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A NEW IDENTIFICATION APPROACH FOR ARX MODELS [1]

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Abstract: Recently, much research has been conducted in the field of identification of the linear models. In general, these methods use a time-domain estimate or a frequency-domain estimate. In this paper. the time-domain estimate and the frequency-domain estimate were combined to identify the autoregressive with exogenous noise (ARX) models. The concept of general prediction error (GPE) criterion is introduced for the time-domain estimate. Optimal frequency estimation is introduced for the frequency-domain estimate. A new identification method, called the empirical frequency-domain optimal parameter (EFOP) estimate, is proposed for the ARX model with noise interference. The algorithm theoretically provides the globally optimum frequency-domain estimate of the model. Some simulations are included to illustrate the new identification method.

Keywords: Autoregressive models, Disturbance parameters, Frequency estimation, Time-domain method, Prediction methods

1. INTRODUCTION

One purpose of system identification is to construct an efficient identifying method for a certain dynamic system. Basically, a model must be constructed from observed data. However, because of noise interference or unmodeled dynamics, the system may not belong to a model class. Therefore, the observed data may be corrupted. A general identification then involves obtaining a model from *a priori* chosen model classes using the corrupted data. Clearly, using the observed data to identify a system results in identification errors.

An identification method, efficient or not, is primarily dependent on the choice of criteria. The identification criteria selection is one of the important decisions to be made during system identification. Therefore, the performance of estimation algorithms when the underlying true system cannot be exactly captured by the chosen model structure has been a topic of recent interest in the field of system identification. The prediction error (PE) criterion (Ljung 1999, Akcay *et al.* 1996, and Weyer 2000) that was derived from the time-domain system is a well-known criterion in system identification. The celebrated Least-Squares (LS) method is the most common choice among the PE criterion. Due to the sensibility and robustness of identification, several different norms of $\mathcal{L}$ for the PE criterion were discussed by Ljung (1999) and Akcay *et al.* (1996).

However, since engineering systems are most often characterized in the frequency-domain, the properties of a closed-loop system can be accurately and intuitively determined by studying the frequency response function. In many cases, it is

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very attractive to describe the signal characteristics in the frequency domain. Estimating the frequency response of a system from observed input-output data is a common requirement. The use of PE criterion may also result in large estimation errors. In particular, the PE criterion fails to provide a better estimation of the system transfer function. For this reason, many recent studies have investigated system identification in the frequency-domain (see McKelvey *et al.* 1996, Gustafsson *et al.* 1997, Ninness 1997, Ninness and Gomez 1998). The design of allpass filters using phase response criteria was considered by Lang and Laakso (1994) and Kumaresan and Rao (1999). However, these approaches are not easily implemented. A new kind of criterion, proposed by Lo and Kwon (2000), is also considered in this paper. A series of identification methods can be derived using this criterion.

System identification must be carried out for a specific purpose. Lo and Kwon (2000) have related the identification of the finite impulse response (FIR) model to the general prediction error (GPE) criterion. The autorregressive exogenous noise interference model is an important dynamic model that serves a certain purpose. The ARMA model involves output feedback signals, but the system signals experience great interference. Some previous identification methods, such as the common LS method or the Extended Least-Squares (ELS) method, could not give the exact estimation if the AR model was disturbed by noise. Lately, there has been a considerable amount of literature written about identification of the ARMA model (for example, Kumaresan and Rao 1999, Broersen 2000, Weyer 2000, Lambert-Lacroix 2000, Makila and Jarvinen 2000, Forssell and Ljung 2000). Many of these studies are focused on how to accurately identify this kind of dynamic model.

In this paper the identification of the ARX model is also considered. First, there is a review of some vital background knowledge that is used in the following sections. This includes the concepts of the GPE criterion and the empirical optimal frequency-domain estimate. The identification method is constructed with the GPE criterion and the empirical optimal frequency-domain estimate. The discrete Fourier transform (DFT) is a useful tool in analyzing the properties of the transfer function and is also used in this paper. Then, the paper discusses the identification of the AR model with several disturbances. From the optimal frequency-domain estimate, several GPE criteria are deduced from the different ARX models. The time-domain estimate and the frequency-domain estimate are then combined to form the empirical frequency-domain optimal parameter (EFOP) estimate for the ARX models. This method provides the globally optimal frequency-domain estimate and minimizes the GPE criterion. It has obvious advantages in anti-disturbance performance and can precisely identify an ARX model with fewer sample numbers. Lastly, several simulation examples are included to illustrate the method's reliability.

