrical inequality type. The approach of positively invariant sets is used to obtain new necessary and sufficient condition of stochastic positive invariance and sufficient condition of stochastic stability. These conditions become those given for stationary continuous-time systems with one mode as known in the literature.

Keywords: positive invariance, continuous-time systems, Markovian systems, constraints.

1. INTRODUCTION

Linear systems with Markovian jumping parameters offers the advantage to model a large varieties of physical phenomena. This class of systems has been used successfully to model manufacturing systems, power systems, economic systems, etc. The reader is refered, for example to (Costa, 1996; Boukas and Yang, 1999) for discrete-time and continuous-time systems. It is well known that all these physical systems admit inputs limitation which are modeled by constraints of inequality type. However, to the best of our knowledge, the problem of stochastic stability of continuous-time linear systems with both Markovian jumping parameters and constrained control have only been investigated for discrete-time systems (Boukas and Benzaouia). The regulator problem for linear systems with constrained control is widely studied after the result established by (Gutman and Hagander, 1985) . The tool of positive invariance was successfully applied to almost all the determiistic systems with constrained control and/or state see for example (Benzaouia and Burgat, 1988 - Benzaouia, 1994) and the references therein. The aim of this paper is to study the regulator problem for continuous-time class of systems with Markovian jumping parameters and constrained control by using the positive invariance approach. In this work, only symmetrical constraints are considered. A necessary and sufficient condition allowing the control law to always be admissible despite the stochastic character of the system is presented. A sufficient condition for stochastic stability of the system is also obtained. This paper is organized as follows: The studied problem is formulated in Section 2. Section 3 deals with definitions and preliminary results as the necessary and sufficient condition of stochastic positive invariance and the sufficient condition of stochastic stability for the free system. In Section 4, these results are used to design a law control ensuring to the control to remain admissible. The stochastic stability is also guaranteed. An algorithm together with an example illustrating this design is presented in Section 5.

2. PROBLEM FORMULATION

Consider the continuous-time linear system with Markovian jumping parameters defined by:

$$(\Sigma): \begin{cases} \dot{x}(t) = A(r(t))x(t) + B(r(t))u(t), \\ x(0) = x_0 \end{cases}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector. The system state is function of $r(t)$ which represents a Markovian continuous-time process taking values in a finite discrete set $S = \{1, 2, \ldots, s\}$ defined with its transition matrix $\Pi = (\lambda_{\alpha\beta})_{\alpha,\beta \in S}$, where $\lambda_{\alpha\beta}$ is a scalar such that: $\lambda_{\alpha\beta} \geq 0$ for $\alpha \neq \beta$ and $\forall \alpha \in S$, $\lambda_{\alpha\alpha} = -\sum_{\beta \neq \alpha} \lambda_{\alpha\beta}$.
The transition probability $P[r(t+h) = \beta | r(t) = \alpha] = \lambda_{\alpha\beta} h + o(h)$if $\beta \neq \alpha$ and $1 + \lambda_{\alpha\alpha} h + o(h)$if $\beta = \alpha$ with, $\lim_{h \to 0} \frac{o(h)}{h} = 0$.
The set of constraints is given for each mode $\alpha \in S$, with $q(\alpha) \in R^{m+}$, by:

$$\Omega(\alpha) = \{u(t) \in R^m / -q(\alpha) \leq u(t) \leq q(\alpha)\} \quad (1)$$

The objective of this work is to built a stabilizing regulator,

$$u(t) = F(r(t))x(t), \quad (2)$$

that satisfies the control constraints (1) and stochastically stabilizes the system (Σ).

