15th Triennial World Congress, Barcelona, Spain

NUMERICALLY RELIABLE DESIGN FOR PROPORTIONAL AND DERIVATIVE STATE-FEEDBACK DECOUPLING CONTROLLER

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Abstract: In this paper we continue the work in Malabre and Velasquez (1994) and Bonilla Estrada and Malabre (2000) and study the row-by-row decoupling problem of linear time-invariant systems by proportional and derivative state feedback. Our contribution, with respect to previous results, is that we develop a numerical method to compute the desired feedback matrices. Our method is only based on orthogonal transformations and hence is numerically reliable. A numerical example is given to illustrate the proposed method. *Copyright@2002 IFAC*

Keywords: Linear Systems, Numerical Solutions, Reliability, Stability of Numerical Method.

1. INTRODUCTION

Consider the linear time-invariant system of the form

$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t), \quad (1)$$

where $x(t) \in \mathbf{R}^n, u(t) \in \mathbf{R}^m$ and $y(t) \in \mathbf{R}^p$ are state, input and output, respectively, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{p\times n}$. Without loss of generality, we always assume that

$$\text{rank}(B) = m,\ \text{rank}(C) = p. \quad (2)$$

For system (1), Tan and Vandewalle (1987) proposed in a proportional and derivative (PD) state feedback

$$u(t) = G\dot{x}(t) + Fx(t) + r(t) \quad (3)$$

such that the output of the closed-loop system exactly maches its reference, namely, $y(t) = r(t)$.

As Malabre and Velasquez (1994) pointed out, this PD feedback was found assuming, first, that system (1) is controllable and square invertible, carrying, next, the system into its m-block diagonal Brunovsky canonical form in Brunovsky (1970), and proposing, finally, the PD state feedback $u_i(t) = \dot{x}_i(t) + c_i x_i(t)$ (where the c_i are the rows of C). Following the same procedure of Tan and Vandewalle (1987), Malabre and Velasquez (1994) studied more general proportional and derivative state-feedback row by row decoupling problem as follows:

Problem 1: *Find feedback (if possible)*

$$u(t) = G\dot{x}(t) + Fx(t) + Mr(t) \tag{4}$$

such that the closed-loop system

$$\begin{aligned}(I - BG)\dot{x}(t) &= (A + BF)x(t) + BMr(t),\\ y(t) &= Cx(t)\end{aligned} \tag{5}$$

satisfies that

$$y(t) = r(t). \tag{6}$$

Hence, in Malabre and Velasquez (1994) the controllability assumption was relaxed and the square invertibility was replaced by a right-invertibility assumption. More recently, in Bonilla Estrada and Malabre (2000), a matrix-based procedure was proposed for designing a solution without requiring the use of canonical forms.

In this paper, we continue the work in Malabre and Velasquez (1994) and Bonilla Estrada and Malabre (2000). We will consider the following two aspects of Problem 1. Question 1 was partly studied in Bonilla Estrada and Malabre (2000), while question 2 has not been studied yet.

(1) In Tan and Vandewalle (1987) and Malabre and Velasquez (1994), a fundamental question arising when dealing with nonproper control laws was not considered, namely, the existence and uniqueness of state solutions for the closed-loop system (5). Since a PD control law of the form (4) is a nonproper transformation, we have to verify that the state solution of closed-loop system (5) exists and is unique, it is well known that this is equivalent to the regularity of the pencil $(I - BG, A + BF)$, i.e.,

$$\mathrm{rank}_g(s(I - BG) - (A + BF)) = n,$$

where and in the following, $\mathrm{rank}_g(D(s))$ denotes the generic rank of function $D(s)$. This question was solved in Bonilla Estrada and Malabre (2000) but only for the particular solutions proposed there.

(2) We need to develop a numerically reliable design method for the construction of the desired feedback (4).

Our method given in this paper is only based on orthogonal transformations and hence is numerically stable and reliable. Furthermore, our method can be implemented using existing tools such as Matlab and LAPACK.

2. MAIN RESULT

It is well-known that any matrix $\Theta \in \mathbf{R}^{m \times n}$ can be factorized as

$$U\Theta = \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \Pi, \tag{7}$$

where U and Π are orthogonal matrix and permutation matrix, respectively, R_1 is nonsingular and upper triangular. The factorization (7) is called the QR factorization of Θ with column pivoting. In (7), let

$$R = \begin{bmatrix} R_1 & R_2 \end{bmatrix} \Pi, \tag{8}$$

then R is of full row rank and

$$U\Theta = \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{9}$$

It is also well-known that any matrix pencil can be transformed into the so-called generalized upper triangular form under orthogonal transformations. This generalized upper triangular form is well analyzed in Van Dooren (1981).