2. PRELIMINARY REVIEWS

Most identification techniques are divided into two classes: the frequency-domain method and the time-domain method. In the latter class, the main identification methods used in engineering are derived from the prediction error (PE) criterion (Ljung 1999, Akcay *et al.* 1996, and Weyer 2000). Much of the identification research was also based on the PE criteria (see for example Ninness 1996; Weyer 2000; Forssell and Ljung 2000; etc.). The GPE criterion is proposed in this paper. The identification technique in this paper is based on this criterion.

Let $\{y(t)\}_1^N$ be an output sequence of a discrete system. $y(t|\theta)$ is the prediction of the output $y(t)$. N and θ are the sample number and the system parameter. $\varepsilon(t,\theta) = y(t) - y(t|\theta)$ is the prediction error of the system at time t. The prediction error vector of the system is defined as

$$\beta(N,\theta) = (\varepsilon(1,\theta), \varepsilon(2,\theta), \cdots, \varepsilon(N,\theta))^T$$

Definition 1. Suppose that $\beta(N,\theta)$ is a prediction error vector of a discrete system. Then a function f is a GPE criteria of the system if the function $f(\beta(N,\theta); N,\theta)$ is a positive definite function of the vector $\beta(N,\theta)$ with respect to the parameters N and θ.

The primary difference between the GPE criterion and the PE criterion was explained by Lo and Kwon (2000). Their definition demonstrates that the PE criterion is contained in the GPE criterion, but they are not equal. For example, the function

$$f(\beta(N,\theta); N,\theta) = \beta^T(N,\theta)Q(N,\theta)\beta(N,\theta) \quad (1)$$

is a GPE criterion of a linear predictive system, which is called a general quadratic criterion, where the matrix $Q(N,\theta) \in \mathbf{R}^{N\times N}$ is a positive definite symmetric matrix.

Suppose that a function $f(\beta(N,\theta); N,\theta)$ is a GPE criterion of a discrete system. The estimation of the system parameter θ is then taken to be the value that minimizes the GPE criterion $f(\beta(N,\theta); N,\theta)$:

$$\widehat{\theta}_N = \arg\min_\theta f(\beta(N,\theta); N,\theta) \quad (2)$$

Estimation (2) is called the EFOP estimate of the parameter θ if the corresponding GPE criterion is constructed with the empirical optimal frequency-domain estimation.

In the GPE criterion, the challenge is to select which positive definite function f makes the system identification more accurate. The GPE criterion in Eq.(1) clearly indicates that problem is required to solve. In fact, the prediction error vector is determined by the given system. The matrix $Q(N,\theta)$ is the only object that can be changed. The other matrix $Q(N,\theta)$ will result from a different estimate. How is a positive definite symmetric matrix $Q(N,\theta)$ chosen in order to obtain a more accurate estimate for the given system in general quadratic criteria? This paper presents a method to construct such a matrix $Q(N,\theta)$ as well as an efficient GPE criteria. This idea is derived from optimizing the transfer function estimate.

3. EFOP ESTIMATE

Consider the ARX model (Jankunas 2000)

$$A(q)y(t) = u(t-1) + v(t) \tag{3}$$

Here $y(t)$, $u(t)$, and $v(t)$ are the output, the input, and the disturbance noise. $A(q)$ is a stable output filter with form:

$$A(q) = 1 + a_1 q^{-1} + a_2 q^{-2} + \cdots + a_n q^{-n}$$

$a_1, a_2, \cdots, a_n$ are unknown system parameters. q^{-1} is the unit delay operator. Note that

$$\theta = (a_1, a_2, \cdots, a_n)^T$$

$$\varphi(t) = (-y(t-1), -y(t-2), \cdots, -y(t-n))^T$$

The noise $v(t)$ may represent observation errors, modeling errors, external disturbances acting on the system, etc. In general, $v(t)$ is expressed by Ljung (1999)