3. PRELIMINARY RESULTS

In this section, a condition of the stability of system (Σ) is given. First the definition of stochastic stability is recalled:

Definition 1. The system (Σ) (avec $u(t) \equiv 0$) is said to be:

(1) stochastically stable(SS) if there exists a scalar η such that: $E\left[\int_0^\infty \|x(t)\|^2 dt\right] \leq \eta(r_0, x_0)$.
(2) mean exponentially stable (MES) if there exist two scalars $\rho > 0, \eta > 0$ such that: $E[\|x(t)\|] \leq \eta e^{-\rho t}$.
(3) mean square stable (MSS) if $\lim_{t \to \infty} E[\|x(t)\|^2] = 0$.

Remark 2. One can note that the (MES) stability implies the (SS) stability of the system.

Theorem 3. The system (Σ) (with $u(t) \equiv 0$) is SS if for all $\alpha \in S$ there exist positive vectors $w(\alpha) \in R^{n+}$ such that:

$$max_i \left(\frac{[\hat{A}(\alpha) w(\alpha)]_i}{w_i(\alpha)} \right) \leq -\mu(\alpha) \quad (3)$$

with,

$$\hat{A}_{ij}(\alpha) = \begin{cases} a_{ii}(\alpha) \text{if, } j = i \\ |a_{ij}(\alpha)| \text{if, } j \neq i \end{cases} \quad (4)$$

$$\mu(\alpha) = \sum_{\beta=1}^{s} \lambda_{\alpha\beta} max_i \left(\frac{w_i(\alpha)}{w_i(\beta)} \right)$$

The proof is given in appendix 1.

Remark 4. Note that condition (3) can equivalently be written under the following form,

$$\hat{A}(\alpha) w(\alpha) \leq -\mu(\alpha) w(\alpha), \forall \alpha \in S \quad (5)$$

This condition will be satisfied only if matrix $A(\alpha)$ is Hurwitz and $\mu(\alpha) \geq 0$. Further, for a system with one mode, one can obtain $\mu(\alpha) = 0$.

Now, a result concerning the application of the positive invariance concept to a continuous-time system with Markovian jumping parameters is presented. Let the system be defined by:

$$E[\dot{z}(t) | r(t), z(t)] = H(r(t)) z(t), \quad (6)$$
$$z(0) = z_0,$$
$$r(0) = \alpha,$$

where $z(t) \in R^m$ is the state vector of the system.

Definition 5. A subset $\mathcal{D}$ of R^m is stochastically positively invariant with respect to (w.r.t) system (6) if for all $z_0 \in \mathcal{D}$ and an initial mode $\alpha \in S$, $E[z(t, z_0, \alpha)] \in \mathcal{D}$.

For each $\alpha \in S$, consider the following domain:

$$\mathcal{D}(\alpha) = \{z \in R^m | -\omega(\alpha) \leq z \leq \omega(\alpha), \omega(\alpha) > 0\}$$

Let

$$\mathcal{D}_c = \bigcap_{\alpha \in S} \mathcal{D}(\alpha) = \{z \in R^m | -\varphi \leq z \leq \varphi\} \quad (7)$$

$$\varphi_i = \min_{\alpha \in S} \omega_i(\alpha)$$

Theorem 6. The set $\mathcal{D}_c$ defined by (7) is stochastically positively invariant w.r.t the system (6) if and only if:

$$\hat{H}(\alpha)\varphi \leq 0, \quad \text{for all} \quad \alpha \in S \quad (8)$$

where $\hat{H}$ is defined by (4).
The proof of this theorem can be found in (Benzaouia *et al.*).

4. MAIN RESULTS

In this section, the obtained results in the previous section will allow us to deal with the problem of continuous-time systems with Markovian jumping parameters and constrained control as presented in the first section. Recall that the control law is given by (2). The control is then admissible, i.e., $u(t) \in \Omega(\alpha)$ if and only if the state belongs in the polyhedral domain $\mathcal{K}(\alpha)$ defined by:

$$\mathcal{K}(\alpha) = \{x \in R^n / -q(\alpha) \leq F(\alpha)x \leq q(\alpha)\} \quad (9)$$

