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GAIN SCHEDULED CONTROL BASED ON INTERPOLATION PRESERVING H_∞ PERFORMANCE[1]

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Abstract: A new approach to the design of a gain scheduled output feedback controller without a varying-parameter rate feedback is presented. First, the controller design is translated into parameterized linear matrix inequalities for the parameter matrices. Then, the sufficient condition is given for the partition of the varying-parameter set based on the concept of H_∞ performance covering. The varying-parameter set is thus partitioned into sufficiently small subsets. After the constant matrices are found for each of the subsets, the required continuous parameter matrices are obtained by using interpolation. The proposed controller overcomes the drawback that the gain scheduled controller may not be found by using the existing gain scheduled linear parameter varying (LPV) synthesis. Moreover, the varying-parameter rate feedback is eliminated and the conservation of the controller design is reduced by means of limiting the upper bound of the varying-parameter rate. Experiment results prove the effectiveness of the proposed controller. *Copyright © 2002 IFAC*

Keywords: linear parameter varying (LPV), interpolation, H_∞ performance, gain scheduling, parameterized linear matrix inequality (PLMI)

1. INTRODUCTION

Gain scheduled control is widely used and generally proved to be efficient (Nichols, *et al*., 1993; Rugh, 1991; Shamma and Athans, 1990) in nonlinear time-varying system engineering design. Its principle is to design local controllers first and then to obtain a global controller by using interpolation. However, its theoretical stability is not guaranteed in the whole varying-parameter range. Recently, due to the development of robust control, there has been more research on gain scheduled control, especially for linear parameter varying (LPV) systems (Apkarian and Tuan, 2000; Apkarian and Adams, 1997; Apkarian and Gahinet, 1995a; Apkarian, *et al*., 1995b; Kajiwara, *et al*., 1999). The existing approaches combine gain scheduling and H_∞ performance. A gain scheduled controller with a linear fractional transformation (LFT) structure based on the small gain theory is presented in (Apkarian and Gahinet, 1995a). The drawback of the LFT description is that the parameters are allowed to be complex numbers. When the known parameters are real numbers, conservation is introduced. For the LPV system with a polytopic structure, a single Lyapunov function is searched in the whole varying-parameter set to guarantee a H_∞ performance for all possible trajectories of the LPV system (Apkarian, *et al*., 1995b; Kajiwara, *et al*., 1999). This, however, is difficult to achieve. Moreover, there is no restriction for the varying-parameter rate in this approach. Since it is allowed to be infinity, a big conservation is introduced. A controller synthesis method is proposed for a special kind of LPV systems with an affine structure with a consideration of the bound of the varying-parameter rate (Apkarian and Tuan, 2000; Apkarian and Adams, 1997). However, varying-parameter rate feedback is needed in the designed controllers, which is unpractical. Above all, all the aforementioned gain scheduled LPV controller designs have a common drawback, i.e., it is not guaranteed to find a gain scheduled controller, which meets the demands.

The framework of the conventional gain scheduled controller is absorbed in this article. On the basis of

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the concept of "H_∞ performance covering", the sufficient condition is given for the partition of the varying-parameter set into sufficiently small subsets. For each of the subsets, the gain scheduled controllers are designed by using interpolation preserving H_∞ performance. After the calculation of the interpolation in each of the subsets, the upper bound of the varying-parameter rate which ensures that the system possesses the H_∞ performance in the whole varying-parameter set is obtained. Thus, although its upper bound is specified, the varying-parameter rate is eliminated in the gain scheduled controller and therefore there is no varying-parameter rate feedback in this article. Furthermore, since the proposed approach is based on the confirmation that a linear time-invariant (LTI) H_∞ controller exists for each fixed point in the varying-parameter set, it overcomes the drawback that the gain scheduled controller, which meets the demands, may not be found by using the existing gain scheduled LPV controller synthesis. This article proves theoretically that the proposed gain scheduled controller makes the system possess a H_∞ performance in the whole varying-parameter set.

2. PROBLEM DESCRIPTION

Consider the following LPV system with $\rho(t) \in \Gamma \subset R^l$ as its varying parameter:

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B_1(\rho(t))w_1(t) + B_2(\rho(t))u(t) \\ z(t) &= C_1(\rho(t))x(t) + u(t) \\ y(t) &= C_2(\rho(t))x(t) + w_2(t) \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $z(t) \in R^m$ and $y(t) \in R^r$ represent the state, the control input, the performance index and the output vector of the system, respectively. $w_1(t) \in R^p$ is the disturbance due to system dynamics uncertainty and external inputs. $w_2(t) \in R^r$ is the system measurement noise. The parameter matrices: $A(\rho(t)) : \Gamma \to R^{n\times n}$, $B_1(\rho(t)) : \Gamma \to R^{n\times p}$, $B_2(\rho(t)) : \Gamma \to R^{n\times m}$, $C_1(\rho(t)) : \Gamma \to R^{m\times n}$ and $C_2(\rho(t)) : \Gamma \to R^{r\times n}$.