Lemma 1. (Van Dooren (1981)) Given a matrix pencil $(\mathcal{E}, \mathcal{A})$, $\mathcal{E}, \mathcal{A} \in \mathbf{R}^{n \times l}$ there exist orthogonal matrices $\mathcal{P} \in \mathbf{R}^{n \times n}$, $\mathcal{Q} \in \mathbf{R}^{l \times l}$ such that $(\mathcal{P}\mathcal{E}\mathcal{Q}, \mathcal{P}\mathcal{A}\mathcal{Q})$ are in the following generalized upper triangular form:

$$\mathcal{P}(s\mathcal{E} - \mathcal{A})\mathcal{Q} = \begin{matrix} \\ n_1 \\ n_2 \end{matrix} \begin{matrix} l_1 & l_2 \\ \left[\begin{matrix} s\mathcal{E}_{11} - \mathcal{A}_{11} \\ 0 \end{matrix}\right. & \left.\begin{matrix} s\mathcal{E}_{12} - \mathcal{A}_{12} \\ s\mathcal{E}_{22} - \mathcal{A}_{22} \end{matrix}\right] \end{matrix}, \tag{10}$$

where

$$\mathrm{rank}(\mathcal{E}_{11}) = n_1,$$

$$\mathrm{rank}(s\mathcal{E}_{22} - \mathcal{A}_{22}) = l_2, \ \forall s \in \mathbf{C}.$$

As a direct consequence of the QR factorization and Lemma 1, we have the following theorem.

Theorem 2. Given system (1). There exist orthogonal matrices $U, V \in \mathbf{R}^{n \times n}$ and $W \in \mathbf{R}^{m \times m}$ such that

$$
U(sI-A)V = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & \tilde{n}_2 & p \end{array} \\ \left[\begin{array}{ccc} sI-A_{11} & -A_{12} & -A_{13} \\ 0 & sE_{22}-A_{22} & sE_{23}-A_{23} \\ -A_{31} & sE_{32}-A_{32} & sE_{33}-A_{33} \end{array}\right] \end{array},
$$

$$
UBW = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \end{array} \begin{array}{c} \begin{array}{cc} n_3 & m-n_3 \end{array} \\ \left[\begin{array}{cc} B_{11} & B_{12} \\ 0 & 0 \\ B_{31} & 0 \end{array}\right] \end{array}, \qquad (11)
$$

$$
CV = p \begin{array}{c} \begin{array}{ccc} n_1 & \tilde{n}_2 & p \end{array} \\ \left[\begin{array}{ccc} 0 & 0 & C_3 \end{array}\right] \end{array},
$$

where

$$
\operatorname{rank}(B_{31}) = n_3, \quad \operatorname{rank}(C_3) = p, \qquad (12)
$$

$$
\operatorname{rank}(sE_{22} - A_{22}) = \tilde{n}_2, \quad \forall s \in \mathbf{C},. \qquad (13)
$$

Proof. We prove Theorem 2 constructively via the following algorithm.

Algorithm 1

Input: Matrices $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}, C \in \mathbf{R}^{p\times n}$ with $\operatorname{rank}(B) = m$ and $\operatorname{rank}(C) = p$.

Output: Orthogonal matrices $U, V \in \mathbf{R}^{n\times n}, W \in \mathbf{R}^{m\times m}$ and the condensed form (11).