Define the following set,

$$\mathcal{K}_c = \bigcap_{\alpha \in S} \mathcal{K}(\alpha) \quad (10)$$

The system in the closed-loop is obtained by,

$$\dot{x}\ (t) = (A(r(t)) + B(r(t))F(r(t))x(t) = A_c(\alpha)x(t) \quad (11)$$

Make the following change of variables,

$$z(t) = F(r(t))x(t) \quad (12)$$

In this case, domain (9) is transformed to the following domain,

$$\mathcal{D}(\alpha) = \{z \in R^m / -q(\alpha) \leq z \leq q(\alpha), q(\alpha) > 0\}$$

Let

$$\mathcal{D}_c = \bigcap_{\alpha \in S} \mathcal{D}(\alpha) \quad (13)$$

Pose, $E(.) = E[\dot{z}\ (t)|z(t), r(t)$ and $L(r(t)) = I_d + hA_c(r(t))$. Thus, one can write,

$$E(.) = \alpha] = \lim_{h \to 0} \frac{1}{h} E\left[z(t+h) - z(t)|z(t), r(t) = \alpha\right]$$

It follows, by using (12) and the developpement of $x(t+h)$,

$$z(t+h) = F(r(t+h))L(r(t))x(t) + o(h)$$

Taking account of the definition of the probability transition, one obtains,

$$E(.) = \lim_{h \to 0} \frac{1}{h} \Big\{ \sum_{\beta=1}^{s} \lambda_{\alpha\beta} h F(\beta) \Big[L(\alpha)x(t) + o(h)\Big] + F(\alpha)\Big[L(\alpha)x(t) + o(h)\Big] - F(\alpha)x(t) \Big\}$$

That is,

$$E(.) = \sum_{\beta=1}^{s} \lambda_{\alpha\beta} F(\beta)x(t) + F(\alpha)A_c(\alpha)x(t)$$

Then, if there exist matrices $H(\alpha) \in R^{m \times m}$ such that:

$$\sum_{\beta=1}^{s} \lambda_{\alpha\beta} F(\beta) + F(\alpha)A_c(\alpha) = H(\alpha)F(\alpha) \quad (14)$$

the dynamical system (11) is transformed by the use of (12) to the following dynamical system:

$$E[\dot{z}(t)|z(t), r(t) = \alpha] = H(\alpha)z(t) \quad (15)$$

At this step, the results of Section 2 can be easily applied to system (15) taking into account the set of constraints (13).

Theorem 7. If there exist matrices $H(\alpha) \in \mathbb{R}^{m \times m}$ and vectors $w(\alpha) \in \mathbb{R}^{n+}$, such that the following hold:

(i):

$$G(\alpha) + F(\alpha)A_c(\alpha) = H(\alpha)F(\alpha), \forall \alpha \in S \quad (16)$$

(ii):

$$\hat{H}(\alpha)\vartheta \leq 0, \forall \alpha \in S, \quad (17)$$

(iii):

$$\hat{A}_c(\alpha)w(\alpha) \leq -\mu(\alpha)w(\alpha), \forall \alpha \in S \quad (18)$$

where,

$$G(\alpha) = \sum_{\beta=1}^{s} \lambda_{\alpha\beta} F(\beta), \vartheta_i = \min_{\alpha \in S} q_i(\alpha) \quad (19)$$

$$\mu(\alpha) = \sum_{\beta=1}^{s} \lambda_{\alpha\beta} max_i \left(\frac{w_i(\alpha)}{w_i(\beta)}\right)$$

then, the closed-loop system (11) is stochastically stable $\forall x_0 \in \mathcal{K}_c$.

