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IMPROVED ESTIMATE OF TIME-DELAY FOR STABILITY OF LINEAR SYSTEMS

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Abstract: The robust stability problem of uncertain linear time-delay systems is investigated using a refined discretized Lyapunov functional approach. The uncertainty under consideration is norm-bounded, and possibly time-varying. A new stability criterion is derived. The computational requirement is reduced for the same discretization mesh. Examples show that the results obtained by this new criterion significantly improve the estimate of the stability limit over some existing results in the literature. *Copyright © 2002 IFAC*

Keywords: Stability; Time-delay; Lyapunov functional; Uncertainty; Linear matrix inequality.

1. INTRODUCTION

The stability problem of time-delay systems has received considerable attention in the last two decades. The practical examples of time-delay systems include engineering, communications and biological systems (Hale and Lunel, 1993). The existence of delay in a practical system may induce instability, oscillation and poor performance (Malek-Zavarei and Jamshidi, 1987). Current efforts on this topic can be divided into two classes: namely frequency-domain based and time-domain based approaches.

In the time-domain approach, the direct Lyapunov method is a powerful tool for studying the stability of linear time-delay systems. There are two different ideas how one can use this method. They are Lyapunov-Krasovskii approach and Lyapunov-Razumikhin approach. Some results are based on a rather *simple* form of Lyapunov-Krasovskii functional, with stability criteria independent of time-delay, see for example, Han and Mehdi (1999a), Kokame *et al.* (1998), Lee *et al.* (1994), Phoojaruenchanachai and Furuta (1992), Shen *et al.* (1991). A model transformation technique is often used to transform a pointwise delay system to a distributed delay system, and delay-dependent stability criteria are obtained by employing Lyapunov-Razumikhin Theorem, see for example, Han and Mehdi (1999b), Li and de Souza (1997), Park (1999), Su (1994), Su and Huang (1992). Although these results are often less conservative than the delay-independent results, they can still be rather conservative. This can be seen by applying these types of criteria to a constant time-delay system without uncertainty and comparing with analytical results (Gu 1997). In addition to the conservatism in applying Razumikhin theorem, the model transformation may introduce additional poles that are not present in the original system, and one of these additional poles may cross the imaginary axis before any of the poles of the original system do as the delay increases from zero (Gu and Niculescu 2000, Kharitonov and Melchor-Aguilar 2000). Moreover, there are no obvious ways to obtain less conservative results even if one is willing to commit more computational effort to the problem. Furthermore, most criteria do not reduce to a necessary and sufficient condition when applied to uncertainty-free systems.

For a linear system with a constant time-delay, it has been proven that the existence of a generalized quadratic form Lyapunov-Krasovskii functional is necessary and sufficient for the stability of an uncertainty-free time-delay system (Huang 1989). The related weight matrices have to satisfy a complicated system of algebraic, ordinary and partial differential equations. A solution construction for this system and the following analysis of this system is a very difficult task. However, a piecewise linear discretization scheme has been proposed to enable one to write the stability criterion in an LMI form (Gu 1997, 1999a). The results have been improved by allowing some parameters to depend on uncertainties, with the resulting formulation no more complicated by eliminating some free variables appearing in the resulting LMI (1999b). Recently, a refinement over Gu (1999b) has been proposed in Gu (2001a) which significantly reduces conservatism and shows significant improvements over the existing results even under very coarse discretization. For uncertainty-free systems, the analytical results can be approached with fine discretization.

In this paper, the robust stability problem of linear time-delay systems with norm-bounded, and possibly time-varying is investigated by using the refined discretized Lyapunov functional approach (Gu 2001a). As compared to Han and Gu (2001), the new criterion requires less computation, converges to the analytical solution much faster for systems without uncertainty, and is much less conservative for uncertain systems.

Notation. Let $\sigma_{\max}(W)$ denote the maximum singular value of matrix W . For a symmetric matrix W , "$W > 0$" denotes that W is positive definite matrix.

2. PROBLEM STATEMENT

Consider the robust stability problem of time-delay system

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]x(t-r) \qquad (1)$$

with initial condition

$$x(t) = \phi(t)\,,\ \forall t \in [-r, 0] \qquad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, r is a constant time delay, $\phi(t)$ is the initial condition, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are known real constant matrices which describe the nominal system of Eq. (1), and $\Delta A(t)$ and $\Delta B(t)$ are real matrix functions representing time-varying parameter uncertainties. The uncertainties are assumed in the form

$$[\Delta A(t) \quad \Delta B(t)] = LF(t)[E_a \quad E_b] \qquad (3)$$

where $F(t) \in \mathbb{R}^{p\times q}$ is an unknown real and possibly time-varying matrix with Lebesgue measurable elements satisfying

$$\sigma_{\max}(F(t)) \le 1 \qquad (4)$$

and L , E_a and E_b are known real constant matrices which characterize how the uncertainty enters the nominal matrices A and B.