Since a nonlinear system in the form of $\dot{x}(t) = f(x(t), w(t), u(t))$ can be linearized to be a LPV system as expressed in (1) by using the Jacobian linearization after the varying parameter $\rho(t)$ is properly selected, the research on the LPV system is of a general significance. For the LPV system expressed in (1), design the following n-order gain scheduled output feedback controller $K(\rho(t))$:

$$K(\rho(t)) : \begin{cases} \dot{x}_K(t) = A_K(\rho(t))x_K(t) + B_K(\rho(t))y(t) \\ u(t) = C_K(\rho(t))x_K(t) \end{cases} \tag{2}$$

Define $x_{cl}(t) = [x(t) \quad x_K(t)]^T$ and $w(t) = [w_1(t) \quad w_2(t)]^T$. Due to (1) and (2), the closed-loop system is expressed as:

$$\begin{cases} \dot{x}_{cl}(t) = A_{cl}(\rho(t))x_{cl}(t) + B_{cl}(\rho(t))w(t) \\ z(t) = C_{cl}(\rho(t))x_{cl}(t) \end{cases} \tag{3}$$

where

$$A_{cl}(\rho(t)) = \begin{bmatrix} A(\rho(t)) & B_2(\rho(t))C_K(\rho(t)) \\ B_K(\rho(t))C_2(\rho(t)) & A_K(\rho(t)) \end{bmatrix}$$

$$B_{cl}(\rho(t)) = \begin{bmatrix} B_1(\rho(t)) & 0 \\ 0 & B_K(\rho(t)) \end{bmatrix}$$

$$C_{cl}(\rho(t)) = [C_1(\rho(t)) \quad C_K(\rho(t))]$$

Definition 1 Given $\gamma > 0$, if for any disturbance $w(t)$, the closed-loop system performance index $z(t)$ satisfies $\int_0^\infty z^T(t)z(t)dt < \gamma^2 \int_0^\infty w^T(t)w(t)dt$, the closed-loop system possesses the H_∞ performance γ.

The problem to be solved in this article is to design the gain scheduled output feedback controller $K(\rho(t))$ so that the closed-loop system possesses the H_∞ performance γ for any $\rho(t)$ in the varying-parameter set Γ. For convenience, the symbol "t" is omitted in the following text.

3. GAIN SCHEDULED CONTROLLER SYNTHESIS

Definition 2 For the closed-loop system (3), where $\rho \in \Gamma$, if there exists a continuous symmetric positive-definite matrix function $P(\rho) : \Gamma \to R^{2n\times 2n}$, which satisfies

$$\begin{aligned} &A_{cl}^T(\rho)P(\rho) + P(\rho)A_{cl}(\rho) + \frac{d}{dt}P(\rho) + \gamma^{-2}C_{cl}^T(\rho)C_{cl}(\rho) \\ &+P(\rho)B_{cl}(\rho)B_{cl}^T(\rho)P(\rho) < 0 \end{aligned} \tag{4}$$

then the closed-loop system possesses H_∞ performance γ.

On the basis of Definition 2, the specific method for gain scheduled controller synthesis is given in this section.

Theorem 1 Given positive definite constant matrices Q_x and Q_y and $\gamma > 0$, if there exist positive definite, continuous and differentiable matrix functions $X(\rho)$ and $Y(\rho)$, where $\rho \in \Gamma$, satisfies

$$\begin{bmatrix} S(\rho) & 0 & \gamma^{-1}B_1(\rho) \\ 0 & -I & 0 \\ \gamma^{-1}B_1^T(\rho) & 0 & -I \end{bmatrix} < 0 \tag{5}$$

$$\begin{bmatrix} R(\rho) & X(\rho)B_1(\rho) & \gamma^{-1}C_1^T(\rho) \\ B_1^T(\rho)X(\rho) & -I & 0 \\ \gamma^{-1}C_1(\rho) & 0 & -I \end{bmatrix} < 0 \tag{6}$$

$$\begin{bmatrix} X(\rho) & \gamma^{-1}I \\ \gamma^{-1}I & Y(\rho) \end{bmatrix} > 0 \tag{7}$$

where

$$\begin{aligned} S(\rho) = {} & [A(\rho) - B_2(\rho)C_1(\rho)]Y(\rho) + Y(\rho)[A(\rho) - B_2(\rho)C_1(\rho)]^T \\ & -B_2(\rho)B_2^T(\rho) + Q_y \end{aligned}$$

$$R(\rho) = X(\rho)A(\rho) + A^T(\rho)X(\rho) - C_2^T(\rho)C_2(\rho) + Q_x$$

and if for any $t \geq 0$ we have

$$\dot{X}(\rho) < Q_x \tag{8}$$

$$\dot{Y}(\rho) > -Q_y \tag{9}$$

, then for any invertible matrix function $N(\rho) : \Gamma \to R^{n\times n}$, there exist the following gain scheduled controller $K(\rho)$:

$F(\rho) = -[B_2^T(\rho)Y^{-1}(\rho) + C_1(\rho)]$

$M(\rho) = [I - \gamma^2 Y(\rho)X(\rho)]N^{-T}(\rho)$

$L(\rho) = -X^{-1}(\rho)C_2^T(\rho)$

$$A_K(\rho) = -N^{-1}(\rho)\{\gamma^2 X(\rho)[A(\rho) + L(\rho)C_2(\rho) + B_2(\rho)F(\rho)]Y(\rho) + A^T(\rho) + [C_1^T(\rho)C_1(\rho) + C_1^T(\rho)F(\rho)]Y(\rho) + X(\rho)B_1(\rho)B_1^T(\rho) + \gamma^2 \dot{X}(\rho)Y(\rho) + \dot{N}(\rho)M(\rho)\}M^{-T}(\rho) \quad (10)$$