Proof: The condition (16) allows us to transform system (11) to system (15). According to Theorem 2.1, condition (17) guarantees the stochastic positive invariance of domain $\mathcal{D}_c$ w.r.t system (15). That is, the stochastic positive invariance of domain $\mathcal{K}_c$ w.r.t system (11) is also guaranteed. Further, condition (18) ensures the stochastic exponential stability of the system in the closed-loop (11). ∇

Comment 8. It is worth noting that condition (17) ensures that domain $\mathcal{D}_c$ is stochastically positively invariant w.r.t system (15), that means that the control will be admissible only in the mean sense. Thus, the trajectories of the system (11) can sometimes leave the set $\mathcal{K}_c$ (the control is saturated), however, according to condition (18), the stochastic stability of the system will be guaranteed.

5. RESOLUTION OF EQUATION $\mathbf{G}(\alpha) + \mathbf{F}(\alpha)\mathbf{A_C}(\alpha) = \mathbf{H}(\alpha)\mathbf{F}(\alpha)$

In this section, an algorithm computing $H(\alpha), \forall \alpha \in S$ is presented. The same idea of decomposing matrix $H(.)$ as in the discrete-time case (Boukas and Benzaouia, 2002) is followed, let $H(\alpha) = H_1(\alpha) + H_2(\alpha)$. The first step is the augmentation of the system by the

introduction of $n-m$ fictitious entries (Benzaouia and Burgat, 1988).
The matrix $H_1(\alpha)$ is chosen such that:

$$H_1(\alpha)\theta_i(\alpha) = \lambda_i(\alpha)\theta_i(\alpha), \quad i = 1, \ldots, n \quad (20)$$

with,

$$\{\theta_1(\alpha), \ldots, \theta_n(\alpha)\} \text{ are linearly independent} \quad (21)$$

and satisfying:

$$B(\alpha)\theta_i(\alpha) \neq 0, \quad (22)$$
$$\sigma(A(\alpha)) \bigcap \sigma(H_1(\alpha)) = \emptyset \quad (23)$$

The looked for matrices $F(\alpha)$ are solution of the following equations,

$$F(\alpha)\Big[A(\alpha) + B(\alpha)F(\alpha)\Big] = H_1(\alpha)F(\alpha) \quad (24)$$

These solutions exist if and only if (Benzaouia, 1994), the eigenvectors of matrices $A_c(\alpha)$, $\{\xi_1(\alpha), \ldots, \xi_n(\alpha)\}$, which are associated to the assigned eigenvalues $\{\lambda_1(\alpha), \ldots, \lambda_n(\alpha)\}$, are linearly independent. $F(\alpha)$ are given by:

$$F(\alpha) = [\theta_1(\alpha), \ldots, \theta_n(\alpha)][\xi_1(\alpha), \ldots, \xi_n(\alpha)]^{-1} \quad (25)$$

One can then compute matrices $G(\alpha)$ for all $\alpha = 1, \ldots, s$. It follows that:

$$[G(\alpha) + F(\alpha)A_c(\alpha)]\,\xi_i(\alpha) = H(\alpha)F(\alpha)\xi_i(\alpha)$$

Since matrix $H(\alpha)$ is decomposed under the following form: $H(\alpha) = H_1(\alpha) + H_2(\alpha)$
keeping in mind (20), one obtains:

$$G(\alpha)\xi_i(\alpha) = H_2(\alpha)\theta_i(\alpha)$$

Then, matrix $H_2(\alpha)$ can easily be deduced and so on for matrix $H(\alpha)$:

$$H_2(\alpha) = G(\alpha)\,[\xi_1(\alpha), \ldots, \xi_n(\alpha)]\,[\theta_1(\alpha), \ldots, \theta_n(\alpha)]^{-1} \quad (26)$$

Algorithm:

1) Augment the matrices $B(\alpha)$ for each mode $\alpha \in S$ with $n - m$ null row and choose matrices $H_1(\alpha)$ according to (21)-(23).
2) Compute the gain matrices $F(\alpha)$ by using (25) and matrices $A_c(\alpha)$ of the system in closed-loop. Note that the obtained matrices $A_c(\alpha)$ are Hurwitz by construction,see (Benzaouia,1994).
3) Choose vectors $w(\alpha) \in \mathcal{R}^{n+}$ satisfying $\mu(alpha) \geq 0$ and test if condition (18) is satisfied. One can choose $w(\alpha) = w_0$ independently of the mode α , the condition (18) becomes: $\widehat{A}_c(\alpha)w_0 \leq 0$. If not go to Step 1 to modify the matrices $H_1(\alpha)$.
4) Test if (17) is satisfied, if not go to Step 1 to modify matrices $H_1(\alpha)$.