Step 1: Perform QR factorizations of B and C^T, respectively, to get orthogonal matrices U_1 and V_1 such that

$$
U_1 B =: \begin{array}{c} n-m \\ m \end{array} \left[\begin{array}{c} 0 \\ B_2^{(1)} \end{array}\right],
$$

$$
CV_1 =: \begin{array}{c} \begin{array}{cc} n-p & p \end{array} \\ \left[\begin{array}{cc} 0 & C_3 \end{array}\right] \end{array},
$$

where

$$
\operatorname{rank}(B_2^{(1)}) = m, \quad \operatorname{rank}(C_3) = p.
$$

Set

$$
U_1(sI-A)V_1 =:
$$

$$
\begin{array}{c} \\ n-m \\ m \end{array} \begin{array}{c} \begin{array}{cc} n-p & p \end{array} \\ \left[\begin{array}{cc} sE_{11}^{(1)} - A_{11}^{(1)} & sE_{12}^{(1)} - A_{12}^{(1)} \\ sE_{21}^{(1)} - A_{21}^{(1)} & sE_{22}^{(1)} - A_{22}^{(1)} \end{array}\right] \end{array}.
$$

Step 2: Compute the generalized upper triangular form of $sE_{11}^{(1)} - A_{11}^{(1)}$ to get orthogonal matrices U_2 and V_2 such that

$$
U_2(sE_{11}^{(1)} - A_{11}^{(1)})V_2 =:
$$

$$
\begin{array}{c} \\ \hat{n}_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & \tilde{n}_2 \end{array} \\ \left[\begin{array}{cc} sE_{11}^{(2)} - A_{11}^{(2)} & sE_{12}^{(2)} - A_{12}^{(2)} \\ 0 & sE_{22} - A_{22} \end{array}\right] \end{array},
$$

where

$$
\operatorname{rank}(E_{11}^{(2)}) = \hat{n}_1,
$$

$$
\operatorname{rank}(sE_{22} - A_{22}) = \tilde{n}_2, \quad \forall s \in \mathbf{C}.
$$

Set

$$
(sE_{21}^{(1)} - A_{21}^{(1)})V_2 =: \begin{array}{c} \begin{array}{cc} n_1 & \tilde{n}_2 \end{array} \\ \left[\begin{array}{cc} sE_{31}^{(2)} - A_{31}^{(2)} & sE_{32}^{(2)} - A_{32}^{(2)} \end{array}\right] \end{array},
$$

$$
U_2(sE_{12}^{(1)} - A_{12}^{(1)}) =: \begin{array}{c} \hat{n}_1 \\ n_2 \end{array} \left[\begin{array}{c} sE_{13}^{(2)} - A_{13}^{(2)} \\ sE_{23} - A_{23} \end{array}\right],
$$

$$
sE_{22}^{(1)} - A_{22}^{(1)} =: sE_{33}^{(2)} - A_{33}^{(2)}.
$$

Step 3. Since

$$
\left[\begin{array}{cc} U_2 & \\ & I \end{array}\right] U_1 V_1 \left[\begin{array}{cc} V_2 & \\ & I \end{array}\right] = \left[\begin{array}{ccc} E_{11}^{(2)} & E_{12}^{(2)} & E_{13}^{(2)} \\ & E_{22} & E_{23} \\ E_{31}^{(2)} & E_{32}^{(2)} & E_{33}^{(2)} \end{array}\right]
$$

which is orthogonal, we can compute an orthogonal matrix U_3 of the form

$$
U_3 = \left[\begin{array}{cc} (E_{11}^{(2)})^T & (E_{31}^{(2)})^T \\ U_{21}^{(3)} & U_{22}^{(3)} \end{array}\right].
$$

Then we have

$$
U_3 \left[\begin{array}{ccc} sE_{11}^{(2)} - A_{11}^{(2)} & sE_{12}^{(2)} - A_{12}^{(2)} & sE_{13}^{(2)} - A_{13}^{(2)} \\ sE_{31}^{(2)} - A_{31}^{(2)} & sE_{32}^{(2)} - A_{32}^{(2)} & sE_{33}^{(2)} - A_{33}^{(2)} \end{array}\right]
$$

$$
=: \begin{array}{c} \\ n_1 \\ n_3 \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & \tilde{n}_2 & p \end{array} \\ \left[\begin{array}{ccc} sI - A_{11} & -A_{12} & -A_{13} \\ -A_{31} & sE_{32} - A_{32} & sE_{33} - A_{33} \end{array}\right] \end{array},
$$

and

$$
U_3 \left[\begin{array}{c} 0 \\ B_2^{(1)} \end{array}\right] =: \begin{array}{c} n_1 \\ n_3 \end{array} \left[\begin{array}{c} B_1 \\ B_3 \end{array}\right].
$$