Define $\mathcal{C}$ as the set of continuous $\mathbb{R}^n$ valued function on the interval $[-r,\ 0]$, and let $x_t \in \mathcal{C}$ be a segment of system trajectory defined as

$$x_t(\theta) = x(t+\theta)\,,\ -r \le \theta \le 0\,. \qquad (5)$$

In this paper, we will attempt to formulate a practically computable criterion for robust stability of uncertain system described by (1) to (4).

3. MAIN RESULT

Choose a Lyapunov-Krasovskii functional $V(\phi)$ of a quadratic form

$$\begin{aligned} &V:\ \mathcal{C} \mapsto \mathbb{R} \\ &V(\phi) = \frac{1}{2}\phi^T(0)P\phi(0) + \phi^T(0)\int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\ &\qquad + \frac{1}{2}\int_{-r}^{0}[\int_{-r}^{0}\phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta]d\xi \qquad (6) \end{aligned}$$

where

$$\begin{aligned} &P \in \mathbb{R}^{n\times n}\,,\ P = P^T \\ &Q : [-r,0] \to \mathbb{R}^{n\times n} \\ &S : [-r,0] \to \mathbb{R}^{n\times n}\,,\ S^T(\xi) = S(\xi) \\ &R : [-r,0]\times[-r,0] \to \mathbb{R}^{n\times n}\,,\ R(\eta,\xi) = R^T(\xi,\eta)\,. \end{aligned}$$

It is well known that (Hale and Lunel 1993)

Theroem 1. The system (1)-(2) is asymptotically stable if there exists a quadratic Lyapunov-

Krasovskii functional V of the form (6) such that for some $\varepsilon > 0$, it satisfies

$$V(\phi) \geq \varepsilon \phi^T(0)\phi(0)$$

and its derivative along the solution of (1) satisfies

$$\dot{V}(\phi) \leq -\varepsilon \phi^T(0)\phi(0)$$

for any $\phi \in \mathcal{C}$, where

$$\dot{V}(\phi) \overset{\Delta}{=} \frac{d}{dt} V(x_t)\Big|_{x_t=\phi}.$$

Choose Q, R and S to be continuous piecewise linear, i.e.,

$$\begin{aligned}
&Q^i(\alpha) \overset{\Delta}{=} Q(\delta_{i-1} + \alpha h) = (1-\alpha)Q_{i-1} + \alpha Q_i \\
&S^i(\alpha) \overset{\Delta}{=} S(\delta_{i-1} + \alpha h) = (1-\alpha)S_{i-1} + \alpha S_i \\
&R(\delta_{i-1} + \alpha h, \delta_{j-1} + \eta h) = R^{ij}(\alpha, \eta) \\
&\overset{\Delta}{=} \begin{cases} (1-\alpha)R_{i-1,j-1} + \eta R_{ij} + (\alpha - \eta)R_{i,j-1}, & \alpha \geq \eta \\ (1-\eta)R_{i-1,j-1} + \alpha R_{ij} + (\eta - \alpha)R_{i-1,j}, & \alpha < \eta \end{cases}
\end{aligned} \tag{7}$$

for $0 \leq \alpha \leq 1$, $0 \leq \eta \leq 1$, where

$$\delta_i = -r + ih, \; i = 0, 1, 2, \cdots, N; \; h = r/N$$

i.e., N is the number of divisions of the interval $[-r, 0]$, and h is the length of each division.

Based on Theorem 1, noting that (31) in Proposition 3 is implied by (46) in Proposition 4 and combining (32) in Proposition 3 and (46) in Proposition 4 in Gu (2001a), the following result was easily obtained.