Example 9. Consider the following linear continuous-time system with Markovian jumping parameters with 2 modes: $A(1) = \begin{bmatrix} 2 & -0.1 \\ 1 & -1 \end{bmatrix}$, $B(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$A(2) = \begin{bmatrix} 3 & 0 \\ 0.1 & -2.1 \end{bmatrix}$, $B(2) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$
with $q(1) = 20, q(2) = 30$
The Markovian process is described by its matrix transition given by:

$$\Pi = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$$

For each mode α, the fictitious entries are introduced as follows:

$$-g(\alpha) \leq v(t) \leq g(\alpha), \alpha = 1, 2$$

Consider the following matrices $H_1(\alpha)$:

$$H_1(1) = \begin{bmatrix} -9 & 1 \\ 0 & -0.6 \end{bmatrix}, H_1(2) = \begin{bmatrix} -5 & -1 \\ 0 & -0.5 \end{bmatrix},$$

The obtained gain matrices $F(\alpha)$ are:

$$F(1) = \begin{bmatrix} -9.4211 & -1.1789 \\ -4.9474 & 4.0074 \end{bmatrix}$$
$$F(2) = \begin{bmatrix} -10.9624 & -0.9188 \\ 17.6317 & -3.2158 \end{bmatrix}$$

Note that the effective gain matrices are to be extracted from the previous matrices. The obtained matrices in the closed-loop are then given by:

$$A_c(1) = \begin{bmatrix} -7.4211 & -1.2789 \\ -8.4211 & -2.1789 \end{bmatrix}$$
$$A_c(2) = \begin{bmatrix} -2.4812 & -0.4594 \\ -10.8624 & -3.0188 \end{bmatrix}$$

One can verify that matrices $H(\alpha)$, $\alpha \in S$ satisfy conditions (17) with the following data

$$g(1) = 40, g(2) = 25, \vartheta = [20, 25]^T.$$

For this choice, one has:

$$\widehat{H}(1)\vartheta = [-145.617, -73.3555]^T < 0$$
$$\widehat{H}(2)\vartheta = [-81.652, -94.3205]^T < 0$$

Further, the condition (18) of Theorem 4.1 is also satisfied with

$$w(1) = \begin{bmatrix} 1.89, & 8 \end{bmatrix}^T, w(2) = \begin{bmatrix} 2, & 7.8 \end{bmatrix}^T$$

That is, $\mu(1) = 0.0513$, $\mu(2) = 0.1746$ and,

$$\widehat{A}_c(1)w_1 + \mu(1)w_1 = \begin{bmatrix} -3.6978, & -1.1051 \end{bmatrix}^T$$
$$\widehat{A}_c(2)w_2 + \mu(2)w_2 = \begin{bmatrix} -1.0299, & -0.4599 \end{bmatrix}^T$$

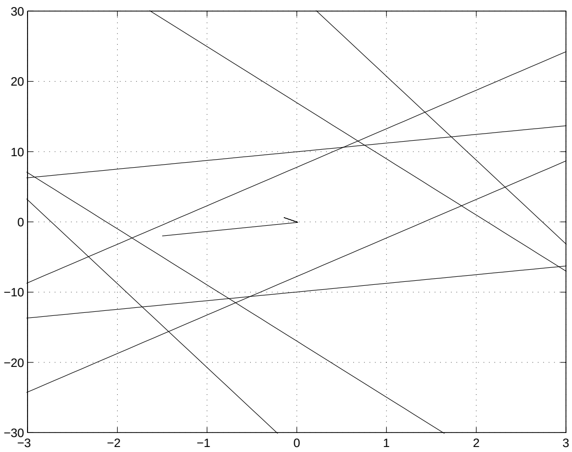

Fig. 1. presents the set $\mathcal{K}_c$ of stochastic invariance positive and stochastic stability of the system with Markovian jumping parameters and constrained control.