It is easy to see that

$$
\operatorname{rank} \left[\begin{array}{cc} I & B_1 \\ 0 & B_3 \end{array}\right] = \operatorname{rank} \left[\begin{array}{cc} E_{11}^{(2)} & 0 \\ E_{41}^{(2)} & B_2^{(1)} \end{array}\right] = \hat{n}_1 + m = n_1 + n_3,
$$

which gives that

$$
\operatorname{rank}(B_3) = n_3.
$$

Step 4. Compute the QR factorization of B_3^T to get orthogonal matrix W such that

$$
B_3 W =: n_3 \begin{array}{c} \begin{array}{cc} n_3 & m-n_3 \end{array} \\ \left[\begin{array}{cc} B_{31} & 0 \end{array}\right] \end{array}.
$$

Because B_3 is of full row rank, thus,

$$
\operatorname{rank}(B_{31}) = n_3.
$$

Step 5. Set

$$B_1W =: \begin{bmatrix} \overset{n_3}{B_{31}} & \overset{m-n_3}{B_{32}} \end{bmatrix},$$

$$U := \begin{bmatrix} (E_{11}^{(2)})^T & & (E_{31}^{(2)})^T \\ & I_{n_2} & \\ U_{21}^{(3)} & & U_{22}^{(3)} \end{bmatrix} \begin{bmatrix} U_2 & \\ & I \end{bmatrix} U_1,$$

$$V := V_1 \begin{bmatrix} V_2 & \\ & I \end{bmatrix}.$$

Then, $U(sI - A)V$, UBW and CV are in the condensed form (11). □

Since the generalized upper triangular form (10) and consequently the condensed form (11) are obtained by only several QR factorizations, hence, the flops required in Algorithm 1 is $O(n^3)$.

Furthermore, Algorithm 1 is implemented by only orthogonal transformations, so, it is numerically stable.

The solvability condition for Problem 1 can be read immediately from the condensed form (11).

Theorem 3. Given system (1) with $\text{rank}(B) = m$ and $\text{rank}(C) = p$. There exists a state feedback of the form (4) such that the pencil $(I - BG, A + BF)$ is regular and (6) holds if and only if

$$\tilde{n}_2 = n_2. \qquad (14)$$

Furthermore, if system (1) is a minimum phase system, then F can be chosen such that the closed-loop system (5) is stable, i.e., the pencil $(I - BG, A + BF)$ is stable.

Proof. Necessity: Since the pencil $(I - BG, A + BF)$ is regular and (6) holds, so,

$$C(s(I - BG) - (A + BF))^{-1}BM = I_p.$$

Consequently, we have

$$n + p \geq \text{rank}_g \begin{bmatrix} s(I - BG) - (A + BF) & B \\ C & 0 \end{bmatrix}$$
$$\geq \text{rank}_g \begin{bmatrix} s(I - BG) - (A + BF) & BM \\ C & 0 \end{bmatrix}$$
$$= n + \text{rank}_g(C(s(I - BG) - (A + BF))^{-1}BM)$$
$$= n + p,$$

which yields that

$$\text{rank}_g \begin{bmatrix} s(I - BG) - (A + BF) & B \\ C & 0 \end{bmatrix} = n + p. \qquad (15)$$

Note that

$$n = n_1 + n_2 + n_3 = n_1 + \tilde{n}_2 + p, \qquad (16)$$

B_{31} is nonsingular, C_3 is of full column rank, the property (13) holds true, and furthermore

$$\text{rank} \begin{bmatrix} s(I - BG) - (A + BF) & B \\ C & 0 \end{bmatrix}$$
$$= \text{rank} \begin{bmatrix} U(sI - A)V & UBW \\ CV & 0 \end{bmatrix}$$
$$= \text{rank} \begin{bmatrix} sI - A_{11} & -A_{12} & -A_{13} & B_{11} & B_{12} \\ 0 & sE_{22} - A_{22} & sE_{23} - A_{23} & 0 & 0 \\ -A_{31} & sE_{32} - A_{32} & sE_{33} - A_{33} & B_{31} & 0 \\ 0 & 0 & C_3 & 0 & 0 \end{bmatrix}$$
$$= \text{rank} \begin{bmatrix} sI - A_{11} + B_{11}B_{31}^{-1}A_{31} & -A_{12} - B_{11}B_{31}^{-1}(sE_{32} - A_{32}) & B_{12} \\ 0 & sE_{22} - A_{22} & 0 \end{bmatrix} + n_3 + p,$$
$$= \text{rank} \begin{bmatrix} sI - A_{11} + B_{11}B_{31}^{-1}A_{31} & B_{12} \end{bmatrix} + \tilde{n}_2 + n_3 + p, \ \forall s \in \mathbf{C},$$

thus, we have

$$\text{rank}_g \begin{bmatrix} s(I - BG) - (A + BF) & B \\ C & 0 \end{bmatrix}$$
$$= n_1 + \tilde{n}_2 + n_3 + p = (n_1 + n_3) + (\tilde{n}_2 + p). (17)$$

Hence, the condition (14) follows directly from (15), (16) and (17).