Lemma 1 (Gu 2001a). For piecewise linear Q, S and R as described by (7) and for $\Delta A(t)$ and $\Delta B(t)$ satisfying (3), system (1)-(2) is robustly stable if the following LMIs hold.

$$\begin{bmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \frac{1}{h}\tilde{S} + \tilde{R} \end{bmatrix} > 0 \tag{8}$$

$$\Xi(t) = \begin{bmatrix} G_{11}(t) & -G_{12}(t) & D_1^s(t) & D_1^a(t) \\ -G_{12}^T(t) & G_{22}(t) & D_2^s(t) & D_2^a(t) \\ D_1^{sT}(t) & D_2^{sT}(t) & \frac{1}{h}S_d + R_d & 0 \\ D_1^{aT}(t) & D_2^{aT}(t) & 0 & \frac{3}{h}S_d \end{bmatrix} > 0 \tag{9}$$

for $\Delta A(t)$ and $\Delta B(t)$ satisfying (3), where

$$\tilde{S} = diag(S_0 \; S_1 \; S_2 \; \cdots \; S_{N-1} \; S_N)$$

$$\tilde{Q} = [Q_0 \; Q_1 \; \cdots \; Q_N]$$

$$\tilde{R} = \begin{bmatrix} R_{00} & R_{01} & \cdots & R_{0N} \\ R_{10} & R_{11} & \cdots & R_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0} & R_{N1} & \cdots & R_{NN} \end{bmatrix}$$

$$G_{11}(t) = -\left\{P[A + \Delta A(t)] + [A + \Delta A(t)]^T P + S_N + Q_N + Q_N^T\right\}$$

$$G_{12}(t) = P[B + \Delta B(t)] - Q_0$$

$$G_{22}(t) = S_0$$

$$S_d = diag(S_{d1}, S_{d2}, \cdots, S_{dN})$$

$$S_{di} = \frac{1}{h}(S_i - S_{i-1})$$

$$R_d = \begin{bmatrix} R_{d11} & R_{d12} & \cdots & R_{d1N} \\ R_{d21} & R_{d22} & \cdots & R_{d2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dN1} & R_{dN2} & \cdots & R_{dNN} \end{bmatrix}$$

$$R_{dij} = \frac{1}{h}(R_{ij} - R_{i-1,j-1}); \; i, j = 1, 2, \cdots, N$$

$$D_j^s(t) = [D_{j1}^s(t) \; D_{j2}^s(t) \; \cdots \; D_{jN}^s(t)]; \; j = 1, 2;$$

$$D_{1i}^s(t) = -\frac{1}{2}[A + \Delta A(t)]^T(Q_i + Q_{i-1}) + \frac{1}{h}(Q_i - Q_{i-1}) - \frac{1}{2}(R_{i,N}^T + R_{i-1,N}^T)$$

$$D_{2i}^s(t) = -\frac{1}{2}[B + \Delta B(t)]^T(Q_i + Q_{i-1}) + \frac{1}{2}(R_{i,0}^T + R_{i-1,0}^T)$$

$$D_j^a(t) = [D_{j1}^a(t) \; D_{j2}^a(t) \; \cdots \; D_{jN}^a(t)]; \; j = 1, 2;$$

$$D_{1i}^a(t) = \frac{1}{2}[A + \Delta A(t)]^T(Q_i - Q_{i-1}) + \frac{1}{2}(R_{i,N}^T - R_{i-1,N}^T)$$

$$D_{2i}^a(t) = \frac{1}{2}[B + \Delta B(t)]^T(Q_i - Q_{i-1}) - \frac{1}{2}(R_{i,0}^T - R_{i-1,0}^T).$$

For polytopic uncertainty, it is known from Gu (2001a) that (9) only needs to be satisfied at all the vertices. For norm-bounded, and possibly time-varying, uncertainty, we may obtain

Theorem 2. The uncertain system described by (1) to (4) is robustly stable if there exist real $n \times n$ matrices $X = X^T$, Y_i, $W_i = W_i^T$ ($i = 0, 1, 2, \cdots, N$) and $Z_{ij} = Z_{ji}^T$ ($i, j = 0, 1, 2, \cdots, N$) and a scalar $\lambda > 0$ such that the following LMIs are satisfied

$$\begin{bmatrix} X & \tilde{Y} \\ \tilde{Y}^T & \frac{1}{h}\tilde{W} + \tilde{Z} \end{bmatrix} > 0 \tag{10}$$

$$\begin{bmatrix} H_{00} & -H_{01} & 0 & \Pi_0^s & \Pi_0^a \\ -H_{01}^T & H_{11} & -H_{12} & \Pi_1^s & \Pi_1^a \\ 0 & -H_{12}^T & H_{22} & \Pi_2^s & \Pi_2^a \\ \Pi_0^{sT} & \Pi_1^{sT} & \Pi_2^{sT} & \frac{1}{h}W_d + Z_d & 0 \\ \Pi_0^{aT} & \Pi_1^{aT} & \Pi_2^{aT} & 0 & \frac{3}{h}W_d \end{bmatrix} > 0 \tag{11}$$

where

$$X = \lambda P, \; Y_i = \lambda Q_i, \; W_i = \lambda S_i, \; (i = 0, 1, 2, \cdots, N)$$
$$Z_{ij} = \lambda R_{ij} \; (i, j = 0, 1, 2, \cdots, N) \tag{12}$$