$B_K(\rho) = N^{-1}(\rho)X(\rho)L(\rho)$

$C_K(\rho) = \gamma^2 F(\rho)Y(\rho)M^{-T}(\rho)$

$$K(\rho) = \begin{bmatrix} A_K(\rho) & B_K(\rho) \\ C_K(\rho) & 0 \end{bmatrix}$$

so that for all $\rho \in \Gamma$, the closed-loop system possesses H_∞ performance γ.

Proof Define $M(\rho) = [I - \gamma^2 Y(\rho)X(\rho)]N^{-T}(\rho)$ (11)

Due to Schur Complement (Gahinet and Apkarian 1994), (7) is equivalent to $X(\rho) - \gamma^{-2}Y(\rho) > 0$, i.e., $Y^{-1}(\rho) - \gamma^2 X(\rho) < 0$ (12)

(12) shows also that $[I - \gamma^2 X(\rho)Y(\rho)]Y^{-1}(\rho)$ is invertible and so does $M(\rho)$.

Define $P(\rho) = \begin{bmatrix} X(\rho) & N(\rho) \\ N^T(\rho) & -\gamma^2 N^T(\rho)Y(\rho)M^{-T}(\rho) \end{bmatrix}$,

$$P^{-1}(\rho) = \begin{bmatrix} \gamma^2 Y(\rho) & M(\rho) \\ M^T(\rho) & -N^{-1}(\rho)X(\rho)M(\rho) \end{bmatrix}$$

$$H(\rho) = A_{cl}^T(\rho)P(\rho) + P(\rho)A_{cl}(\rho) + \dot{P}(\rho) + \gamma^{-2}C_{cl}^T(\rho)C_{cl}(\rho) + P(\rho)B_{cl}(\rho)B_{cl}^T(\rho)P(\rho) \quad (13)$$

Due to Definition 2, in order that the closed-loop system (3) possesses H_∞ performance γ, we must have $P(\rho) > 0$ and $H(\rho) < 0$ for all $\rho \in \Gamma$. Since $P(\rho)P^{-1}(\rho) = I$, $N(\rho)M^T(\rho) = I - \gamma^2 X(\rho)Y(\rho)$.

Since

$$\begin{aligned} & X(\rho) + N(\rho)[\gamma^2 N^T(\rho)Y(\rho)M^{-T}(\rho)]^{-1}N^T(\rho) \\ &= X(\rho) + \gamma^{-2}N(\rho)M^T(\rho)Y^{-1}(\rho) \\ &= X(\rho) + \gamma^{-2}[I - \gamma^2 X(\rho)Y(\rho)]Y^{-1}(\rho) = \gamma^{-2}Y^{-1}(\rho) > 0 \quad (14) \end{aligned}$$

Also,

$$\begin{aligned} -\gamma^2 N^T(\rho)Y(\rho)M^{-T}(\rho) &= -\gamma^2 N^T(\rho)Y(\rho)[I - \gamma^2 X(\rho)Y(\rho)]^{-1}N(\rho) \\ &= -\gamma^2 N^T(\rho)[Y^{-1}(\rho) - \gamma^2 X(\rho)]^{-1}N(\rho) \end{aligned}$$

From (12), we have $-\gamma^2 N^T(\rho)Y(\rho)M^{-T}(\rho) > 0$ (15)

Thus, due to Schur Complement, (14) and (15) are equivalent to $P(\rho) > 0$.

In order to prove $H(\rho) < 0$, define

$P_1(\rho) = \begin{bmatrix} \gamma^2 Y(\rho) & I \\ M^T(\rho) & 0 \end{bmatrix}$ and $P_2(\rho) = P(\rho)P_1(\rho) = \begin{bmatrix} I & X(\rho) \\ 0 & N^T(\rho) \end{bmatrix}$

Let $\overline{H}(\rho) = P_1^T(\rho)H(\rho)P_1(\rho)$ (16)

Since $P_1(\rho)$ is invertible, $H(\rho) < 0$ if and only if $\overline{H}(\rho) < 0$.

Substituting (13) into (16), we have

$$\overline{H}(\rho) = P_1^T(\rho)A_{cl}^T(\rho)P_2(\rho) + P_2^T(\rho)A_{cl}(\rho)P_1(\rho) + P_1^T(\rho)\dot{P}(\rho)P_1(\rho) + \gamma^{-2}P_1^T(\rho)C_{cl}^T(\rho)C_{cl}(\rho)P_1(\rho) + P_2^T(\rho)B_{cl}(\rho)B_{cl}^T(\rho)P_2(\rho) \quad (17)$$

Define $\overline{H}(\rho) = \begin{bmatrix} \overline{H}_{11}(\rho) & \overline{H}_{12}(\rho) \\ \overline{H}_{12}^T(\rho) & \overline{H}_{22}(\rho) \end{bmatrix}$