Sufficiency: Because the condition (14) holds, we have

$$n_2 = \tilde{n}_2, \ p = n_3.$$

Since B_{31} is nonsingular, so, there exist $G_{12} \in \mathbf{R}^{n_3 \times n_2}$, $G_{13} \in \mathbf{R}^{n_3 \times n_3}$, $F_{11} \in R^{n_3 \times n_1}$, $F_{12} \in \mathbf{R}^{n_3 \times n_2}$ and $F_{13} \in \mathbf{R}^{n_3 \times n_3}$ such that

$$B_{31} \begin{bmatrix} G_{12} & G_{13} \end{bmatrix} = \begin{bmatrix} E_{32} & E_{33} \end{bmatrix}, \qquad (18)$$

$$B_{31} \begin{bmatrix} F_{11} & F_{12} & F_{13} \end{bmatrix} = -\begin{bmatrix} A_{31} & A_{32} & A_{33} + B_{31}C_3 \end{bmatrix}. \qquad (19)$$

Let

$$G = W \begin{bmatrix} 0 & G_{12} & G_{13} \\ 0 & 0 & 0 \end{bmatrix} V^T,$$

$$F = W \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & 0 & 0 \end{bmatrix} V^T, \ M = W \begin{bmatrix} I_p \\ 0 \end{bmatrix}, \qquad (20)$$

where F_{21}, F_{12}, F_{13} are arbitrary. Then we have

$$U(s(I - BG) - (A + BF))V =$$

$$\begin{bmatrix} sI - A_{11} + B_{11}B_{31}^{-1}A_{31} - B_{12}F_{21} & -sB_{11}B_{31}^{-1}E_{32} - A_{12} + B_{11}B_{31}^{-1}A_{32} & -sB_{11}B_{31}^{-1}E_{33} - A_{13} + B_{11}B_{31}^{-1}A_{33} \\ 0 & sE_{22} - A_{22} & sE_{23} - A_{23} \\ 0 & 0 & B_{31}C_3 \end{bmatrix}.$$

Note that the property (13) holds and $B_{31}C_3$ is nonsingular, we know that the pencil $(I-BG, A+BF)$ is regular. Moreover, $C(s(I-BG)-(A+BF))^{-1}BM = I_p$, equivalently, in the closed-loop system (5), $y(t) = r(t)$.

Furthermore, if system (1) is of minimum phase, then for any $s \in \mathbf{C}/\mathbf{C}^-$ we have

$$\text{rank} \begin{bmatrix} sI - A_{11} & B_{11} & B_{12} \\ -A_{31} & B_{31} & 0 \end{bmatrix} = n_1 + n_3,$$

where $\mathbf{C}^-$ denotes the open left-half complex plane. Equivalently, for any $s \in \mathbf{C}/\mathbf{C}^-$,

$$\text{rank} \begin{bmatrix} sI - A_{11} + B_{11}B_{31}^{-1}A_{31} & B_{12} \end{bmatrix} = n_1.$$

Hence, there exists a F_{21} such that

$$A_{11} + B_{12}F_{21} - B_{11}B_{31}^{-1}A_{31} \text{ is stable.} \quad (21)$$

Consequently, the pencil $(I - BG, A + BF)$ is stable. □

Obviously, Theorem 3 leads to the following algorithm.

Algorithm 2

Input: Matrices $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}, C \in \mathbf{R}^{p\times n}$ with $\text{rank}(B) = m$ and $\text{rank}(C) = p$.

Output: Matrices F, G and M (if possible) such that $(I - BG, A + BF)$ is regular and (6) is true. Moreover, if system (1) is a minimum phase system, then the pencil $(I-BG, A+BF)$ is stable.

Step 1. Perform Algorithm 1 to compute the condensed form (11).