$$v(t) = H_0(q)w(t)$$

where $H_0(q)$ is a monic inversely stable filter and $w(t)$ is white noise independent of the input signal $u(t)$. Let $\{u(t)\}_1^N$ be a finite deterministic input sequence and $\{y(t)\}_1^N$ be an output response sequence of model (3) based on the input signal $\{u(t)\}_1^N$. $G(\omega^k)$ is the transfer function of model (3). $\widehat{G}_N(\omega^k)$ is the empirical transfer function estimate of the transfer function $G(\omega^k)$ (Ljung 1999). Note that

$$D_N(k,\theta) = \widehat{G}_N(\omega^k) - G(\omega^k), \quad k = 1, 2, \cdots, N.$$

This paper will now discuss the transfer function estimate during two normal trials of the filter $H_0(q)$. Although some hypotheses are imposed upon the filter $H_0(q)$, we still emphasize that those hypotheses are only necessary to deduce the algorithms. In fact, the simulations illustrate that these identification algorithms are more efficient with complicated color noise.

Case 1: Suppose that filter $H_0(q) = 1$. $\{u(t)\}_1^N$ is a finite input sequence of model (3). Let $r(t,\theta) = A(q)u(t), t = 1, 2, \cdots, N$ and

$$r_e(t,\theta) = \begin{cases} r(t,\theta), & if \quad 1 \le t \le N \\ r(t+N,\theta), & if \quad 1-N \le t \le 0 \end{cases}$$

$$\psi_1(t,\theta) = (r_e(t-1,\theta), r_e(t-2,\theta), \cdots, r_e(t-N,\theta))^T$$

$$Q_1(N,\theta) = \left(\sum_{t=1}^{N} r^2(t,\theta)\right)^{-2} \frac{1}{N} \sum_{t=1}^{N} \psi_1(t,\theta)\psi_1^T(t,\theta)$$

Theorem 1. Suppose that the noise $\{w(t)\}_1^N$ of model (3) is white noise with variance λ. Then

$$F_1(D_N(k,\theta)) = \beta^T(N,\theta)Q_1(N,\theta)\beta(N,\theta) \tag{4}$$

is the optimal frequency-domain estimate (Luo and Kwon 2000) of model (3). □

Theorem 1 shows that the estimation error of the transfer function $D_N(k,\theta)$ can attain the empirical optimum using performance index (4) This form is the same as display (1). Therefore, function $F_1(D_N(k,\theta))$ is a GPE criterion of the globally optimal frequency-domain estimate of model (3). From (2) and the above relation the EFOP estimate of ARX model (3) under the case $H_0(q) = 1$ is of the form

$$\widehat{\theta}_{EFOP1} = \arg\min_{\theta} \beta(N,\theta)^T Q_1(N,\theta)\beta(N,\theta) \tag{5}$$

Case 2: Suppose that the filter $H_0(q) = A(q)$. Then model (3) becomes an output error (OE) model (Ljung 1999).

Theorem 2. Suppose that the noise $\{w(t)\}_1^N$ of model (3) is white noise with variance λ. Then the optimal frequency-domain estimate $F_2(D_N(k,\theta))$ of model (3) is:

$$F_2(D_N(k,\theta)) = \beta^T(N,\theta)Q_2(N,\theta)\beta(N,\theta) \tag{6}$$

where $\beta(N,\theta)$ is the prediction error vector of model (3). $Q_2(N,\theta)$ is defined as:

$$Q_2(N,\theta) = \left(\sum_{t=1}^{N} u^2(t)\right)^{-2} \frac{1}{N} \sum_{t=1}^{N} \psi_2(t,\theta)\psi_2^T(t,\theta)$$

and

$$\psi_2(t,\theta) = (u_e(t-1,\theta), u_e(t-2,\theta), \cdots, u_e(t-N,\theta))^T$$

$$u_e(t,\theta) = \begin{cases} u(t), & if \quad 1 \le t \le N \\ u(t+N), & if \quad 1-N \le t \le 0 \end{cases}$$