Substituting (17) into $P_1(\rho)$, $P_2(\rho)$, $A_{cl}(\rho)$, $B_{cl}(\rho)$ and $C_{cl}(\rho)$ and combining (5) and (6), we have

$$\gamma^{-2}\overline{H}_{11}(\rho) < -Q_y - \dot{Y}(\rho) \quad (18)$$

$$\overline{H}_{22}(\rho) < -Q_x + \dot{X}(\rho) \quad (19)$$

$$\overline{H}_{12}(\rho) = \gamma^2 X(\rho)[A(\rho) + L(\rho)C_2(\rho) + B_2(\rho)F(\rho)]Y(\rho) + A^T(\rho) + N(\rho)A_K(\rho)M^T(\rho) + [C_1^T(\rho)C_1(\rho) + C_1^T(\rho)F(\rho)] + X(\rho)B_1(\rho)B_1^T(\rho) + \gamma^2 \dot{X}(\rho)Y(\rho) + \dot{N}(\rho)M^T(\rho)$$

When $A_K(\rho)$ takes (10), $\overline{H}_{12}(\rho) = 0$. Combining (18) and (19), we have

$$\overline{H}(\rho) < \begin{bmatrix} -\gamma^2[Q_y + \dot{Y}(\rho)] & 0 \\ 0 & -Q_x + \dot{X}(\rho) \end{bmatrix}$$

When (8) and (9) hold, $\overline{H}(\rho) < 0$, i.e., $H(\rho) < 0$. Therefore, for all $\rho \in \Gamma$, the closed-loop system possesses H_∞ performance γ.

4. ELIMINATION OF THE VARYING-PARAMETER RATE FEEDBACK IN THE CONTROLLER

Corollary 1 In Theorem 1, if $N(\rho)$ satisfies

$$\frac{\partial N(\rho)}{\partial \rho_i} = -\gamma^2 \frac{\partial X(\rho)}{\partial \rho_i} Y(\rho)[I - \gamma^2 X(\rho)Y(\rho)]^{-1}N(\rho) \quad (i = 1, \cdots, l) \quad (20)$$

then the $A_K(\rho)$ in the gain scheduled controller $K(\rho)$ becomes

$$A_K(\rho) = -N^{-1}(\rho)\{\gamma^2 X(\rho)[A(\rho) + L(\rho)C_2(\rho) + B_2(\rho)F(\rho)]Y(\rho) + A^T(\rho) + [C_1^T(\rho)C_1(\rho) + C_1^T(\rho)F(\rho)]Y(\rho) + X(\rho)B_1(\rho)B_1^T(\rho)\}M^{-T}(\rho) \quad (21)$$

and the controller $K(\rho)$ can guarantee that the closed-loop system possesses H_∞ performance γ for all $\rho \in \Gamma$.

Proof In the proof of Theorem 1, assume $A_K(\rho)$ takes (21). Then $\overline{H}_{12}(\rho) = \gamma^2 \dot{X}(\rho)Y(\rho) + \dot{N}(\rho)M^T(\rho)$.

From (11), $M^{-T}(\rho) = [I - \gamma^2 X(\rho)Y(\rho)]^{-1}N(\rho)$.

Substituting the above into (20),

$$\frac{\partial N(\rho)}{\partial \rho_i} = -\gamma^2 \frac{\partial X(\rho)}{\partial \rho_i} Y(\rho)M^{-T}(\rho) \qquad (i = 1, \cdots, l)$$

Further,

$$\frac{\partial N(\rho)}{\partial \rho_1}\dot{\rho}_1 + \cdots + \frac{\partial N(\rho)}{\partial \rho_l}\dot{\rho}_l = -\gamma^2[\frac{\partial X(\rho)}{\partial \rho_1}\dot{\rho}_1 + \cdots + \frac{\partial X(\rho)}{\partial \rho_l}\dot{\rho}_l]Y(\rho)M^{-T}(\rho)$$

i.e., $\dot{N}(\rho)M^T(\rho) = -\gamma^2 \dot{X}(\rho)Y(\rho)$. Thus, $\overline{H}_{12}(\rho) = 0$. Then,

$$\overline{H}(\rho) < \begin{bmatrix} -\gamma^2[Q_y + \dot{Y}(\rho)] & 0 \\ 0 & -Q_x + \dot{X}(\rho) \end{bmatrix}$$

We have $\overline{H}(\rho) < 0$, i.e., $H(\rho) < 0$ from (8) and (9) in Theorem 1. Therefore, the closed-loop system possesses H_∞ performance γ for all $\rho \in \Gamma$.

Remark For scalar varying parameters, (20) is a first-order linear differentiation equation, whose solution is easy to be obtained. Given an invertible initial condition N_0, suppose an invertible matrix $T(\rho, \rho_0)$ is the transition matrix of the differentiation equation. Then for any ρ in the varying-parameter set, the solution of the differential equation can be expressed as $N(\rho) = T(\rho, \rho_0)N_0$.