Step 2. Check the condition (14). If $\tilde{n}_2 \neq n_2$, print "Problem 1 is unsolvable". Otherwise, continue.

Step 3. Compute the SVD of B_{31} to get orthogonal matrices P and Q such that

$$B_{31} = P\Sigma Q, \ \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_p \end{bmatrix}, \ \sigma_1 \geq \cdots \geq \sigma_p,$$

and consequently compute

$$\begin{bmatrix} G_{12} & G_{13} \end{bmatrix} = Q^T \begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_p \end{bmatrix} P^T \begin{bmatrix} E_{32} & E_{33} \end{bmatrix},$$

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \end{bmatrix} = -Q^T \begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_p \end{bmatrix} P^T \begin{bmatrix} A_{31} & A_{32} & A_{33} - B_{31}C_3 \end{bmatrix}.$$

Then compute matrices F, G and M by (20), in which, if system (1) is of minimum phase, then F_{21} is chosen such that (21) is satisfied, otherwise, take $F_{21} = 0$. Output F, G and M. □

Algorithm 2 above is implemented using only orthogonal transformations such as QR factorizations and SVD and hence is numerically reliable.

In the following we give an example to show how Algorithm 2 works. All computations are carried out by Matlab 5.0 with IEEE standard (machine precision is about 10^{-16}). and the rank of any matrix D involved is determined by Matlab command $rank(D, \epsilon)$ with $\epsilon = 10^{-10}$.

Example 1. Let

$$A = \begin{bmatrix} 1.206344159438 & 1.321960096349 & 0.365862989013 & -1.173607249816 & -0.367241830856 & -0.547729656741 \\ 0.506482306553 & 0.366181839670 & -0.070193999292 & 0.472483295591 & -0.160109780244 & 0.186924676262 \\ 0.1365278627717 & 0.024271298021 & -0.055466913467 & 0.108953190549 & 0.450060455414 & 0.159973220468 \\ 0.420151039289 & 0.094843585451 & -0.183531151211 & 0.466255223560 & 0.011420903083 & -0.185891774863 \\ -0.190990609830 & 0.215875457226 & -0.172245698140 & 0.094465864922 & -0.019804218221 & -0.104017150961 \\ 0.191733656518 & 0.338611911504 & -0.393623303377 & -0.208692060043 & -0.102823318310 & -0.257403185168 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.63590385006071 & -0.46797205902043 & 0.75859364642821 \\ 0.20198206633489 & -0.34844892749690 & 0.10173082031191 \\ 0.01108929191805 & -0.68633025953431 & -0.38880214265178 \\ 0.18694765209622 & -0.10150567734668 & 0.20051818208883 \\ 0.36564819906248 & -0.19590733500115 & -0.09189634842298 \\ -0.27363210336882 & -0.17709023065489 & -0.39867801744857 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.1592698467275 & -0.2380038254552 & 0.1745758951970 & 0.4594147934043 & -0.0262624756805 & 0.4321893414889 \\ -0.2052107572932 & 0.0366962032256 & 0.0342299110418 & 0.6852435391356 & -0.0520288124304 & 0.1012192706910 \end{bmatrix}.$$

First we perform Algorithm 1 to compute the form (11) and we get $n_1 = \tilde{n}_2 = n_2 = 2$ and [1]

$$UBW = \left[\begin{array}{cc|c} 0.0498 & 0.6408 & 0.8439 \\ 0.0784 & 0.1909 & 0.1739 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0.1708 & 0.4398 & 0 \\ 0.9943 & 0.3400 & 0 \end{array}\right],$$

$$CV = \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0.5548 & 0.4508 \\ 0 & 0 & 0 & 0 & 0.1210 & 0.7159 \end{array}\right],$$

$$U(sI-A)V = \left[\begin{array}{ll|ll|ll} s-0.6288 & -0.5751 & -0.3840 & -0.0158 & -0.6315 & -0.3533 \\ -0.1338 & s-0.4514 & -0.6831 & -0.0164 & -0.7176 & -0.1536 \\ \hline 0 & 0 & -0.0928 & -0.6108s-0.1901 & -0.7918s-0.6927 & -0.6756 \\ 0 & 0 & 0 & -0.5869 & -0.0841 & s-0.6992 \\ \hline -0.6299 & -0.3127 & s-0.6124 & -0.0576 & -0.4544 & -0.7275 \\ -0.3705 & -0.0129 & -0.6085 & -0.7918s-0.3676 & 0.6108s-0.4418 & -0.4784 \end{array}\right]$$