and

$$G_{ij}^0 = G_{ij}(t)\big|_{F(t)=0}\ ;\ i, j = 1,\ 2$$
$$D_j^{0s} = D_j^s(t)\big|_{F(t)=0}\ ;\ D_j^{0a} = D_j^a(t)\big|_{F(t)=0}\ ;\ j = 1,\ 2$$
$$\tilde{Z} = \lambda\tilde{R}\ ;\ \tilde{Y} = \lambda\tilde{Q}\ ;\ \tilde{W} = \lambda\tilde{S}$$
$$H_{00} = I\ ;\ H_{01} = L^T X\ ;\ H_{11} = \lambda G_{11}^0 - E_a^T E_a$$
$$H_{12} = \lambda G_{12}^0 + E_a^T E_b\ ;\ H_{22} = \lambda G_{22}^0 - E_b^T E_b$$
$$W_d = \lambda S_d\ ;\ Z_d = \lambda R_d \qquad (13)$$
$$\Pi_j^s = [\Pi_{j1}^s\ \ \Pi_{j2}^s\ \ \cdots\ \ \Pi_{jN}^s]\ ;\ j = 0,\ 1,\ 2$$
$$\Pi_{0i}^s = -\frac{1}{2}L^T(Y_i + Y_{i-1})\ ;\ \Pi_{1i}^s = \lambda D_{1i}^s\ ;\ \Pi_{2i}^s = \lambda D_{2i}^s$$
$$\Pi_j^a = [\Pi_{j1}^a\ \ \Pi_{j2}^a\ \ \cdots\ \ \Pi_{jN}^a]\ ;\ j = 0,\ 1,\ 2$$
$$\Pi_{0i}^a = \frac{1}{2}L^T(Y_i - Y_{i-1})$$
$$\Pi_{1i}^a = \lambda D_{1i}^a\ ;\ \Pi_{2i}^a = \lambda D_{2i}^a\ .$$

Proof. See the full version of the paper (Han and Yu 2001).

Remark 1. Through some transformation, rewrite (14) in Han and Gu (2001) as

$$\begin{bmatrix} H_{00} & -H_{01} & 0 & \Pi_0^s & \Pi_0^a \\ -H_{01}^T & H_{11} & -H_{12} & \Pi_1^s & \Pi_1^a \\ 0 & -H_{12}^T & H_{22} & \Pi_2^s & \Pi_2^a \\ \Pi_0^{sT} & \Pi_1^{sT} & \Pi_2^{sT} & \frac{1}{h}W_d + Z_d & 0 \\ \Pi_0^{aT} & \Pi_1^{aT} & \Pi_2^{aT} & 0 & \frac{1}{h}W_d \end{bmatrix} > 0 \qquad (14)$$

where

$$\Pi_j^s = -\frac{1}{2}(\Gamma_j^1 + \Gamma_j^0),\ \Pi_j^a = -\frac{1}{2}(\Gamma_j^1 - \Gamma_j^0)\ \ ; j = 0,1,2$$

where Γ_j^k ($j = 0,1,2$; $k = 0,1$) are defined in Han and Gu (2001). By Corollary 1 in Han and Gu (2001), it can be concluded that the uncertain system described by (1) to (4) is robustly stable if there exist real $n\times n$ matrices $X = X^T$, Y_i, $W_i = W_i^T$ ($i = 0,1,2,\cdots,N$) and $Z_{ij} = Z_{ji}^T$ ($i, j = 0,1,2,\cdots,N$) such that (14),

$$Z_{00} > 0\ ,\ Z_d > 0 \qquad (15)$$

$$\begin{bmatrix} X & \tilde{Y} \\ \tilde{Y}^T & \frac{1}{h}\hat{W} + \tilde{Z} \end{bmatrix} > 0 \qquad (16)$$

where

$$\hat{W} = diag\left(2W_0\ W_1\ W_2\ \cdots\ W_{N-2}\ W_{N-1}\ 2W_N\right).$$

Theorem 2 is less restrictive than this result in view of the fact that (14) is equivalent to (11) if the coefficient 3 in the entry (5,5) is replaced by 1. In addition, the constraints $Z_{00} > 0$ and $Z_d > 0$ are no longer needed. Condition (10) is less restrictive than (16) due to the coefficient 2 in the first and last entries of $\hat{W}$. Therefore, Theorem 2 is much less conservative, and requires less computation.

4. EXAMPLES

To illustrate the improvement of the method over that in Han and Gu (2001), the following examples are presented.