5. INTERPOLATION PRESERVING H_∞ PERFORMANCE

Finding matrix functions $X(\rho)$ and $Y(\rho)$ is the key in Theorem 1. In references (Apkarian and Tuan, 2000; Apkarian and Adams, 1997), the drawback of this approach is that it only applies to a special kind of LPV systems with an affine structure and it does not guarantee to find $X(\rho)$ and $Y(\rho)$, which meet the demands. The proposed approach in this article absorbs the framework of the conventional gain scheduling.

Definition 3 Suppose that for the fixed $\rho_i \in \Gamma$ ($i = 0,\cdots,m$) there exist constant matrices X_i and Y_i which satisfy Theorem 1. Let U_i be an open neighborhood containing ρ_i and for each fixed $\rho \in U_i$, X_i and Y_i satisfy Theorem 1. If $\Gamma \subset \bigcup_{i=0}^{m} U_i$ and $U_i \bigcap U_{i+1}$ are non-empty sets, then the matrix sets $\{X_i, Y_i, i = 0,\cdots,m\}$ satisfy the condition of H_∞ performance covering.

Under the condition of H_∞ performance covering, along with the changes of ρ in different neighborhoods U_i, the transitions of X_i and Y_i can make the system obtain H_∞ performance. However, the transitions create jump transfers of the controller. In our approach, continuous matrix functions $X(\rho)$ and $Y(\rho)$ in the whole $\rho \in \Gamma$ are obtained by using interpolation under the condition of H_∞ performance covering, as is proposed in this article. The interpolation technique is presented below for scalar varying parameters as an example. It is easy to generalize it to vector varying parameters.

Theorem 2 For positive constants δ_x and δ_y, let $Q_x = \delta_x I$ and $Q_y = \delta_y I$. Suppose there exist constant matrices $(X_0, Y_0),\cdots,(X_m, Y_m)$ corresponding to $\rho_0 < \cdots < \rho_m \in \Gamma$, which satisfy the condition of H_∞ performance covering, i.e., for the open neighborhoods U_i containing ρ_i, X_i and Y_i satisfy Theorem 1. Then there exist regions $[r_{i-1}, r_i] \subset U_{i-1} \bigcap U_i \bigcap [\rho_{i-1}, \rho_i]$, $i = 1,\cdots,m$ and continuous matrix functions $X(\rho)$ and $Y(\rho)$:

$$X(\rho) = \begin{cases} X_{i-1} & \rho \in [\rho_{i-1}, r_{i-1}] \\ \dfrac{r_{i+1}-\rho}{r_{i+1}-r_i} X_{i-1} + \dfrac{\rho - r_i}{r_{i+1}-r_i} X_i & \rho \in (r_{i-1}, r_i) \\ X_i & \rho \in [r_i, \rho_i] \end{cases} \quad (22)$$

$$Y(\rho) = \begin{cases} Y_{i-1} & \rho \in [\rho_{i-1}, r_{i-1}] \\ \dfrac{r_{i+1}-\rho}{r_{i+1}-r_i} Y_{i-1} + \dfrac{\rho - r_i}{r_{i+1}-r_i} Y_i & \rho \in (r_{i-1}, r_i) \\ Y_i & \rho \in [r_i, \rho_i] \end{cases} \quad (23)$$

When

$$|\dot{\rho}| < \min_{i=1,\cdots,m} \left[\frac{(r_i - r_{i-1})\delta_x}{\|X_i - X_{i-1}\|}, \frac{(r_i - r_{i-1})\delta_y}{\|Y_i - Y_{i-1}\|} \right] \quad (24)$$

, the closed-loop system possesses H_∞ performance γ for all $\rho \in \Gamma$.

Proof From Definition 3, when $\rho \in [\rho_{i-1}, r_i) \subset U_{i-1}$, X_{i-1} and Y_{i-1} satisfy (5)~(7) and when $\rho \in (r_{i-1}, \rho_i] \subset U_i$, X_i and Y_i satisfy (5)~(7). Thus, for the continuous matrix functions $X(\rho)$ and $Y(\rho)$ expressed in (22) and (23), when $\rho \in [\rho_{i-1}, r_{i-1}]$, $X(\rho)$ and $Y(\rho)$ satisfy (5)~(7). Also, when $\rho \in [r_i, \rho_i]$, $X(\rho)$ and $Y(\rho)$ satisfy (5)~(7). When $\rho \in (r_{i-1}, r_i)$, both (X_{i-1}, Y_{i-1}) and (X_i, Y_i) satisfy (5)~(7) since this region belongs to U_{i-1} as well as U_i. Then $\dfrac{r_{i+1}-\rho}{r_{i+1}-r_i} X_{i-1} + \dfrac{\rho - r_i}{r_{i+1}-r_i} X_i$ and $\dfrac{r_{i+1}-\rho}{r_{i+1}-r_i} Y_{i-1} + \dfrac{\rho - r_i}{r_{i+1}-r_i} Y_i$ also satisfy (5)~(7). As a summary, when $\rho \in [\rho_{i-1}, \rho_i]$, $X(\rho)$ and $Y(\rho)$ satisfy (5)~(7). Furthermore, the continuous $X(\rho)$ and $Y(\rho)$ are regarded as differentiable, since sufficiently close smooth approximation of (22) and (23) can always be found (See APPENDIX Theorem 3) such that for $\rho \in [\rho_{i-1}, \rho_i]$ (5)~(7) are still satisfied. Due to Theorem 3, when $\rho \in [\rho_{i-1}, \rho_i]$,

$$\|\dot{X}(\rho)\| \le \frac{\|X_i - X_{i-1}\|}{|r_i - r_{i-1}|} |\dot{\rho}|, \quad \|\dot{Y}(\rho)\| \le \frac{\|Y_i - Y_{i-1}\|}{|r_i - r_{i-1}|} |\dot{\rho}|$$