$$U = \begin{bmatrix} -0.497093157263 & -0.227569427573 & -0.677662067520 & -0.152184717061 & -0.457033593785 & 0.099220849248 \\ -0.221027101709 & -0.613525610441 & 0.398513058987 & 0.023686554143 & -0.186797105349 & -0.616819646288 \\ -0.316193968413 & 0.733249596854 & 0.057603433355 & -0.208648907910 & -0.154379269104 & -0.540074922596 \\ -0.276034729894 & 0.143504403010 & -0.006244804156 & 0.947325383701 & -0.074973189735 & 0.011222960731 \\ -0.514781969544 & -0.101289851310 & -0.144463827083 & -0.068874029552 & 0.8337416009042 & -0.063256647822 \\ -0.512709605309 & 0.057361677266 & 0.598109115139 & -0.174858164590 & -0.177899242827 & 0.560250360705 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.497093157263 & -0.221027101709 & -0.514781969544 & 0.599087194009 & -0.062803654443 & -0.276034729894 \\ -0.227569427573 & -0.613525610441 & -0.101289851310 & -0.493285365406 & -0.545540907341 & 0.143504403010 \\ -0.677662067520 & 0.398513058987 & -0.144463827083 & -0.508758892489 & 0.319714026061 & -0.006244804156 \\ -0.152184717061 & 0.023686554143 & -0.068874029552 & 0.265892639640 & 0.058402471531 & 0.947325383701 \\ -0.457033593785 & -0.186797105349 & 0.833741600904 & 0.235152755572 & 0.013575009184 & -0.074973189735 \\ 0.099220849248 & -0.616819646288 & -0.063256647822 & -0.113722281442 & 0.769823766449 & 0.011222960731 \end{bmatrix},$$

$$W = \begin{bmatrix} -0.61216551899505 & 0.14650196881176 & -0.77703960676905 \\ -0.48149542191810 & 0.71043067736433 & 0.51327420676630 \\ -0.62722845600740 & -0.68834978448020 & 0.36436113703389 \end{bmatrix}.$$

Because $n_2 = \tilde{n}_2$, and the system (1) is of minimum phase, so we perform Step 2 of Algorithm 2 to compute F, G and M and we obtain

$$F = \begin{bmatrix} 0.418317822628 & 2.044609623520 & -1.968931972122 & -0.204101716411 & 0.323010855939 & 2.313683418246 \\ -0.256217923715 & -1.343467383497 & 1.304658255032 & 0.122539811124 & -0.211490657721 & -1.526510278017 \\ -0.209499754476 & -0.969230984727 & 0.919821934914 & 0.112724474103 & -0.147919148344 & -1.076564856972 \end{bmatrix},$$

$$G = \begin{bmatrix} -0.082497412241 & -0.139022779013 & -0.598904807338 & 0.071416743549 & 0.915998387458 & -0.493754406301 \\ -0.730836474798 & -0.282802902832 & -0.856861959981 & -0.004440326677 & 2.069342598530 & -0.637271486316 \\ 0.853592009408 & 0.101902278569 & -0.070171076357 & 0.158559016330 & -0.961609563658 & -0.155262752608 \end{bmatrix},$$

$$M = \begin{bmatrix} -0.61216551899505 & 0.14650196881176 \\ -0.48149542191810 & 0.71043067736433 \\ -0.62722845600740 & -0.68834978448020 \end{bmatrix}.$$

It has been verified that for matrices F, G and M above, the pencil $(I - BG, A + BF)$ is regular and $C(s(I - BG) - (A + BF))^{-1}BM = I$. Moreover, the pencil $(I - BG, A + BF)$ is stable (its finite eigenvalues are $-1.6743 + 0.7785i$ and $-1.6743 - 0.7785i$.

[1] Because of the space limitation, the elements in the matrices $U(sI-A)V$, UBW and CV are described with only 4 decimal digits.

3. CONCLUSIONS

In this paper we have studied the row-by-row decoupling problem of linear time-invariant systems by proportional and derivative state feedback and developed a numerical method to compute the desired feedback matrices. Our method is only based on orthogonal transformations and hence is numerically reliable. A numerical example has been given to illustrate the proposed method.

4. REFERENCES

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