□

Theorem 2 proves that the function (6) is the empirical optimal frequency-domain estimate of model (3) in the case 2. We can attain the EFOP estimate of model (3) by minimizing GPE criterion (6). But it may be complicated in the calculation. If the relative noises are smaller than the output signals, the EFOP estimate can be simplified. Note that

$$Q_3(N) = (\sum_{t=1}^{N} u^2(t))^{-2}\frac{1}{N}\sum_{t=1}^{N}\psi_3(t,\theta)\psi_3^T(t,\theta)$$

$$\psi_3(t) = (y_e(t-1), y_e(t-2), \cdots, y_e(t-N))^T$$

and

$$y_e(t) = \begin{cases} y(t), & if \quad 1 \le t \le N \\ y(t+N), & if \quad 1-N \le t \le 0 \end{cases}$$

The parameter estimation depends on the choice of matrix $Q(N)$. The different matrix $Q(N)$ will correspond to the different estimation. A concrete form of $Q(N)$ by some Toeplitz matrices is given here. For the AR model, the rectangular-Block Toeplitz form Is also considered by Buzenac-Settineri and Najim (2000), Lambert-Lacroix (2000), and Byrnes *et al.* (2000, 2001), but they are completely unrelated to this discussion.

Remark 1: Although the form of expression (6), a GPE criterion for system (3), appears similar to the Minimizing Variance Estimation (MVE), there is also an essential difference between them. In the sense of giving the smallest covariance matrix in MVE, the best choice of the matrix $Q(N)$ should be the noise covariance matrix. "Notice that it requires knowledge of the noise covariance matrix, which might not be a realistic assumption" (Ljung 1999). Therefore, the matrix $Q(N)$ in the MVE could not be obtained in practice. Even if some knowledge about the system noise is known, the form of MVE is still different from expression (6). For example, if the noise is a white noise with variance σ, the matrix $Q(N)$ in MVE should be $\sigma^{-1}I$. Then MVE just becomes the LS Estimation. However, the matrix $Q(N)$ in this paper is a Toeplitz matrix, which is generated by a system output signal. The simulations will also compare the efficiency of these two methods. From (6) it is not difficult to conclude the next theorem.

Theorem 3. Suppose that the disturbance $\{w(t)\}_1^N$ of model (3) is white noise with variance λ and the relative noises are smaller than the output signals. The EFOP parameter estimation $\widehat{\theta}_{EFO2}$ is given as

$$\begin{cases} \widehat{\theta}_{EFOP2} = (\Phi^T(N)P(N)\Phi(N))^{-1} \\ \qquad\qquad \Phi^T(N)P(N)Y(N) \\ P(N) = \sum_{t=1}^{N}\psi_3(t)\psi_3^T(t) \end{cases}$$

where

$$\Phi(N) = (\varphi(1), \varphi(2), \cdots, \varphi(N))^T$$
$$Y(N) = (y(1) - u(N), \cdots, y(N) - u(N-1))^T$$

□

Remark 2: Theorem 3 and relationship (5) demonstrate that the reason why the EFOP estimate method has a good anti-disturbance performance is because the output signals or input signals are stressed in the EFOP estimation methods. In particular, these algorithms have an advantage for identifying the systems affected by color noise interference, although these methods are deduced under the hypothesis that noise $w(t)$ is a white noise. The simulations illustrate the fact is valid when noise $w(t)$ is also a color noise. Moreover, we can demonstrate that Theorem 3 could adapt to case 1 of system (3). In many cases the algorithm described by Theorem 3 is still available to identify the ARX model if the weighting signals $y_e(t)$ in Theorem 3 are replaced by the signals $u_e(t)$.

4. SIMULATIONS

The following simulations are carried out for the ARX models with the disturbance noise sequence $\{v(t)\}_1^N$. To illustrate the behavior of the EFOP estimates, several simulation trials are conducted for comparison with the previous algorithms. For a real system, the output $\{y(t)\}_1^N$ is generated by the system with a given input sequence $\{u(t)\}_1^N$. Let

$$\eta = (\sum_{t=1}^{N} v^2(t) / \sum_{k=1}^{N} y^2(k))^{\frac{1}{2}}$$

be the noise-to-signal ratio, which expresses the disturbed extent of the model signal. Note that θ is the real model parameter, while $\widehat{\theta}_{EFO}$, $\widehat{\theta}_{LS}$ and $\widehat{\theta}_{ELS}$ are the EFOP estimate, the LS estimate, and the extended least-squares (ELS) estimate, respectively.