From (24), $\dfrac{\|X_i - X_{i-1}\|}{|r_i - r_{i-1}|} |\dot{\rho}| < \delta_x$ and $\dfrac{\|Y_i - Y_{i-1}\|}{|r_i - r_{i-1}|} |\dot{\rho}| < \delta_y$, i.e., $\|\dot{X}(\rho)\| \le \delta_x$ and $\|\dot{Y}(\rho)\| \le \delta_y$. Thus (8) and (9) in Theorem 1 are also satisfied and therefore the closed-loop system possesses H_∞ performance γ for all $\rho \in \Gamma$.

6. EXPERIMENTS

Experiments are done for a self-developed planar two-joint direct-drive manipulator. Its dynamics equation is as follows (Yu and Chen, 1999)

$$\begin{bmatrix} a & b\cos(\theta_2 - \theta_1) \\ b\cos(\theta_2 - \theta_1) & c \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -b\dot{\theta}_2^2 \sin(\theta_2 - \theta_1) \\ b\dot{\theta}_1^2 \sin(\theta_2 - \theta_1) \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \quad (25)$$

where $a = 5.6794 kg \cdot m^2$, $b = 1.4730 kg \cdot m^2$ and $c = 1.7985 kg \cdot m^2$.

Linearize the robotic dynamics equation (25) around the equilibrium manifold $X_e = (\theta_{1e} \;\; \theta_{2e} \;\; \dot{\theta}_{1e} \;\; \dot{\theta}_{2e})^T = (\theta_{1e} \;\; \theta_{2e} \;\; 0 \;\; 0)^T$ and $\tau_e = (0 \;\; 0)^T$ by using the Jacobian approach:

$$\dot{x} = Ax + Bu \quad (26)$$

where $x = \left(\hat{\theta}_1 \;\; \hat{\theta}_2 \;\; \hat{\dot{\theta}}_1 \;\; \hat{\dot{\theta}}_2 \right)^T$, $u = (\hat{\tau}_1 \;\; \hat{\tau}_2)^T$, $\hat{\theta}_1 = \theta_1 - \theta_{1e}$, $\hat{\theta}_2 = \theta_2 - \theta_{2e}$, $\hat{\dot{\theta}}_1 = \dot{\theta}_1 - \dot{\theta}_{1e} = \dot{\theta}_{1e}$, $\hat{\dot{\theta}}_2 = \dot{\theta}_2 - \dot{\theta}_{2e} = \dot{\theta}_{2e}$,

$\hat{\tau}_1 = \tau_1 - \tau_{1e} = \tau_1$, $\hat{\tau}_2 = \tau_2 - \tau_{2e} = \tau_2$, $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{c}{ac - b^2\cos^2(\theta_{2e} - \theta_{1e})} & \dfrac{-b\cos(\theta_{2e} - \theta_{1e})}{ac - b^2\cos^2(\theta_{2e} - \theta_{1e})} \\ \dfrac{-b\cos(\theta_{2e} - \theta_{1e})}{ac - b^2\cos^2(\theta_{2e} - \theta_{1e})} & \dfrac{a}{ac - b^2\cos^2(\theta_{2e} - \theta_{1e})} \end{bmatrix}.$$

It is seen from (26) that matrix B is the function of $\cos(\theta_{2e} - \theta_{1e})$, where $\theta_{2e} - \theta_{1e}$ is the angle between joint 1 and joint 2, which decides the dynamic characteristics of (26). Practically, the measured values of θ_1 and θ_2 can be regarded as the equilibrium points to linearize system (25). Therefore, a varying parameter $\rho = \cos(\theta_2 - \theta_1)$ is defined, where $\theta_2 - \theta_1 \in [-\pi, 0]$ and $\rho \in [-1, 1]$. Assume that the output vector $y(t)$ is the joint positions of the manipulator and $w_2(t)$ is the position measurement noise. Since there are modeling errors such as the high-frequency unmodelled part and dynamic uncertainty and external disturbance, the disturbance term $w_1(t)$ is the equivalence of all the above factors. The performance index $z(t)$ represents the disturbance-rejection performance for disturbances $w_1(t)$ and $w_2(t)$. Then, (26) can be expressed in a same form as (1), where $A(\rho(t)) = A$, $B_1(\rho(t)) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_2(\rho(t)) = B$, $C_1(\rho(t))$ is the performance weighting matrix. In this article, we choose $C_1(\rho(t)) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. $C_2(\rho(t))$ is the output measurement matrix and $C_2(\rho(t)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

The interpolation preserving H_∞ performance proposed in Section 5 supplies an efficient approach to find the matrix functions $X(\rho)$ and $Y(\rho)$. Let $\delta_x = 0.015$, $\delta_y = 0.85$ and $\gamma = 1$ in Theorem 2. Under the condition of H_∞ performance covering, by using MATLAB *LMI Control Toolbox* (Gahinet, *et al*., 1995), calculate X_0 and Y_0 for $\rho \in [-1, -0.4]$, X_1 and Y_1 for $\rho \in [-0.6, 0]$, X_2 and Y_2 for $\rho \in [-0.2, 0.4]$, X_3 and Y_3 for $\rho \in [0.2, 0.8]$ and X_4 and Y_4 for $\rho \in [0.6, 1]$. Then they can be interpolated according to Theorem 2.