6. CONCLUSION

In this paper, necessary and sufficient conditions for domain $\mathcal{D}_c$ to be stochastically positively invariant w.r.t the system (6) are established for linear continuous-time systems with Markovian jumping parameters. A new sufficient condition of stochastic stability is also obtained by using the non quadratic Lyapunov function as is usually the case in the problems with constraints of inequality type. These two results are applied to give a new sufficient condition of stochastic stability for systems with Markovian jumping parameters and symmetrical constrained control. An Algorithm is also presented based on the resolution of the obtained algebraic equation to compute matrices $H(\alpha)$.

7. REFERENCES

Benzaouia, A. and Burgat, C ”Regulator problem for linear discrete-time systems with non symmetrical constrained control”, Int. J. Control,Vol. 48,No. 6, pp. 2441-2451, 1988.

Benzaouia, A ”The regulator problem for linear discrete-time systems with non symmetrical constrained control”, Proc. of the 30th CDC, Brighton, pp.1742-1743, 991.

Benzaouia, A., and Hmamed, A ”Regulator problem of linear continuous-time systems with constrained control”, IEEE Trans.Aut. Control, Vol.38, No.10, pp. 1556-1560, 1993.

Benzaouia, A ”Resolution of equation XA+XBX=HX and the pole assignment problem”, IEEE Trans. Aut. Control, Vol. 40, No. 10, pp. 2091-2095,1994.

Benzaouia, A , Boukas, E.K and Daraoui, N ”Stability of continuous-time Linear Systems with Markovian Jumping Parameters and Constraints” IEEE Trans. Aut. Control, Submitted.

Boukas, E. K. and Yang, H ”Stability of discrete-time systems with Markovian jumping parameters”, Math. Control Signals Systems, No. 8, pp. 390-402,1995.

Boukas, E. K. and Yang, H ”Exponential stability of stochastic Markovian jumping parameters”, Automatica, Vol. 35, No. 8, pp. 1437-1441,1999.

Boukas, E. K. and Benzaouia, A ”Stability of Discrete-time Linear Systems with Markovian Jumping Parameters and Constrained Control” IEEE Trans. Aut. Control, To appear, 2002.

Costa, O. L. V ”Mean square stabilizing solutions for discrete-time coupled algebraic Riccati equations”, IEEE Trans. Aut. Control, Vol. 41, No. 4, pp.593-598, 1996.

Appendix 1

Consider the following Lyapunov function candidate:

$$V(x(t), r(t)) = max_i \frac{|x_i(t)|}{w_i(r(t))}, i = 1, \ldots, n \tag{27}$$

The infinitesimal generator is computed by:

$$\begin{aligned}\mathcal{A}V(x(t), \alpha) &= \lim_{h\to 0} \frac{1}{h} \left\{E\left[V(x(t+h), r(t+h))|r(t) = \alpha\right] - V(x(t), \alpha)\right\} \\ &= \lim_{h\to 0} \frac{1}{h} \left\{E\left[max_i \frac{|x_i(t+h)|}{w_i(r(t+h))}|r(t) = \alpha\right] - V(x(t), \alpha)\right\}\end{aligned}$$

Recall that, $x(t+h) = x(t) + hA(r(t))x(t) + o(h) = L(r(t))x(t) + o(h)$ with $L(r(t)) = I_d + hA(r(t))$.
Moreover,

$$E\left[max_i \frac{|x_i(t+h)|}{w_i(r(t+h))}|r(t) = \alpha\right] = \sum_{\beta\in S} max_i \frac{|L(\alpha)x(t) + o(h)|}{w_i(\beta)} P\left[r(t+h) = \beta|r(t) = \alpha\right]$$