Example 1. The output error model is given by

$$\begin{aligned} y(t) + a_1 y(t-1) + a_2 y(t-2) &= u(t-1) \\ +w(t) + a_1 w(t-1) + a_2 w(t-2) & \quad (7) \end{aligned}$$

The real system parameters are $a_1 = -1.7, a_2 = 0.72$ and the experimental sample number N is 2000. The input signal $\{u(t)\}_1^N$ is generated by a sine generator and $\{w(t)\}_1^N$ is an approximately white noise with variance $\lambda = 5$. The output of the system is then generated by (7) with a noise-to-signal ratio $\eta = 0.351$. The parameter is estimated by the EFOP2 method, the LS method, and for the recursive pseudolinear regression of system (7) the ELS estimate. $\overline{\theta}_{EFOP2}$ denotes the average estimate from the 200th EFOP estimate value to the 2000th EFOP2 estimate. $\overline{\theta}_{LS}$ and $\overline{\theta}_{ELS}$ denote the average estimate from the 200th LS estimate value to the 2000th LS estimate and the average estimate from the 200th ELS estimate value to the 2000th ELS estimate, respectively. The calculated values are given by the following forms:

$$\overline{\theta}_{EFOP2} = \begin{pmatrix} -1.6879 \pm 0.0125 \\ 0.7092 \pm 0.0123 \end{pmatrix}$$

$$\overline{\theta}_{LS} = \begin{pmatrix} -0.5717 \pm 0.0170 \\ -0.3446 \pm 0.0196 \end{pmatrix}$$

$$\overline{\theta}_{ELS} = \begin{pmatrix} -1.3905 \pm 0.2652 \\ 0.4062 \pm 0.2732 \end{pmatrix}$$

where the calculational error is defined by the standard deviation. The calculational results show that the EFOP method identifies real system (7) more efficiently than the LS method or the ELS method. Furthermore, with the sample number N increasing, the values of the EFOP estimate are closer to the real parameters. These are also validated by simulation figure 1.

From simulation figure 1 the EFOP method can be intuitively compared with the LS method and ELS method. Although in this example the disturbance is an approximately white noise the EFOP estimate effectively identified the ARX color noise interference model. The next identification is an example of case 1 of system (3) with a color noise.

Example 2. The system is

$$y(t) - 0.8y(t) = u(t-1) + v(t) \qquad (8)$$

The input signal is generated by a pulse generator. The experimental sample number N is also N = 2000. The real system parameter is $a_1 = -0.8$. The interfered noise $\{v(t)\}_1^N$ is assumed by the form of an color noise sequence

$$v(t) = \frac{1 - 1.3q^{-1} + 0.5q^{-2}}{1 + 0.8q^{-1}} w(t)$$

where $\{w(t)\}_1^N$ is an approximate white noise. Then the output of system (7) is given with a noise-to-signal ratio $\eta = 0.387$. The parameter is estimated by the EFOP1 method, EFOP2 method, and the LS method for this disturbance. $\overline{\theta}_{EFOPi}$ (j=1,2) denotes the average estimate from the 200th EFOPi estimate value to the 2000th EFOPi estimate. $\overline{\theta}_{LS}$ denotes the average estimate from the 200th LS estimate value to the 2000th LS estimate. The simulation results can be shown by figure 2.