In accordance with Sections 3 and 4, the gain scheduled controller $K(\rho)$ without varying-parameter rate feedback is then obtained. $N(\rho)$ can be found by Corollary 1 and Remark (selecting $N_0 = I_4$). Calculation gives $\|X_i - X_{i-1}\| < 0.0061$ and $\|Y_i - Y_{i-1}\| < 0.3865$, $i = 1,2,3,4$. Due to (24) in Theorem 2, in order to guarantee that the robotic system possesses H_∞ performance γ in $\rho \in [-1, 1]$, the upper bound of the varying-parameter rate is calculated to be $|\dot{\rho}| < \frac{0.2 \times \delta_y}{0.3865} = 0.4398\, rad/s$. Since $\dot{\rho} = [-\sin(\theta_2 - \theta_1)](\dot{\theta}_2 - \dot{\theta}_1)$, the calculated upper bound of the varying-parameter rate can meet the velocity demand in the practical robotic motion.

In experiment, the end of the manipulator tracks a circle with a diameter of $0.50m$ with a velocity of $0.50 m \cdot s^{-1}$. The coordinate of the circuit origin is $(0.40m, 0.30m)$. A comparison is made between the proposed gain scheduled controller and the conventional gain scheduled PID control. As shown in Fig. 1, it is seen that under the functioning of the conventional PID controller, the maximum errors for joints 1 and 2 are 0.1°, respectively, with evident small vibrations, while under the functioning of the proposed gain scheduled controller, the maximum errors for joints 1 and 2 are 0.04° and 0.03°, respectively, without any small vibration.

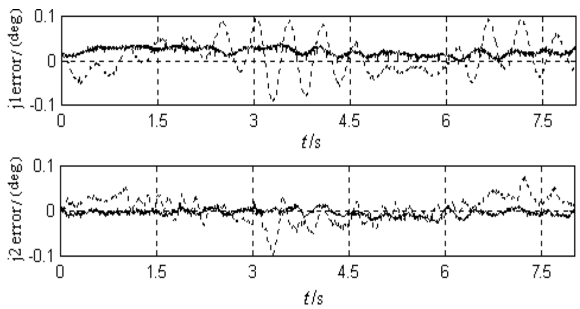

Fig. 1 Experiment results. The conventional gain scheduled PID control:—— ; The gain scheduled control based on the interpolation preserving H_∞ performance:------.

APPENDIX

Theorem 3 If $W(\rho): R \to R^{n \times n}$, $\rho \in [\rho_0, \rho_m]$ is continuous and has $m+1$ corner points at $r_0, \cdots, r_m$, i.e.,

$$W(\rho) = \begin{cases} W_{i-1} & \rho \in [\rho_{i-1}, r_{i-1}] \\ \dfrac{r_{i+1} - \rho}{r_{i+1} - r_i} W_{i-1} + \dfrac{\rho - r_i}{r_{i+1} - r_i} W_i & \rho \in (r_{i-1}, r_i) \qquad i = 1, \cdots, m \\ W_i & \rho \in [r_i, \rho_i] \end{cases} \tag{A1}$$

which can be rewritten as

$$W(\rho) = \begin{cases} V_0 \rho + Q_0 & \rho \in [\rho_0, r_0) \\ V_i \rho + Q_i & \rho \in [r_{i-1}, r_i], \quad i = 1, \cdots, m \\ V_{m+1} \rho + Q_{m+1} & \rho \in (r_m, \rho_m] \end{cases} \tag{A2}$$

and satisfies $V_{i-1} r_{i-1} + Q_{i-1} = V_i r_{i-1} + Q_i$, $i = 1, \cdots, m+1$ where $(V_0, Q_0), \cdots, (V_{m+1}, Q_{m+1})$ are easy to be obtained, then for any $\varepsilon > 0$, there exist a continuous and differentiable function $\hat{W}(\rho): R \to R^{n \times n}$ and $\delta > 0$ such that

$$\left\| W(\rho) - \hat{W}(\rho) \right\| < \varepsilon \qquad \rho \in (r_{i-1} - \frac{\delta}{2}, r_{i-1} + \frac{\delta}{2}) \tag{A3}$$

$$W(\rho) = \hat{W}(\rho) \qquad \rho \notin (r_{i-1} - \frac{\delta}{2}, r_{i-1} + \frac{\delta}{2}) \tag{A4}$$

$$\left\|\frac{d}{dt}\hat{W}(\rho)\right\| \le \max_{\rho\ne r_{i-1}}\left\|\frac{d}{dt}W(\rho)\right\|,\ i=1,\cdots,m+1 \tag{A5}$$

Proof Select

$$\hat{W}(\rho)=\begin{cases} W(r_{i-1}-\frac{\delta}{2})+\delta\int_0^{\frac{\rho-r_{i-1}}{\delta}+\frac{1}{2}}[(1-\sigma)V_{i-1}+\sigma V_i]d\sigma & \rho\in(r_{i-1}-\frac{\delta}{2},r_{i-1}+\frac{\delta}{2}) \\ W(\rho) & \rho\notin(r_{i-1}-\frac{\delta}{2},r_{i-1}+\frac{\delta}{2}) \end{cases} \tag{A6}$$

It can be verified that $\hat{W}(\rho)$ is continuous and differentiable in $\rho\in[\rho_0,\rho_m]$.