Using the definition of the transition probability, one obtains,

$$\begin{aligned}\mathcal{A}V(x(t), \alpha) &= \lim_{h\to 0} \frac{1}{h} \left\{\sum_{\beta=1}^{s} \lambda_{\alpha\beta} max_i \frac{|x_i(t+h)|}{w_i(\beta)} h + max_i \frac{|(L(\alpha)x(t))_i + o(h)|}{w_i(\alpha)} - V(x(t), \alpha)\right\} \\ &\leq \sum_{\beta=1}^{s} \lambda_{\alpha\beta} max_i \frac{|x_i(t)|}{w_i(\beta)} + \lim_{h\to 0} \frac{1}{h} \left[max_i \frac{|(L(\alpha)x(t))_i|}{w_i(\alpha)} - max_i \frac{|x_i(t)|}{w_i(\alpha)}\right]\end{aligned}$$

Let $l_{ij}(\alpha), i = 1, \ldots, n, j = 1, \ldots, n$ be the component of matrix $L(\alpha)$, It follows that,

$$\mathcal{A}V(x(t), \alpha) \leq \left\{\sum_{\beta=1}^{s} \lambda_{\alpha\beta} max_i \left(\frac{w_i(\alpha)}{w_i(\beta)}\right) + \lim_{h\to 0} \frac{1}{h} \left[max_i \sum_{k=1}^{n} \frac{|l_{ik}(\alpha)|w_k(\alpha)}{w_i(\alpha)} - 1\right]\right\} V(x(t), \alpha)$$

Since,

$$\sum_{k=1}^{n} \frac{|l_{ik}(\alpha)|w_k(\alpha)}{w_i(\alpha)} = \sum_{k\neq i} h \frac{|a_{ik}(\alpha)|}{w_i(\alpha)} w_k(\alpha) + |1 + ha_{ii}(\alpha)| = \sum_{k\neq i} h \frac{|a_{ik}(\alpha)|}{w_i(\alpha)} w_k(\alpha) + 1 + ha_{ii}(\alpha) \text{ for a small h}$$

This leads to, $\mathcal{A}V(x(t), \alpha) \leq \underbrace{\left\{\sum_{\beta=1}^{s} \lambda_{\alpha\beta} max_i \left(\frac{w_i(\alpha)}{w_i(\beta)}\right) + max_i \frac{\left(\hat{A}(\alpha)w(\alpha)\right)_i}{w_i(\alpha)}\right\}}_{-\rho(\alpha)} V(x(t), \alpha)$

That is, $E\left[\mathcal{A}V(x(t), \alpha)\right] \leq -\rho E\left[V(x(t), \alpha)\right]$ with $\rho = \max_{\alpha\in S} \rho(\alpha)$
One can use the fact that: $V(x(t), r(t)) = V(x_0, r_0) + \int_0^t \mathcal{A}V(x(t), r(s))ds$
which implies,

$$\begin{aligned}E\left[V(x(t), r(t)\right] &\leq E\left[V(x_0, r_0\right] + \int_0^t E\left[\mathcal{A}V(x(s), r(s))\right] ds \\ &\leq E\left[V(x_0, r_0\right] - \rho \int_0^t E\left[V(x(s), r(s)\right] ds\end{aligned}$$

By virtue of Gronwall lemma, one has:

$$E\left[V(x(t), r(t)\right] \leq e^{-\rho t} E\left[V(x_0, r_0)\right] \tag{28}$$

(28) implies that $E\left[\|x(t)\|\right] \leq \eta e^{-\rho t}$ with $\|x(t)\| = \max_i |x_i(t)|$and $\eta = \max_{\alpha\in S} \min_i w_i(\alpha) E\left[V(x_0, r_0)\right]$ ∇