Although the estimation $\widehat{\theta}_{LS}$ result tends to be steady, with increasing sample number N, the LS method produces a quite significant error between θ and $\widehat{\theta}_{LS}$. The estimation $\widehat{\theta}_{EFOP1}$ oscillates more, but near the true parameter θ. Furthermore, the $\widehat{\theta}_{EFOP2}$ yields a better estimation of the parameter. This result is also illustrated by the following calculated values.

$$\overline{\theta}_{LS} = -0.5378 \pm 0.0113$$
$$\overline{\theta}_{EFOP1} = -0.7809 \pm 0.0143$$
$$\overline{\theta}_{EFOP2} = -0.7952 \pm 0.0042$$

From simulation examples above we can see that EFOP estimates identify the ARX model well and do not need many sample numbers. Moreover, for systems whose sample numbers are difficult to attain, they have an obvious advantage.

5. CONCLUSIONS

In this paper considers the identification of ARX models with noise interference. The GPE criterion and the empirical optimal frequency-domain estimate are introduced and some frequency properties are analyzed. Through minimizing the error of the empirical transfer function estimation, several GPE criteria are constructed for the corresponding ARX models. The empirical frequency-domain optimal parameter estimates for the ARX models are obtained by minimizing the GPE criterion. The advantage of the EFOP estimate is that it has a good anti-disturbance performance and the interfered model can be precisely identified with fewer sample numbers. An easy method to find the approximate estimation of the EFOP is deduced. Although our method is derived from special noise, it more effectively identifies the ARMA model disturbed by color noise. Lastly, several simulation examples are included to illustrate the method's reliability.

REFERENCES

Akcay, H., Hjalmarsson, H., & Ljung, L. (1996). On the choice of norms in system identification. *IEEE Transactions on Automatic Control*, **41**, 1367-1372.

Broersen, P. M. T. (2000). Autoregressive model orders for Durbin's MA and ARMA estimators. *IEEE Transactions on Signal Processing*, **48**(8), 2454-2457.

Buzenac-Serrineri, V. & Najim, M. (2000). OL-RIV: A new fast adaptive algorithm for Rectangular Block Toeplitz systems. *IEEE Transactions on Signal Processing*, **48**(9), 2519-2534.

DeFatta, D. J., Lucas, J. G., & Hodgkiss, W. S. (1988). *Digital Signal Processing: A System Design Approach*. John Wiley & Sons.

Gustafsson, F., Gunnarsson, S.,& Ljung, L. (1997). Shaping frequency-dependent time resolution when estimating spectral properties with parametric methods. *IEEE Transactions on Signal Processing*, **45**(4), 1025-1035.

Jankunas, A. (2000). Optimal adaptive control for estimation of parameters of ARX models. *IEEE Transactions on Automatic Control*, **45**(5), 964-968.

Kumaresan, R. & Rao, A. (1999). On designing stable allpasss filters using AR modeling. *IEEE Transactions on Signal Processing*, **47**(1), 229-231.