When $\rho\in(r_{i-1}-\frac{\delta}{2},r_{i-1}+\frac{\delta}{2})$,

$$\left\|W(\rho)-\hat{W}(\rho)\right\| \le \left\|W(\rho)-W(r_{i-1}-\frac{\delta}{2})\right\|+\delta\int_0^{\frac{\rho-r_{i-1}}{\delta}+\frac{1}{2}}\left\|(1-\sigma)V_{i-1}+\sigma V_i\right\|d\sigma \tag{A7}$$

Let us prove first $\left\|W(\rho)-W(r_{i-1}-\frac{\delta}{2})\right\|\le\delta\max\{\|V_{i-1}\|,\|V_i\|\}$

When $\rho\in(r_{i-1}-\frac{\delta}{2},r_{i-1}]$,

$$\left\|W(\rho)-W(r_{i-1}-\frac{\delta}{2})\right\|=\left\|V_{i-1}\rho+Q_{i-1}-V_{i-1}r_{i-1}+\frac{\delta}{2}V_{i-1}-Q_{i-1}\right\|$$

$$\le\|V_{i-1}\|\|\rho-r_{r-1}|+\frac{\delta}{2}\|V_{i-1}\| \le\delta\|V_{i-1}\|$$

When $\rho\in[r_{i-1},r_{i-1}+\frac{\delta}{2})$,

$$\left\|W(\rho)-W(r_{i-1}-\frac{\delta}{2})\right\|=\left\|V_i\rho+Q_i-V_{i-1}r_{i-1}+\frac{\delta}{2}V_{i-1}-Q_{i-1}\right\|$$

$$=\left\|V_i\rho+Q_i-V_ir_{i-1}-Q_i+\frac{\delta}{2}V_{i-1}\right\|$$

$$\le\frac{\delta}{2}\|V_i\|+\frac{\delta}{2}\|V_{i-1}\| \le\delta\max\{\|V_i\|,\|V_{i-1}\|\}$$

Therefore, when $\rho\in(r_{i-1}-\frac{\delta}{2},r_{i-1}+\frac{\delta}{2})$,

$$\left\|W(\rho)-W(r_{i-1}-\frac{\delta}{2})\right\|\le\delta\max\{\|V_{i-1}\|,\|V_i\|\}$$

Also, since $\frac{\rho-r_{i-1}}{\delta}+\frac{1}{2}\in(0,1)$,

$$\delta\int_0^{\frac{\rho-r_{i-1}}{\delta}+\frac{1}{2}}\|(1-\sigma)V_{i-1}+\sigma V_i\|d\sigma\le\delta\max\{\|V_{i-1}\|,\|V_i\|\}.$$

Due to (A7), $\left\|W(\rho)-\hat{W}(\rho)\right\|\le2\delta\max\{\|V_{i-1}\|,\|V_i\|\}$.

Let $\delta<\frac{\varepsilon}{2\delta\max\{\|V_{i-1}\|,\|V_i\|\}}$. Then $\left\|W(\rho)-\hat{W}(\rho)\right\|<\varepsilon$, $\rho\in(r_{i-1}-\frac{\delta}{2},r_{i-1}+\frac{\delta}{2})$, which is exactly (A3).

When $\rho\notin(r_{i-1}-\frac{\delta}{2},r_{i-1}+\frac{\delta}{2})$, (A4) does exist due to (A6).

When $\rho\in(r_{i-1}-\frac{\delta}{2},r_{i-1}+\frac{\delta}{2})$,

$\frac{d}{d\rho}\hat{W}(\rho)=[1-(\frac{\rho-r_{i-1}}{\delta}+\frac{1}{2})]V_{i-1}+(\frac{\rho-r_{i-1}}{\delta}+\frac{1}{2})V_i$ due to (A6).

Then, $\left\|\frac{d}{d\rho}\hat{W}(\rho)\right\|\le\max\{\|V_{i-1}\|,\|V_i\|\}=\max_{\substack{\rho\ne r_{i-1}\\ \rho\in(r_{i-1}-\frac{\delta}{2},r_{i-1}+\frac{\delta}{2})}}\left\|\frac{d}{d\rho}W(\rho)\right\|$.

Therefore, when $\rho\in[\rho_0,\rho_m]$,

$\left\|\frac{d}{d\rho}\hat{W}(\rho)\right\|\le\max_{\substack{\rho\ne r_{i-1}\\ i=1,\cdots,m+1}}\left\|\frac{d}{d\rho}W(\rho)\right\|$, which is exactly (A5).

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