Example 1. Consider the uncertain time-delay system

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]x(t - r) \qquad (17)$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\ B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

and $\Delta A(t)$ and $\Delta B(t)$ are unknown matrices satisfying $\|\Delta A(t)\| \le \alpha$ and $\|\Delta B(t)\| \le \alpha$, $\forall t$. The above system is of the form of Eq. (1) with $L = \alpha I$ and $E_a = E_b = I$.

For $\alpha = 0.2$, the maximum time-delay $r_{\max}$ for stability is estimated by using the method in Han and Gu (2001) for different *N*. The results are listed in the following table

N	1	2	5	10
Han & Gu (2001)	2.78	2.97	3.09	3.12

Using the method in this paper, the results are listed in the following table

N	1	2	3
This paper	3.098	3.132	3.133

It is easy to see that the results in this paper significantly improve the ones in Han and Gu (2001). The results here are much less conservative than those as surveyed in de Souza and Li (1999).

Now we consider the effect of the uncertainty bound α on the maximum time-delay for stability $r_{\max}$. The following table illustrates the numerical results for $N = 1$ and different α. We can see that as $\alpha \to 0$, the stability limit for delay approaches the uncertainty-free case in Gu (1999a) and Gu (2001a). As α increases, $r_{\max}$ decreases.

α	0.00	0.10	0.20
Han & Gu (2001)	5.30	3.77	2.78
This paper	6.05	4.26	3.09
α	0.30	0.40	0.50
Han & Gu (2001)	2.12	1.67	1.33
This paper	2.33	1.81	1.43
α	0.60	0.70	0.80
Han & Gu (2001)	1.08	0.87	0.69
This paper	1.14	0.91	0.70

It is clear that the method in this paper provides a larger estimate of $r_{\max}$ than the one in Han and Gu (2001).

Example 2. Consider the uncertain time-delay system

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]x(t-r) \quad (18)$$

where A and B are non-diagonal matrices and are given as follows

$$A = \begin{bmatrix} -3 & -2.5 \\ 1 & 0.5 \end{bmatrix}, \; B = \begin{bmatrix} 1.5 & 2.5 \\ -0.5 & -1.5 \end{bmatrix}$$

and $\Delta A(t)$ and $\Delta B(t)$ are unknown matrices satisfying $\|\Delta A(t)\| \le \alpha$ and $\|\Delta B(t)\| \le \alpha$, $\forall t$. This system is of the form of Eq. (1) with $L = \alpha I$ and $E_a = E_b = I$.

For the nominal system (18), i.e. $\alpha = 0.0$, the maximum time-delay for stability, $r_{\max}$, is 2.4184 according to Chen *et al.* (1994). Using the method in Han and Gu (2001), $r_{\max}$ is estimated in the following table for different *N*.

N	1	2	5	10
Han, Gu (2001)	2.2085	2.3278	2.3991	2.4133

Using the method in this paper, the results for different *N* are listed in the following

N	1	2	3
This paper	2.4021	2.4174	2.4183

It's clear that the convergence to the analytical solution is greatly accelerated.

For $\alpha = 0.2$, the maximum time-delay $r_{\max}$ for stability is estimated by using the method in Han and Gu (2001) for different *N*. The results are listed in the following table

N	1	2	5	10
Han, Gu (2001)	1.2206	1.2657	1.2884	1.2925

Using the method in this paper, the results are listed in the following table

N	1	2	3
This paper	1.2904	1.2937	1.2939

It is clear that the new method is much less conservative.

The effect of the uncertainty bound α on the maximum time-delay for stability $r_{\max}$ is also studied. The numerical results for $N = 1$ and different α are estimated in the following table. It again shows that as $\alpha \to 0$, the stability limit for delay approaches the uncertainty-free case in the above tables. As α increases, $r_{\max}$ decreases.

α	0.00	0.05	0.10
Han & Gu (2001)	2.2084	1.8758	1.4999
This paper	2.4021	2.0265	1.7276
α	0.15	0.20	0.25
Han & Gu (2001)	1.3963	1.2206	1.0743
This paper	1.4869	1.2904	1.1274
α	0.30	0.35	0.40
Han & Gu (2001)	0.9499	0.8357	0.7263
This paper	0.9897	0.8708	0.7647

It is clear that the results in this paper indeed signficantly improve the ones derived in Han and Gu (2001).

5. CONCLUSION

The problem of robust stability of linear time-delay systems with norm-bounded, and possibly time-varying, uncertainty has been addressed. A new stability criterion has been obtained. Numerical examples have shown significant improvements over some existing results.

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