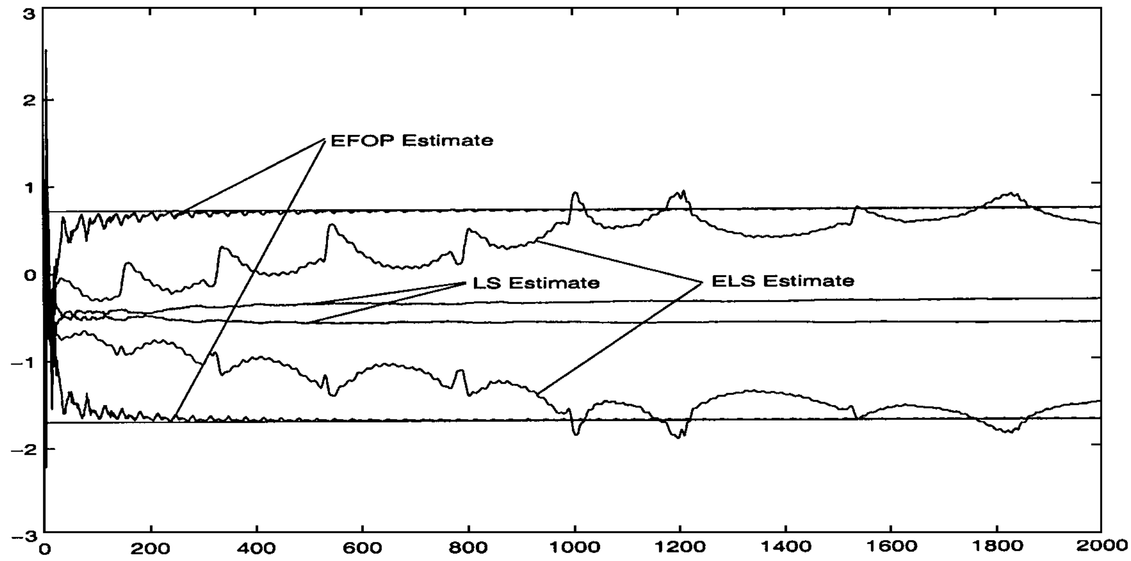

Fig. 1. Simulation of Example 1

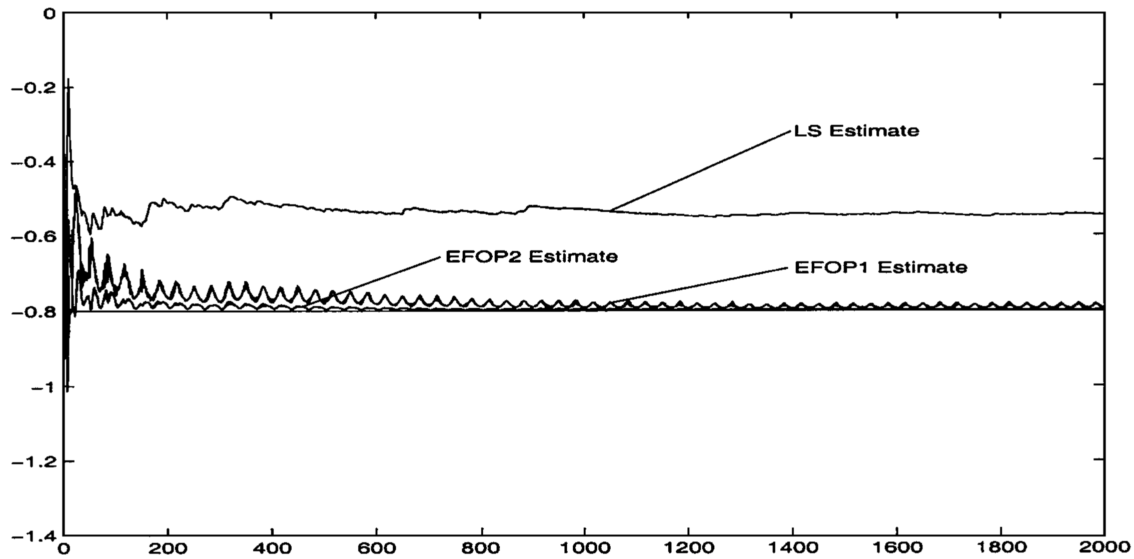

Fig. 2. Simulation of Example 2

Lambert-Lacroix, S. (2000). On periodic autoregressive processes estimation. *IEEE Transactions on Signal Processing,* **48**(6), 1800-1803.

Lang, M. & Laakso, T. I. (1994). Simple and robust method for the design of allpass filters using Least-Squares phase error criterion. *IEEE Transactions on Circuits and Systems-II: Analog and Digital Signal Processing,* **41**(1), 40-48.

Ljung, L. (1999). *System Identification-Theory for the User.* Upper Saddle River, NJ: Prentice Hall.

McKelvey, T., Akcay, H., & Ljung, L. (1996). Subspace-based identification of infinite dimensional multivariable systems from frequency-response data. *Automatica,* **37**, 885-902.

Ninness, B. (1996). Integral constraints on the accuracy of Least-squares estimation. *Automatica,* **32**(3) 391-397.

Ninness, B. & Gomez, J. C. (1998). Frequency domain analysis of tracking and noise performance of adaptive algorithms. *IEEE Transactions on Signal Processing,* **45**, 1314-1332.

Pintelon, R., Schoukens, J., & Vandersteen, G. (1997). Frequency domain system identification using arbitrary signals. *IEEE Transactions on Automatic Control,* **42**, 1717-1720.

Weyer, E. (2000). Finite sample properties of system identification of ARX models under mixing conditions. *Automatica,* **36**, 1291-1299.